Maximal L_p-L_q regularity for the heat equation with various boundary conditions in the half space

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Abstract

We prove resolvent L_p estimates and maximal L_p-L_q regularity estimates for the heat equation with Dirichlet, Neumann and Robin boundary conditions in the half space. Each solution is constructed by a Fourier multiplier of x'-direction and an integral of x_N-direction. We decompose the solution such that the symbols of the Fourier multipliers are bounded and holomorphic. We see that the operator norms are dominated by a homogeneous function of order -1 for x_N-direction. The basis are Weis's operator-valued Fourier multiplier theorem and a boundedness of a kernel operator.

Keywords : resolvent estimate, maximal regularity, heat equation.

1 Introduction

This paper is concerned with resolvent L_p estimates and maximal L_p-L_q regularity for the heat equation with three types of boundary conditions in the half-space with $1 < p, q < \infty$. The boundary conditions are Dirichlet, Neumann and Robin. The resolvent estimate is used for the generation of analytic semigroups, and the maximal regularity is used to solve quasi-linear evolution equations such as free boundary problems called Stefan problems. Let $\Omega \subset \mathbb{R}^N$ be a domain with three disjoint boundaries Γ_D, Γ_N and Γ_R. We allow that one or two of them are empty. We keep in mind the following linear problem;

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,\infty), \\ u = h_D & \text{on } \Gamma_D \times (0,\infty), \\ \partial_\nu u = h_N & \text{on } \Gamma_N \times (0,\infty), \\ \alpha u + \beta \partial_\nu u = h_R & \text{on } \Gamma_R \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases}$$

Here unknowns are u, while f, h_D, h_N, h_R and u_0 are given functions, $\partial_\nu = \frac{\partial}{\partial \nu} = \nu \cdot \nabla$ with the unit outward normal vector ν, and $\alpha, \beta > 0$. Note that the end-point case $(\alpha, \beta) = (1, 0)$ in Robin boundary condition implies Dirichlet boundary condition and the case $(\alpha, \beta) = (0, 1)$ implies Neumann boundary condition. Not only this non-stationary heat equation but also the following generalized resolvent problem are analyzed;

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega, \\ u = h_D & \text{on } \Gamma_D, \\ \partial_\nu u = h_N & \text{on } \Gamma_N, \\ \alpha u + \beta \partial_\nu u = h_R & \text{on } \Gamma_R. \end{cases}$$

This resolvent equation is derived from Laplace transform of the equation (1).

In this paper we do not treat the domain with curved boundaries so that the domain is the half-space. However the domain will be allowed more general domains like a bounded domain by cut-off techniques and localizations. We do not use such procedures since that is common and the analysis of the half space is the most important steps. Instead of them, we consider the problem with non-homogeneous data, which is a key to treat non-linear problems. After a reduction to $f = 0$, we consider the solution operator from boundary data h to the solutions u. Although these solutions are given by a Fourier multiplier of $h(x', 0)$ which is independent of x_N-variable, we shall use $h(x', x_N)$ by using an integral. We decompose the symbols of the solution operators into new symbols and new independent variables. Since the new symbol of the Fourier multiplier operator are bounded and holomorphic, we are able to use Fourier multiplier theorem with the connection to Mikhlin conditions. We confirm that the operator norm is dominated by a homogeneous function of order -1 in x_N. Therefore this shows resolvent estimates by a theorem proved in the paper [11]. Note that the new decomposed independent variables become suitable right-hand side of the generalized resolvent estimates. Moreover we also get maximal L_p-L_q regularity estimates by the same method. There are a lot of technical ideas to get the maximal regularity in the half-space. However we emphasize that we do not need such elaborate calculations. The basis is developed by a book [18] covering various subjects to harmonic analysis and the maximal regularity.

As previous works, we refer the paper by Shibata et al. [16, 27]. His method is based on a sufficient condition for $L_p(\mathbb{R}, X)$-boundedness of Fourier multiplier operators due to Weis [30] in terms of $\mathcal{R}$-bounded of the symbols under X is $\mathcal{HT}$ space. For the Stokes equations, there are a lot of results, e.g. for model problems with Neumann or free boundary conditions [24, 25, 27], Robin conditions [22, 28], two-phase problems [26]. Our method has already used for the Stokes equations with various boundary conditions [11, 12] in the half space. Recently we proved the same results for the layer domain, which is applied for the Stokes equations with Dirichlet-Neumann boundary condition in [13], Neumann-Neumann boundary condition in [14], and for the heat equation with various boundary conditions in [15].

