Representative volume element approximations in elastoplastic spring networks

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In this manuscript we report on results obtained in [Hab+23; NV18] regarding stochastic homogenization of networks consisting of elastoplastic springs. The model under consideration comes in the form of an evolutionary rate-independent system (ERIS) with random (stationary and ergodic) coefficients. As usual in the homogenization of random heterogeneous media, the effective properties are defined via statistical averages that, in practice, need to be approximated. A common approach in mathematical and computational homogenization to tackle this difficulty is based on the method of the representative volume element (RVE) approximation, see, e.g., [Fey99; Mie02; MTY04; NW19; SH13; Wat+08]. In this approach an approximation of the effective properties is obtained by considering a large volume of the random material, the so-called RVE, and one expects that by ergodicity converges as the size of the RVE tends to infinity. Mathematically it is a challenging problem to prove the convergence of RVEs and to obtain optimal bounds on the rate of convergence. The latter is one of the main problems considered in the field of quantitative stochastic homogenization. In recent years, optimal bounds have mainly been found for linear elliptic PDEs (see, e.g., [AKM19; Fis18; GNO14; GNO15; GNO20; GO11; GO12]), including the system of linear elasticity with strongly correlated material properties [GNO21]. Only view quantitative results are available for nonlinear elliptic systems, such as [AS16; CG23; FN21].

The homogenization theory for ERIS, as considered in this report, is less developed: While qualitative stochastic homogenization results for various ERIS have been obtained in recent years [Hei17; HS18; NV18], the quantitative theory is still widely open. As we shall outline below, in [Hab+23] we establish qualitative convergence of the RVE approximation for an ERIS describing a network of elastoplastic springs, and we explore the convergence rates numerically.

In the following we give a brief description of some of the results obtained in [Hab+23; NV18]. Although the results are more general, to simplify the presentation we shall consider the simplest setting, namely a two-dimensional network with independent and identically distributed material properties: Let $(\mathbb{Z}^2, \mathsf{E})$ denote the two-dimensional, triangular lattice graph with edge set $\mathsf{E} = \{\mathsf{e} = (\underline{\mathsf{e}}, \overline{\mathsf{e}}) :, \overline{\mathsf{e}} - \underline{\mathsf{e}} \in \mathsf{E}_0\}$ where $\mathsf{E}_0 = \{e_1, e_2, e_3 := e_1 + e_2\}$. Each edge $\mathsf{e} \in \mathsf{E}$ represents an elastoplastic spring that connects the vertices $\underline{\mathsf{e}}$ and $\overline{\mathsf{e}}$, and each spring is specified by three material parameters $(a, h, \sigma_{\text{yield}})$ describing the elastic modulus, the hardening parameter and the yield stress of the spring. In fact, we assume that these material parameters are random variables, and we make the assumption that the random fields $(a(\mathsf{e}), h(\mathsf{e}), \sigma_{\text{yield}}(\mathsf{e}))_{\mathsf{e}\in\mathsf{E}}$ are independent and identically distributed. Furthermore, we assume ellipticity in the sense that there exists $0 < \lambda \leq \Lambda < \infty$ such that almost surely we have

$$\lambda \leq a(\mathsf{e}), h(\mathsf{e}) \leq \Lambda, \qquad 0 \leq \sigma_{\text{yield}}(\mathsf{e}) \leq \Lambda \qquad \text{for all } \mathsf{e} \in \mathsf{E}.$$

Let $O \subset \mathbb{R}^2$ be a bounded Lipschitz describing the macroscopic domain occupied by the rescaled lattice $(\varepsilon\mathbb{Z}^d, \varepsilon\mathsf{E})$ with lattice spacing $0 < \varepsilon \ll 1$. We describe a state of the lattice by $y = (u, p)$ where u denotes the displacement field and p the plastic strain. For simplicity we

