On a critical fast diffusion stochastic equation with Stratonovich-type Brownian perturbation

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1 Introduction

In this talk, we consider a nonlinear diffusion process of the following form

$$dX\left(t,\xi\right)=\Delta_{\xi}\log\left(X\left(t,\xi\right)\right)dt \tag{1}$$

where X is the positive density for the time-space coordinates $\left(t,\xi\right)$. This equation describe the process that has been observed in several experiments in physics involving plasma [7][8]. Most of the natural phenomena exhibit variability which cannot be modeled by using deterministic approaches. More accurately, natural systems can be represented as stochastic models and the deterministic description can be considered as a particular case of the pertinent stochastic models. The propose of this work is to analyze such equations within the framework of stochastic evolution equations with multiplicative noise, where (1) is the underlying motivating example.

Let us now introduce the suitable framework for this problem. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis. We consider a Stratonovich stochastic partial differential equation in $\mathbb{R}^d$ with $d\geq 3$, of the form

$$\begin{cases} dX_t-\Delta\log X_t\,dt=\sigma(X_t)\circ dW_t, & [0,T]\times\mathbb{R}^d,\\ X_0=x, \end{cases} \tag{2}$$

where W_t is a $\dot{H}^{-1}(\mathbb{R}^d)$-valued cylindrical Wiener process, and the diffusion coffecient satisfies $\sigma(y)\in\mathscr{L}_2(Q^{\frac{1}{2}}\dot{H}^{-1}(\mathbb{R}^d);\mathbb{H})$ for $y\in\mathbb{H}$ where Q is a non negative trace class operator on $\dot{H}^{-1}(\mathbb{R}^d)$, and $\mathbb{H}=\{\dot{H}^{-1}(\mathbb{R}^d),H^{-1}(\mathbb{R}^d),L^2(\mathbb{R}^d)\}$. Here, the spaces $\dot{H}^{-1}(\mathbb{R}^d)$ and $H^{-1}(\mathbb{R}^d)$ are usual homogeneous and inhomogeneous Sobolev spaces respectively. We will briefly recall the definition of the spaces later.

The stochastic theory of nonlinear diffusion equations was, recently, intensively studied for the drift of the form $-\Delta\Psi$ where $\Psi:\ \mathbb{R}\rightarrow\mathbb{R}$ defined by $\Psi\left(r\right)=r^m$ is a maximal monotone operator with additive and multiplicative noise. In the case $m\geq 1$, the corresponding equation describes the slow diffusions (dynamics of fluids in porous media) and their existence, uniqueness and positivity and behavior of the solution have already been studied. The case $m \in (0,1)$ is relevant in the mathematical modeling of dynamics of an ideal gas in a porous media. The case $m \in (-1,0)$ describes the super fast diffusion (behavior of a cloud of electrons). For the case $m \leq -1$, it has been proved that, even in the deterministic case, there is no solution with finite mass (see [10]). For details about the previous results see [4] and the references therein.

The case

$$\Psi(r) = \log r$$

for positive solutions, can be seen as corresponding situations $m = 0$ since

$$\Delta(r^m) = m \operatorname{div}(r^{m-1}\nabla r)$$

and therefore, formally

$$\Delta(\log r) = \operatorname{div}(r^{-1}\nabla r).$$

Our work treats the fast logarithmic diffusion in an unbounded domain with a stochastic Stratonovich type noise, and provides an answer to the case $m = 0$ left as an open problem in [3].

2 Preliminaries and Main result

The space $H^1(\mathbb{R}^d)$ is the inhomogeneous Sobolev space on $\mathbb{R}^d$ (functions belonging, together with their first-order partial derivatives, to $L^2(\mathbb{R}^d)$). The space $\dot{H}^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ is the homogeneous Sobolev space of (real-valued) tempered distributions u over $\mathbb{R}^d$ having an $L^1_{loc}(\mathbb{R}^d)$ Fourier distribution $\hat{u}$ and such that

$$\|u\|_s^2 = \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi < \infty.$$

The properties of these spaces have been summarized in [1]. In particular, note that the space $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space provided $s < \frac{d}{2}$. For $|s| < \frac{d}{2}$, the spaces $\dot{H}^s(\mathbb{R}^d)$ and $\dot{H}^{-s}(\mathbb{R}^d)$ are dual:

$$\langle u, v\rangle_{(\dot{H}^s(\mathbb{R}^d), \dot{H}^{-s}(\mathbb{R}^d))} = \int_{\mathbb{R}^d} u(\xi)v(\xi)d\xi,$$

for any $(u,v) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{-s}(\mathbb{R}^d)$. Furthermore, the following continuous embeddings hold true for $0 \leq s < \frac{d}{2}$:

$$\dot{H}^s\left(\mathbb{R}^d\right) \subset L^{\frac{2d}{d-2s}}\left(\mathbb{R}^d\right), \qquad L^{\frac{2d}{d+2s}}\left(\mathbb{R}^d\right) \subset \dot{H}^{-s}\left(\mathbb{R}^d\right) \tag{3}$$