The structure of the paper is as follows. First we introduce some notations and state our main theorems in section 2. Then, in section 3, we prepare some known results. Since the equations are inhomogeneous, we transform the equation into homogeneous except for boundary data h. This is as usual and is stated in section 4. In section 5, we solve the equations in the half space by partial Fourier transforms. Three types of boundary conditions are treated similarly. The solution formula is Fourier multiplier type concerned with $e^{-\sqrt{\lambda+|\xi'|^2}x_N}$. From so called Volevich's trick, the solutions are given by an integral form whose integrands are Fourier multiplier operators which act h and $\partial_N h$. In the last section 6, we prove the main theorem. We decompose the symbols while paying attention to the desired estimates. Resolvent estimate is straightforward from the theorem prepared in section 3 and the estimates of $e^{-\sqrt{\lambda+|\xi'|^2}x_N}$ with complex variables. Maximal regularity estimates are also same as resolvent estimates by the prepared sufficient condition.

2 Main theorem

We formulate the resolvent and the non-stationary problems in the half-space. Let $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ be the half-space and its flat boundary and let Q_+ and Q_0 be the corresponding time-space domain;

$$\mathbb{R}^N_+ := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \quad \mathbb{R}^N_0 := \{x = (x', 0) = (x_1, \dots, x_{N-1}, 0) \in \mathbb{R}^N\},$$
$$Q_+ := \mathbb{R}^N_+ \times (0, \infty), \qquad Q_0 := \mathbb{R}^N_0 \times (0, \infty).$$

The resolvent problem is as follows;

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \mathbb{R}^N_+, \\ \alpha u - \beta \partial_N u = h & \text{on } \mathbb{R}^N_0. \end{cases} \tag{1}$$

Here $\partial_N = \frac{\partial}{\partial x_N}$ and $\alpha, \beta \geq 0$ ($(\alpha, \beta) \neq (0, 0)$). The case $(\alpha, \beta) = (1, 0)$ implies Dirichlet, and the case $(\alpha, \beta) = (0, 1)$ implies Neumann.

The non-stationary problem is as follows;

$$\begin{cases} \partial_t U - \Delta U = F & \text{in } Q_+, \\ \alpha U - \beta \partial_N U = H & \text{on } Q_0. \end{cases} \tag{2}$$

Given a domain D, Lebesgue and Sobolev spaces are denoted by $L_q(D)$ and $W^m_q(D)$ with norms $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W^m_q(D)}$. Same manner is applied in the X-valued spaces $L_p(\mathbb{R}, X)$ and $W^m_p(\mathbb{R}, X)$. For a scalar function f, we use the following symbols;

$$\nabla f = (\partial_1 f, \dots, \partial_N f), \quad \nabla^2 f = (\partial_i \partial_j f \mid i, j = 1, \dots, N).$$

Even though $\mathbf{g} = (g_1, \dots, g_{\tilde{N}}) \in X^{\tilde{N}}$ for some $\tilde{N}$, we use the notations $\mathbf{g} \in X$ and $\|\mathbf{g}\|_X$ as $\sum_{j=1}^{\tilde{N}} \|g_j\|_X$ for simplicity. Namely, we use e.g. $\|(f, \nabla f, \nabla^2 f)\|_{L_q(D)} = \sum_{\alpha \in \mathbb{N}^N_0, |\alpha| \leq 2} \|\partial^\alpha_x f\|_{L_q(D)}$.

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse;

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}[g](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)d\xi.$$

Although we usually consider time interval $\mathbb{R}_+$, we regard functions on $\mathbb{R}$ to use Fourier transform. To do so and to consider Laplace transforms as Fourier transforms, we introduce some function spaces;

$$L_{p,0,\gamma_0}(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbb{R}, X), \ f(t) = 0 \text{ for } t < 0\},$$
$$W^m_{p,0,\gamma_0}(\mathbb{R}, X) := \{f \in L_{p,0,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t} \partial^j_t f(t) \in L_p(\mathbb{R}, X), \ j = 1, \dots, m\},$$

for some $\gamma_0 \geq 0$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}_\lambda$ denote two-sided Laplace transform and its inverse, defined as

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)dt = \mathcal{F}_{t\to\tau}[e^{-\gamma t} f](\tau),$$
$$\mathcal{L}^{-1}_\lambda[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\lambda)d\tau = e^{\gamma t} \mathcal{F}^{-1}_{\tau\to t}[g(\gamma + i\cdot)](t),$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s > 0$ and X-valued function f, we use the following Bessel potential spaces to treat fractional orders;

$$H^s_{p,0,\gamma_0}(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid \Lambda^s_\gamma f := \mathcal{L}^{-1}_\lambda[|\lambda|^s \mathcal{L}[f](\lambda)](t) \in L_{p,0,\gamma}(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_0\}.$$

Let $\Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \varepsilon\}$. We are ready to state our main results.