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assume that the displacement vanishes at the boundary of O. A convenient way to implement this is to extend y to the entire lattice by setting it zero outside of O. This leads to the following state space for $y=(u,p)$,

$$Y_\varepsilon := L^2_0(\varepsilon\mathbb{Z}^2\cap O)^2 \times L^2_0(\varepsilon\mathsf{E}\cap O)$$

where $L^2_0(\varepsilon\mathbb{Z}^2\cap O)^2 = \{u:\varepsilon\mathbb{Z}^2\to\mathbb{R}^2 \,:\, u(x)=0 \text{ for } x\notin O\}$ and $L^2_0(\varepsilon\mathsf{E}\cap O)^2 = \{p:\varepsilon\mathsf{E}\to\mathbb{R} : p(\mathsf{e})=0$ for all e with $\{\underline{\mathsf{e}},\overline{\mathsf{e}}\}\cap O=\emptyset\}$. Y_ε equipped with the norm

$$\|(u,p)\|_{Y_\varepsilon} := \Big(\sum_{\varepsilon\mathbb{Z}^2}|u(x)|^2\sum_{\varepsilon\mathbb{Z}^2}|u(x)|^2+\sum_{\varepsilon\mathsf{E}}|p(\mathsf{e})|^2\Big)^{\frac12}$$

is a Hilbert space. For a displacement u and an edge $\mathsf{e}\in\varepsilon\mathsf{E}$ we define the discrete measure of strain $\partial u(\mathsf{e}) := \frac{u(\overline{\mathsf{e}})-u(\underline{\mathsf{e}})\cdot(\overline{\mathsf{e}}-\underline{\mathsf{e}})}{|\overline{\mathsf{e}}-\underline{\mathsf{e}}|^2}$. The evolution of the spring network is then modeled by a ERIS of the form

$$0\in\partial\mathcal{R}_\varepsilon(\dot y_\varepsilon(t))+D_y\mathcal{E}_\varepsilon(t,y_\varepsilon(t)),$$

where $y_\varepsilon\in W^{1,1}(0,T;Y_\varepsilon)$ describes the evolution of the state,

$$\mathcal{E}_\varepsilon(t,y) := \frac12\varepsilon^2\sum_{\mathsf{e}\in\varepsilon\mathsf{E}}\Big(a(\mathsf{e}/\varepsilon)\big(\partial u(\mathsf{e})-p(\mathsf{e})\big)^2+h(\mathsf{e}/\varepsilon)p(\mathsf{e})^2\Big)-\varepsilon^2\sum_{x\in\varepsilon\mathbb{Z}^2}l(t,x)\cdot u(x)$$

is an energy functional with loading l, and

$$\mathcal{R}_\varepsilon(\dot y) := \varepsilon^2\sum_{\mathsf{e}\in\varepsilon\mathsf{E}}\sigma_{\text{yield}}(\mathsf{e}/\varepsilon)|\dot p(\mathsf{e})|$$

is a dissipation functional. We understand the system $(Y_\varepsilon,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ as an ERIS in the spirit of [MR15] and appeal to the concept of energetic solutions.

In [NV18, Theorem 4.10] we prove that as $\varepsilon\to0$ the ERIS $(Y_\varepsilon,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ Γ-converges to a homogenized, continuum model that comes in the form of an ERIS with

- a state space

  $$Y_{\text{hom}} := H^1_0(O)^d\times(L^2(O;L^2(\Omega))^k\times L^2(O;L^2_s(\Omega)),\qquad y_{\text{hom}}=(u,p,\chi_s),$$

  where $(\Omega,\mathbb{P})$ denotes the probability space that is used to model the random coefficients $(a,h,\sigma_{\text{yield}})$ and $L^2_s(\Omega)$ denotes a space of corrector fields;

- an energy functional

  $$\mathcal{E}_{\text{hom}}(t,y_{\text{hom}}) := \frac12\int_O\mathbb{E}\big[A\begin{pmatrix}P_s\nabla u+\chi_s\\ p\end{pmatrix}\cdot\begin{pmatrix}P_s\nabla u+\chi_s\\ p\end{pmatrix}\big]-\int_O l(t,x)\cdot u(x)\,dx,$$

  where A denotes the random, symmetric 3×3-matrix defined by

  $$A\begin{pmatrix}\mathsf{d}\\ p\end{pmatrix}\cdot\begin{pmatrix}\mathsf{d}\\ p\end{pmatrix} := \sum_{i=1}^3\bar a\big((0,e_i)\big)(\mathsf{d}_i-p_i)^2+h\big((0,e_i)\big)p_i^2,\qquad \mathsf{d},p\in\mathbb{R}^3,$$