The space $H^{-1}(\mathbb{R}^d)$ is thus usually endowed with the norm

$$\|u\|^2_{H^{-1}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+|\xi|^2)^{-1}|\hat{u}(\xi)|^2 d\xi,$$

or $\|u\|^2_{H^{-1}(\mathbb{R}^d)} = \|(I-\Delta)^{-\frac{1}{2}}u\|^2_{L^2(\mathbb{R}^d)}$.

In order to mention our main result, we here prepare the space $C_{\mathbb{P}}([0,T];\dot{H}^s(\mathbb{R}^d))$ which denotes, for any $s\in\mathbb{R}$, the space of all $\dot{H}^s(\mathbb{R}^d)$-valued $(\mathscr{F}_t)_{t\geq 0}$ progressively measurable processes $X:\Omega\times[0,T]\to\dot{H}^s(\mathbb{R}^d)$ such that

$$\mathbb{E}\int_0^T \|X(t)\|^2_{\dot{H}^s}dt<+\infty,$$

and for all compacts $\mathscr{O}$, the realization of X on $\mathscr{O}$ has a continuous modification in $C([0,T];L^2(\Omega,\dot{H}^s(\mathscr{O})))$. $\mathbb{E}$ denotes the integration with respect to the probability measure $d\mathbb{P}$.

We shall consider a solution to the equation (2) in the sense of the definition below.

DEFINITION 1. *Fix any $T>0$. Let $x\in L^2\left(\mathbb{R}^d\right)\cap\dot{H}^{-1}(\mathbb{R}^d)$. An $\dot{H}^{-1}(\mathbb{R}^d)$-valued adapted process X is called strong solution to (2) if the following conditions hold*

- $X(t,\xi,\omega)>0$, *$dt\times d\xi\times d\mathbb{P}$-a.e. on* $[0,T]\times\mathbb{R}^d\times\Omega$,
- $X\in L^2\left(\Omega\times(0,T)\times\mathbb{R}^d;\mathbb{R}\right)\cap C_{\mathbb{P}}([0,T];\dot{H}^{-1}(\mathbb{R}^d))$,
- $\log\left(X\left(\cdot\right)\right)\in L^2\left([0,T]\times\Omega;\dot{H}^1(\mathbb{R}^d)\right)$, *and*

$$\langle X\left(t\right),e_k\rangle_{L^2}=\langle x,e_k\rangle_{L^2}-\int_0^t\int_{\mathbb{R}^d}\nabla\ln\left(X\right)\cdot\nabla e_k d\xi ds+\left\langle\int_0^t\sigma(X\left(s\right))\circ dW_s,e_k\right\rangle_{L^2}$$

for all $k\in\mathbb{N}$ and all $t\in[0,T]$ where $\{e_k\}_k$ is the orthonormal basis in $\dot{H}^{-1}(\mathbb{R}^d)$.

Remark. One can easily see that our definition of solution is one of the usual ones for porous media equation. It is a strong solution from the stochastic point of view and a weak solution from the PDE point of view.

We give some assumptions on the noise. Let us consider a sequence $\{\mu_k\}_{k\geq 1}\subset\mathbb{R}^{\mathbb{N}}_+$ such that

$$\sum_{k\geq 1}\mu_k(\|e_k\|^2_{L^\infty}+\|e_k\|^2_{L^d}+1)<\infty$$

where $\{e_k\}_k$ is an orthonormal basis for $\dot{H}^{-1}(\mathbb{R}^d)$ (a density argument applies for the finiteness of the above series.) To this sequence μ_k, we associate Q a non-negative trace class operator such that $Qe_k=\mu_k e_k$ on $\dot{H}^{-1}(\mathbb{R}^d)$. In (2), W_t is a $\dot{H}^{-1}(\mathbb{R}^d)$-valued cylindrical Wiener process :

$$W_t=\sum_{k\in\mathbb{N}}\beta_k(t)e_k,$$

where $\{\beta_k\}_k$ is a sequence of mutually independent Brownian motion on a probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_t,\mathbb{P})$. The noise coefficient $\sigma(x)$ depends linearly on $x\in\mathbb{H}\in\{\dot{H}^{-1},H^{-1},L^2\}$, and is defined by

$$\sigma(x)(Q^{\frac{1}{2}}u):=\sum_{k\in\mathbb{N}}\sqrt{\mu_k}\langle e_k,u\rangle_{\dot{H}^{-1}}e_k x.$$

This is well-defined (see our original paper [6]).