Theorem 2.1 (resolvent L_q estimate)**.** *Let $0<\varepsilon<\pi/2$ and $1<q<\infty$. Then for any $\lambda\in\Sigma_\varepsilon$,*

$$f\in L_q(\mathbb{R}^N_+),\quad h\in\begin{cases}W^2_q(\mathbb{R}^N_+) \text{ if } \beta=0,\\ W^1_q(\mathbb{R}^N_+) \text{ if } \beta>0,\end{cases}$$

problem (1) *admits a unique solution $u\in W^2_q(\mathbb{R}^N_+)$ with the resolvent estimate;*

$$\|(\lambda u,\lambda^{1/2}\nabla u,\nabla^2 u)\|_{L_q(\mathbb{R}^N_+)}\le\begin{cases}C\|(f,\lambda h,\lambda^{1/2}\nabla h,\nabla^2 h)\|_{L_q(\mathbb{R}^N_+)} & \text{if } \beta=0,\\ C\|(f,\lambda^{1/2}h,\nabla h)\|_{L_q(\mathbb{R}^N_+)} & \text{if } \beta>0\end{cases}$$

for some constants $C=C_{N,q,\varepsilon,\alpha,\beta}$.

Theorem 2.2 (maximal L_p-L_q estimate)**.** *Let $1<p,q<\infty$ and $\gamma_0\ge 0$. Then for any*

$$F\in L_{p,0,\gamma_0}(\mathbb{R},L_q(\mathbb{R}^N_+)),\quad H\in\begin{cases}W^1_{p,0,\gamma_0}(\mathbb{R},L_q(\mathbb{R}^N_+))\cap L_{p,0,\gamma_0}(\mathbb{R},W^2_q(\mathbb{R}^N_+)) & \text{if } \beta=0,\\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R},L_q(\mathbb{R}^N_+))\cap L_{p,0,\gamma_0}(\mathbb{R},W^1_q(\mathbb{R}^N_+)) & \text{if } \beta>0,\end{cases}$$

problem (2) *with $U_0=0$ admits a unique solution $U\in W^1_{p,0,\gamma_0}(\mathbb{R},L_q(\mathbb{R}^N_+))\cap L_{p,0,\gamma_0}(\mathbb{R},W^2_q(\mathbb{R}^N_+))$ with the maximal L_p-L_q regularity;*

$$\|e^{-\gamma t}(\partial_t U,\gamma U,\Lambda_\gamma^{1/2}\nabla U,\nabla^2 U)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))}\le\begin{cases}C\|e^{-\gamma t}(F,\partial_t H,\Lambda_\gamma^{1/2}\nabla H,\nabla^2 H)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} & \text{if } \beta=0,\\ C\|e^{-\gamma t}(F,\Lambda_\gamma^{1/2}H,\nabla H)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} & \text{if } \beta>0\end{cases}$$

for any $\gamma\ge\gamma_0$ with some constants $C=C_{N,p,q,\gamma_0,\alpha,\beta}$.

3 Preliminaries

In this section we prepare some theorems which are used later. First we recall a theorem regarding the generation of analytic semigroups.

Theorem 3.1 ([16, Theorem 3.1.8])**.** *Let $1<q<\infty$, $D(A)\subset W^2_q(\Omega)$ be a subspace, $D(A)\subset L_q(\Omega)$ be dense, $A:D(A)\to L_q(\Omega)$ is a linear operator satisfying $\|Au\|_{L_q(\Omega)}\le C\|u\|_{W^2_q(\Omega)}$ for all $u\in D(A)$ and the following resolvent estimate hold; there exists $0<\varepsilon<\pi/2$ and $C>0$ such that $\Sigma_\varepsilon\subset\rho(A)$ and for any $\lambda\in\Sigma_\varepsilon$, $f\in L_q(\Omega)$, we have*

$$\|(\lambda u,\lambda^{1/2}\nabla u,\nabla^2 u)\|_{L_q(\Omega)}\le C\|f\|_{L_q(\Omega)},\quad u:=(\lambda-A)^{-1}f(\in D(A)).$$