- a dissipation functional

  $$\mathcal{R}_{\text{hom}}(\dot y_{\text{hom}}) := \int_O\mathbb{E}[\rho(\omega,\dot p)]\,dx,$$

  where $\rho:\mathbb{R}^3\to\mathbb{R}$ denotes a random, positively-one-homogeneous density and is defined by

  $$\rho(v) := \sum_{i=1}^3\sigma_{\text{yield}}\big((0,e_i)\big)|v_i|.$$

We note that the state y_{hom} of the homogenized system consists of the displacement u, the plastic strain p, and additional variable χ_s. Both, p and χ_s are internal variables that describe the microscopic configuration of the network.

Let $y_{\text{hom}} = (u, p, \chi_s)$ denote the unique solution with initial condition $y_{\text{hom}}(0) = 0$ to the homogenized ERIS. In [Hab+23, Theorem 3.6] we prove that the displacement component u of the solution y_{hom} is the unique solution in $W^{1,1}(0,T; H_0^1(O))$ of the force balance equation

$$-\nabla \cdot \sigma_{\text{hom}}(t) = l(t) \qquad \text{in } O \text{ for a.e. } t \in (0,T)$$

where $\sigma_{\text{hom}}(t)$ is given by the hysteretic stress-strain relation

$$\sigma_{\text{hom}}(t) = \mathcal{W}_{\text{hom}}\Big[\operatorname{sym} \nabla u_{\text{hom}}\Big](t).$$

Here, $\mathcal{W}_{\text{hom}} : W^{1,1}(0,T; \mathbb{R}^{2\times 2}_{sym}) \to W^{1,1}(0,T; \mathbb{R}^{2\times 2}_{sym})$ is a generalized Prandtl-Ishlinskii hysteresis operator that describes the homogenized mechanical properties of the network. It itself is defined with help of an ERIS with a very large state space. Its size is comparable to the infinite product space $\otimes_{\mathsf{E}} L^2(\Omega_0, \mathbb{P}_0)$ where $(\Omega_0, \mathbb{P}_0)$ is the probability space that we use to model distribution of the triple $(a, h, \sigma_{\text{yield}})T$ for a single edge. In view of this, in practice, the hysteresis operator $\mathcal{W}_{\text{hom}}$ needs to be approximated. In [Hab+23, Section 4.2] we introduce an RVE-approximation for $\mathcal{W}_{\text{hom}}$: Roughly speaking, given a prescribed curve of strains $t \mapsto F(t) \in \mathbb{R}^{2\times 2}_{sym}$, we approximate $t \mapsto \mathcal{W}_{\text{hom}}[F](t)$ by considering the ERIS of the random network (with lattice spacing $\varepsilon = 1$) restricted to a large box of size $L \gg 1$ with boundary conditions that enforce that the strain differs from the prescribed F only by the strain of a periodic displacement. The spatial average of the stress tensor of the associated solution then yields an approximation $\mathcal{W}_L[F]$ for $\mathcal{W}_{\text{hom}}[F]$. In [Hab+23, Theorem 4.6] we prove that the RVE-approximation $\mathcal{W}_L$ converges to $\mathcal{W}_{\text{hom}}$ as $L \to \infty$.

In contrast to $\mathcal{W}_{\text{hom}}$, which is a deterministic operator, the approximation $\mathcal{W}_L$ is a random operator in the sense that different samples for the random field $(a(\mathsf{e}), h(\mathsf{e}), p(\mathsf{e}))_{\mathsf{e}\in\mathsf{E}}$ lead to a different $\mathcal{W}_L$. In [Hab+23, Section 6] we numerically explore the random fluctations of $\mathcal{W}_L$ as $L \to \infty$. We mainly focus on the case where $F(t)$ describes monotone uniaxial loading. Numerically we make the interesting observation that the variance of W_L decays with the rate of the central limit theorem. For the system of linear elasticity with material properties that feature a rapid decay of correlations, this behavior is well understood even on the level of rigorous proofs. However, it is an interesting open problem to rigorously understand the scaling behavior in the case of linear elastoplasticity.

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