We state now the main result of this work.

Theorem For each $x\in L^2\left(\mathbb{R}^d\right)\cap\dot{H}^{-1}(\mathbb{R}^d)$ such that $x\ln x-x\in L^1\left(\mathbb{R}^d\right)$ and $x>0$ a.e. on $\mathbb{R}^d$, there is a unique positive solution X to (2).

Here, we note the important properties of the stochastic term to prove the theorem:

- Let $\mathbb{H} = \{\dot{H}^{-1}, H^{-1}, L^2\}$. Then

$$\mathbb{E}\left\|\int_0^t \sigma(X(s))dW(s)\right\|_{\mathbb{H}}^2 = \sum_k \mu_k \mathbb{E}\int_0^t \|X(s)e_k\|_{\mathbb{H}}^2 ds$$
$$\leq \left(\sum_k \mu_k \left(\|e_k\|_{L^\infty}^2 + C_{\dot{H}^1 \subset L^{\frac{2d}{d-2}}} \|\nabla e_k\|_{L^d}^2\right)\right)\mathbb{E}\int_0^t \|X(s)\|_{\mathbb{H}}^2 ds.$$

- The Stratonovich product may be written as Itô product with a correlation:

$$\sigma(X(t)) \circ dW(t) = \sigma(X(t))dW(t) + \frac{1}{2}(\sigma \otimes \sigma)(X(t))dt,$$

where

$$(\sigma \otimes \sigma)(x) := \sum_k \mu_k e_k^2 x, \quad x \in \mathbb{H}$$

which has the property

$$\|\sigma \otimes \sigma\|_{\mathscr{L}(\mathbb{H},\mathbb{H})}^2 \leq \sum_k \mu_k \|e_k\|_{L^\infty}^2 \left(\|e_k\|_{L^\infty}^2 + C_{\dot{H}^1 \subset L^{\frac{2d}{d-2}}} \|e_k\|_{L^d}^2\right).$$

3 Ideas

Our problem has several technical difficulties which will be treated by using several specific approximation with the parameters $\varepsilon, \nu, \lambda$. More precisely,

(i) The first main set of difficulties comes from the properties of the logarithm: We have a problem due to the fact that zero does not belong to $D(\log)$ and we can not assume that $D(\log) = \mathbb{R}$. Another problem which is specific to the logarithm diffusion is the fact that we can not assume any polynomial growth hypothesis, nor the strong monotonicity assumption. All those technical difficulties impose the choice of a particular form of the first approximation in the parameter λ and the use of a Stratonovich multiplicative noise. Namely, having fixed $\lambda > 0$, we set

$$\tilde{\Psi}_\lambda(r) := \Psi_\lambda(r) - \Psi_\lambda(0) + \lambda r,$$

for all $r \in \mathbb{R}$.

The first approximating equation is then as follows.

$$\begin{cases} dX_\lambda = \triangle \tilde{\Psi}_\lambda(X_\lambda)\ dt\ + \sigma(X_\lambda) \circ dW(t),\ t \geq 0; \\ X_\lambda(0) = x. \end{cases}$$

(ii) The second main set of difficulties comes from the unboundedness of the domain which does not allow us to use the Poincaré inequality. This technical problem impose the use of a second approximation. We introduce the operator $\triangle - \nu I$ for $\nu > 0$ and consider

$$\begin{cases} dX_{\lambda,\nu}(t) & = (\triangle - \nu I)\,\tilde{\Psi}_\lambda(X_{\lambda,\nu}(t))\ dt\ + \sigma(X_{\lambda,\nu}(t)) \circ dW(t),\ t \geq 0; \\ X_{\lambda,\nu}(0) & = x. \end{cases}$$

(iii) Finally set $A_\nu = (\triangle - \nu I)\,\tilde{\Psi}_\lambda$. The third approximation with Yoshida approximation $A_\nu^\varepsilon := \frac{1}{\varepsilon}\left(I - (I + \varepsilon A_\nu)^{-1}\right)$ $(\varepsilon > 0)$ is necessary to get some estimates, by Itô calculus, in appropriated spaces.

At the last, we remark that the important contribution of the Stratonovich noise is the following uniform bound.

$$\sup_{\lambda\in(0,1]} \|\Psi_\lambda(X_\lambda)\|^2_{L^2\left(\Omega\times[0,T];\dot{H}^1\left(\mathbb{R}^d\right)\right)} \leq C\left(1 + \|x\|^2_{L^2} + \int_{\mathbb{R}^d} (x\ln x - x)\,(\xi)\,d\xi\right).$$

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