Then the operator A generates (C_0)-semigroup $\{T(t)\}_{t\ge0}$ on $L_q(\Omega)$, which satisfies, by letting $u(t):=T(t)u_0$ for $u_0\in L_q(\Omega)$,

$$\begin{aligned}&u(t)\in C^1((0,\infty),L_q(\Omega))\cap C^0((0,\infty),D(A))\cap C^0([0,\infty),L_q(\Omega)),\\ &u'(t)=Au(t)\ (t>0),\\ &\|(tu'(t),u,t^{1/2}\nabla u,t\nabla^2 u)\|_{L_q(\Omega)}\le C\|u_0\|_{L_q(\Omega)},\\ &\|u'(t)\|_{L_q(\Omega)}\le C\|u_0\|_{W^2_q(\Omega)}.\end{aligned}$$

Moreover, $\{T(t)\}_{t\ge0}$ can be analytically extended to a sector $\Sigma_{\pi/2+\varepsilon}$ and

$$T(t)T(s)=T(s)T(t)=T(t+s)\ (t,s\in\Sigma_{\pi/2+\varepsilon}),\quad \lim_{\Sigma_{\pi/+\varepsilon}\ni t\to0}\|T(t)u_0-u_0\|_{L_q(\Omega)}=0.$$

Following this theorem, theorem 2.1 derives the analytic semigroup $\{T(t)\}_{t\geq 0}$ on $L_q(\mathbb{R}^N_+)$ whose generator A has the domain $D(A) := \{u \in W^2_q(\mathbb{R}^N_+) \mid \alpha u - \beta \partial_N u = 0 \text{ on } \mathbb{R}^N_0\}$ by setting $h = 0$. This just solves the heat equations with various boundary condition since we are able to take $\alpha, \beta \geq 0$ $((\alpha, \beta) \neq (0, 0))$.

For non-trivial initial data, we have the following lemma.

Theorem 3.2 ([16, Lemma 3.2.1])**.** *Let $1 < p, q < \infty$. Then for any $u_0 \in (X, D(A))_{1-1/p,p}(\subset B^{2(1-1/p)}_{q,p}(\Omega))$, $u(t) = T(t)u_0$ satisfies*

$$\|(u', \nabla^2 u)\|_{L_p(0,\infty,L_q(\Omega))} \leq C\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.$$

This theorem implies that it is enough to consider zero-initial data for maximal regularity theorem.

Next, we consider some sufficient conditions to get L_q estimate and L_p-L_q estimate. We begin with a Fourier multiplier theorem on the whole space. Let $\tilde{\Sigma}_\eta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \eta\} \cup \{z \in \mathbb{C} \setminus \{0\} \mid \pi - \eta < |\arg z|\}$ for $\eta \in (0, \pi/2)$.

Theorem 3.3 ([18, Theorem 4.3.9, Proposition 4.3.10])**.** *Let $m : \tilde{\Sigma}^N_\eta \to \mathbb{C}$ be bounded and holomorphic for some $0 < \eta < \pi/2$ then the Fourier multiplier $\mathcal{F}^{-1}m\mathcal{F}$ is a bounded linear operator on $L_q(\mathbb{R}^N)$ for any $1 < q < \infty$.*

We prepare a theorem to prove the main theorems concerning the half space. This gives an easy way to show a boundedness of an operator. Let us difine the operators T and $\tilde{T}_\gamma$ by

$$\begin{aligned} T[m]f(x) &= \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m(\xi', x_N + y_N)\mathcal{F}_{x'}f](x, y_N)dy_N, \\ \tilde{T}_\gamma[m_\lambda]g(x,t) &= \mathcal{L}^{-1}_\lambda \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m_\lambda(\xi', x_N + y_N)\mathcal{F}_{x'}\mathcal{L}g](x, y_N, \lambda)dy_N, \\ &= [e^{\gamma t}\mathcal{F}^{-1}_{\tau\to t} T[m_\lambda]\mathcal{F}_{t\to\tau}(e^{-\gamma t}g)](x,t), \end{aligned}$$

where $\lambda = \gamma + i\tau \in \Sigma_\varepsilon$, $m, m_\lambda : \mathbb{R}^N_+ \to \mathbb{C}$ are multipliers, and $f : \mathbb{R}^N_+ \to \mathbb{C}$ and $g : \mathbb{R}^N_+ \times \mathbb{R} \to \mathbb{C}$.

Theorem 3.4 ([11, Theorem 6.1])**.** (i) *Let m satisfy the following two conditions:*
(a) *There exists $\eta \in (0, \pi/2)$ such that $\{m(\cdot, x_N), x_N > 0\} \subset H^\infty(\tilde{\Sigma}^{N-1}_\eta)$.*
(b) *There exist $\eta \in (0, \pi/2)$ and $C > 0$ such that $\sup_{\xi' \in \tilde{\Sigma}^{N-1}_\eta} |m(\xi', x_N)| \leq Cx_N^{-1}$ for all $x_N > 0$.*
Then $T[m]$ is a bounded linear operator on $L_q(\mathbb{R}^N_+)$ for every $1 < q < \infty$.
(ii) *Let $\gamma_0 \geq 0$ and let m_λ satisfy the following two conditions:*
(c) *There exists $\eta \in (0, \pi/2 - \varepsilon)$ such that for each $x_N > 0$ and $\gamma \geq \gamma_0$,*

$$\tilde{\Sigma}^N_\eta \ni (\tau, \xi') \mapsto m_\lambda(\xi', x_N) \in \mathbb{C}$$

is bounded and holomorphic.
(d) *There exist $\eta \in (0, \pi/2 - \varepsilon)$ and $C > 0$ such that $\sup\{|m_\lambda(\xi', x_N)| \mid (\tau, \xi') \in \tilde{\Sigma}^N_\eta\} \leq Cx_N^{-1}$ for all $\gamma \geq \gamma_0$ and $x_N > 0$.*
Then $\tilde{T}_\gamma[m_\lambda]$ satisfies

$$\|e^{-\gamma t}\tilde{T}_\gamma g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} \leq C\|e^{-\gamma t}g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))}$$

for every $\gamma \geq \gamma_0$ and $1 < p, q < \infty$.

4 Reduction to the problem only with boundary data

In this section we show that it is enough to consider the case $f = 0$ or $F = 0$ by subtracting solutions of inhomogeneous data.

4.1 Whole space

We start considering with the whole space problem

$$\lambda u - \Delta u = f \qquad \text{in } \mathbb{R}^N, \tag{3}$$
$$\partial_t U - \Delta U = F \quad \text{in } \mathbb{R}^N \times (0,\infty) \tag{4}$$

subject to the initial condition $U(x,0) = 0$. The following theorem is prepared.

Theorem 4.1. *Let $1 < p, q < \infty, 0 < \varepsilon < \pi/2$ and $\gamma_0 \geq 0$.*
(1) *For any $\lambda \in \Sigma_\varepsilon, f \in L_q(\mathbb{R}^N)$, problem* (3) *admits a unique solution $u \in W_q^2(\mathbb{R}^N)$ that satisfies the following estimates:*

$$\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\mathbb{R}^N)} \leq C_{N,q,\varepsilon}\|f\|_{L_q(\mathbb{R}^N)}.$$

(2) *For any $F \in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^N))$, problem* (4) *admits a unique solution*

$$U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^N)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\mathbb{R}^N)))$$

that satisfies the estimate:

$$\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C_{N,p,q,\gamma_0}\|e^{-\gamma t}F\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))}\}$$

for any $\gamma \geq \gamma_0$.

The proof is given in appendix.

4.2 Half space

For $f \in L_q(\mathbb{R}^N_+)$, let f^o be odd extension to $\mathbb{R}^N$, given by

$$f^o(x) := \begin{cases} f(x) & \text{for } x_N > 0, \\ -f(x', -x_N) & \text{for } x_N < 0. \end{cases}$$

We have $f^o \in L_q(\mathbb{R}^N)$. The function $v := \mathcal{F}^{-1}(\lambda + \sum_{j=1}^N \xi_j^2)^{-1}\mathcal{F}(f^o)$ belongs to $W_q^2(\mathbb{R}^N)$ and solves heat equation $(\lambda - \Delta)v = f^o$ in $\mathbb{R}^N$. We see $v|_{\mathbb{R}^N_0} = 0$ as follows;

$$v(x', 0) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'}\left(\int_{-\infty}^{\infty}(\lambda + \sum_{j=1}^N \xi_j^2)^{-1}\mathcal{F}_x[f^o](\xi)d\xi_N\right)d\xi'$$

$$\mathcal{F}_{x'}[v|_{x_N=0}](\xi') = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}(\lambda + \sum_{j=1}^N \xi_j^2)^{-1}\mathcal{F}_x[f^o](\xi)d\xi_N$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-iy_N\xi_N}(\lambda+\sum_{j=1}^N \xi_j^2)^{-1}[\mathcal{F}_{x'}f^o](\xi',y_N)dy_N\right)d\xi_N\\
&= \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} (e^{-iy_N\xi_N}-e^{iy_N\xi_N})(\lambda+\sum_{j=1}^N \xi_j^2)^{-1}[\mathcal{F}_{x'}f](\xi',y_N)dy_N\right)d\xi_N\\
&= \frac{1}{(2\pi)^N}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} (e^{-iy_N\xi_N}-e^{iy_N\xi_N})(\lambda+\sum_{j=1}^N \xi_j^2)^{-1}d\xi_N\right)[\mathcal{F}_{x'}f](\xi',y_N)dy_N\\
&= 0
\end{aligned}
$$

since $\xi_N \mapsto (e^{-iy_N\xi_N}-e^{iy_N\xi_N})(\lambda+\sum_{j=1}^N \xi_j^2)^{-1}$ is the odd function. Similarly, $V|_{\mathbb{R}^N_0}=0$ hold for non-stationary problems with zero-initial value.

Moreover we see, for $\gamma \ge \gamma_0 \ge 0$,

$$
\begin{aligned}
&\|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\mathbb{R}^N)} \le C\|f^o\|_{L_q(\mathbb{R}^N)} \le 2C\|f\|_{L_q(\mathbb{R}^N_+)},\\
&\|e^{-\gamma t}(\partial_t V, \gamma V, \Lambda_\gamma^{1/2}\nabla V, \nabla^2 V)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \le C\|e^{-\gamma t}F^o\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \le 2C\|e^{-\gamma t}F\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))}.
\end{aligned}
$$

Setting $u=v+w$ in (1), and $U=V+W$ in (2) with $U_0=0$, respectively, we would like to find the solutions w and W of

$$
\begin{cases}
\lambda w - \Delta w = 0 & \text{in } \mathbb{R}^N_+,\\
\alpha w - \beta\partial_N w = h + \beta\partial_N v =: \bar{h} & \text{on } \mathbb{R}^N_0.
\end{cases} \tag{5}
$$

and

$$
\begin{cases}
\lambda W - \Delta W = 0 & \text{in } Q_+,\\
\alpha W - \beta\partial_N W = H + \beta\partial_N V =: \bar{H} & \text{on } Q_0.
\end{cases} \tag{6}
$$

Here we have $(\bar{h},\bar{H})=(h,H)$ when $\beta=0$, and

$$
\begin{aligned}
\|(\lambda^{1/2}\bar{h},\nabla\bar{h})\|_{L_q(\mathbb{R}^N_+)} &\le \|(\lambda^{1/2}h,\nabla h)\|_{L_q(\mathbb{R}^N_+)} + \beta\|(\lambda^{1/2}\partial_N v, \nabla\partial_N v)\|_{L_q(\mathbb{R}^N_+)}\\
&\le \|(\lambda^{1/2}h,\nabla h)\|_{L_q(\mathbb{R}^N_+)} + C\|f\|_{L_q(\mathbb{R}^N_+)},\\
\|e^{-\gamma t}(\Lambda_\gamma^{1/2}\bar{H},\nabla\bar{H})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} &\le \|e^{-\gamma t}(\Lambda_\gamma^{1/2}H,\nabla H)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} + C\|e^{-\gamma t}F\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))}
\end{aligned}
$$

when $\beta>0$.

In this section we conclude that $f=0$ and $F=0$ are enough to consider in theorems 2.1 and 2.2.

5 Solution formulas from boundary data

We give the solution of the resolvent problem (1) with $f=0$ and $\lambda\in\Sigma_\varepsilon$ by Fourier multipliers for each boundary condition. We apply partial Fourier transform with respect to tangential direction $x'\in\mathbb{R}^{N-1}$ so that we use the notations

$$
\hat{v}(\xi',x_N) := \mathcal{F}_{x'}v(\xi',x_N) := \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'}v(x',x_N)dx',
$$

$$\mathcal{F}^{-1}_{\xi'} w(x', x_N) = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} w(\xi', x_N) d\xi'$$

for functions $v, w : \mathbb{R}^N_+ \to \mathbb{C}$. We use $A := \sqrt{\sum_{j=1}^{N-1} \xi_j^2}$ and $B := \sqrt{\lambda + A^2}$ with positive real parts.

By partial Fourier transform, we have the following second order ordinary differential equations;

$$\begin{cases} (B^2 - \partial_N^2)\hat{u} = 0 & \text{in } x_N > 0, \\ \alpha\hat{u} - \beta\partial_N \hat{u} = \hat{h} & \text{on } x_N = 0. \end{cases}$$

The solution is of the form

$$\hat{u}(\xi', x_N) = \frac{e^{-Bx_N}}{\alpha + \beta B}\hat{h}.$$

Let $\phi_{\alpha,\beta}(\lambda, \xi', x_N) = (\alpha+\beta B)^{-1} e^{-Bx_N}$, which derives the solution formula $u(x) = [\mathcal{F}^{-1}_{\xi'} \phi_{\alpha,\beta} \mathcal{F}_{x'} h](x)$. In the next step, we use the Volevich trick $a(\xi', 0) = -\int_0^\infty \partial_N a(\xi', y_N) dy_N$ for a suitable decaying function a. We obtain the solution formula;

$$\begin{aligned} u(x) = -\Bigg\{ &\int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[(\partial_N \phi_{\alpha,\beta}(\lambda, \xi', x_N + y_N)) \mathcal{F}_{x'} h \right](x, y_N) dy_N \\ &+ \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\phi_{\alpha,\beta}(\lambda, \xi', x_N + y_N) \mathcal{F}_{x'} (\partial_N h) \right](x, y_N) dy_N \Bigg\}. \end{aligned}$$

Since Laplace-transformed non-stationary heat equations (2) with $F = 0$ on $\mathbb{R}$ are the resolvent problem (1), we have the following formula;

$$\begin{aligned} U(x,t) = -\mathcal{L}^{-1}_\lambda \Bigg\{ &\int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[(\partial_N \phi_{\alpha,\beta}(\lambda, \xi', x_N + y_N)) \mathcal{F}_{x'} \mathcal{L} H \right](x, y_N, \lambda) dy_N \\ &+ \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\phi_{\alpha,\beta}(\lambda, \xi', x_N + y_N) \mathcal{F}_{x'} \mathcal{L}(\partial_N H) \right](x, y_N, \lambda) dy_N \Bigg\}. \end{aligned}$$

6 Proof of resolvent estimates and maximal regularity estimates

In the previous section, we obtained the solution formula. We use the following identity;

$$B^2 = \lambda + \sum_{m=1}^{N-1} \xi_m^2, \qquad 1 = \frac{B^2}{B^2} = \frac{\lambda^{1/2}}{B^2}\lambda^{1/2} - \sum_{m=1}^{N-1} \frac{i\xi_m}{B^2}(i\xi_m).$$

We consider two cases; one is $\beta = 0$, and the other is $\beta > 0$. We decompose the solution so that the independent variables become the right-hand side of the estimates;
For the case $\beta = 0$, we consider as follows;

$$\begin{aligned} u(x) = -\Bigg\{ &\int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[B^{-2} \partial_N \phi_{\alpha,0}(\lambda, \xi', x_N + y_N) \mathcal{F}_{x'}((\lambda - \Delta')h) \right](x, y_N) dy_N \\ &+ \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[\lambda^{1/2} B^{-2} \phi_{\alpha,0}(\lambda, \xi', x_N + y_N) \mathcal{F}_{x'}(\lambda^{1/2} \partial_N h) \right](x, y_N) dy_N \\ &- \sum_{m=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \left[i\xi_m B^{-2} \phi_{\alpha,0}(\lambda, \xi', x_N + y_N) \mathcal{F}_{x'}(\partial_m \partial_N h) \right](x, y_N) dy_N \Bigg\}. \end{aligned}$$

For the case $\beta > 0$, we consider as follows;

$$u(x) = -\left\{\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}B^{-2}\partial_N\phi_{\alpha,\beta}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}h)\right](x,y_N)dy_N \right.$$
$$-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_m B^{-2}\partial_N\phi_{\alpha,\beta}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m h)\right](x,y_N)dy_N$$
$$\left. +\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\phi_{\alpha,\beta}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_N h)\right](x,y_N)dy_N\right\}.$$

Let $S_u(\lambda,\xi',x_N)$ be any of symbols;

$$S_u(\lambda,\xi',x_N) := \begin{cases} B^{-2}\partial_N\phi_{\alpha,0}(\lambda,\xi',x_N) & \text{or,}\\ \lambda^{1/2}B^{-2}\phi_{\alpha,0}(\lambda,\xi',x_N) & \text{or,}\\ i\xi_m B^{-2}\phi_{\alpha,0}(\lambda,\xi',x_N) & \text{or,}\\ \lambda^{1/2}B^{-2}\partial_N\phi_{\alpha,\beta}(\lambda,\xi',x_N) & \text{or,}\\ i\xi_m B^{-2}\partial_N\phi_{\alpha,\beta}(\lambda,\xi',x_N) & \text{or,}\\ \phi_{\alpha,\beta}(\lambda,\xi',x_N). \end{cases}$$

We are able to prove that all of the symbols are bounded in the sense that

$$\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde{\Sigma}_\eta^{N-1}\\ \ell,\ell'=1,\dots,N-1}}\left\{(|\lambda|+|\lambda|^{1/2}|\xi_\ell|+|\xi_\ell||\xi_{\ell'}|)|S_u|+(|\lambda|^{1/2}+|\xi_\ell|)|\partial_N S_u|+|\partial_N^2 S_u|\right\}$$
$$< Cx_N^{-1} \tag{7}$$

for any $0<\varepsilon<\pi/2$ and $0<\eta<\min\{\pi/4,\varepsilon\}$ because of the estimates

Lemma 6.1 ([11, Lemma 6.3]). *Let $0<\varepsilon<\pi/2$, $0<\eta<\min\{\pi/4,\varepsilon/2\}$ and $m=0,1,2,3$. Then for any $(\lambda,\xi',x_N)\in\Sigma_\varepsilon\times\tilde{\Sigma}_\eta^{N-1}\times(0,\infty)$, letting $A:=\sqrt{\sum_{j=1}^{N-1}\xi_j^2}$, $B:=\sqrt{\lambda+A^2}$ and $\tilde{A}:=\sqrt{\sum_{j=1}^{N-1}|\xi_j|^2}$, we have*

$$c\tilde{A}\le \operatorname{Re}A\le|A|\le\tilde{A}, \tag{a}$$
$$c(|\lambda|^{1/2}+\tilde{A})\le\operatorname{Re}B\le|B|\le|\lambda|^{1/2}+\tilde{A}, \tag{b}$$
$$c(\alpha+\beta(|\lambda|^{1/2}+\tilde{A}))\le|\alpha+\beta B|\le\alpha+\beta(|\lambda|^{1/2}+\tilde{A}), \tag{c}$$
$$|\partial_N^m e^{-Bx_N}|\le C(|\lambda|^{1/2}+\tilde{A})^m e^{-c(|\lambda|^{1/2}+\tilde{A})x_N}\le C(|\lambda|^{1/2}+\tilde{A})^{-1+m}x_N^{-1} \tag{d}$$

with positive constants c and C, which are independent of λ,ξ',x_N.

The inequality (7) corresponds to the estimates λu, $\lambda^{1/2}\partial_\ell u$, $\partial_\ell\partial_{\ell'}u$, $\lambda^{1/2}\partial_N u$, $\partial_\ell\partial_N u$ and $\partial_N^2 u$ respectively.

We also see that the new symbol S_u, multiplied λ, ξ_ℓ and ∂_N, are holomorphic in $(\tau,\xi')\in\tilde{\Sigma}_\eta^N$. Therefore we are able to use theorem 3.4.

Theorem 6.2. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then for any $\lambda \in \Sigma_\varepsilon$, $h \in \begin{cases} W_q^2(\mathbb{R}_+^N) & \text{if } \beta = 0, \\ W_q^1(\mathbb{R}_+^N) & \text{if } \beta > 0, \end{cases}$ problem (1) with $f = 0$ admits a solution $u \in W_q^2(\mathbb{R}_+^N)$ with the resolvent estimate;*

$$\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\mathbb{R}_+^N)} \leq \begin{cases} C\|(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h)\|_{L_q(\mathbb{R}_+^N)} & \text{if } \beta = 0, \\ C\|(\lambda^{1/2} h, \nabla h)\|_{L_q(\mathbb{R}_+^N)} & \text{if } \beta > 0 \end{cases}$$

for some constant $C = C_{N,q,\varepsilon,\alpha,\beta}$.

This theorem and the estimates in section 4 derive the existence part of theorem 2.1. The uniqueness is proved by a duality argument. For any $\psi \in C_0^\infty(\mathbb{R}_+^N)$, take $v \in W_q^2(\mathbb{R}_+^N)$ by

$$\begin{cases} \lambda v - \Delta v = \psi & \text{in } \mathbb{R}_+^N, \\ \alpha v - \beta \partial_N v = 0 & \text{on } \mathbb{R}_0^N. \end{cases}$$

Let u be a solution of (1). We see

$$\begin{aligned} \int_{\mathbb{R}_+^N} u\psi dx &= \int_{\mathbb{R}_+^N} u(\lambda v - \Delta v) dx \\ &= \int_{\mathbb{R}_+^N} (\lambda u - \Delta u) v dx + (v\partial_N u - u\partial_N v)|_{\mathbb{R}_0^N} \quad \text{(integration by parts)} \\ &= 0. \end{aligned}$$

By fundamental lemma of calculus of variations, this shows $u = 0$, which implies the solution is unique.

For the non-stationary problem, we have, by theorem 3.4 again,

Theorem 6.3. *Let $1 < p, q < \infty$ and $\gamma_0 \geq 0$. Then for any*

$$H \in \begin{cases} W_{p,0,\gamma_0}^1(\mathbb{R}, L_q(\mathbb{R}_+^N)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\mathbb{R}_+^N)) & \text{if } \beta = 0, \\ H_{p,0,\gamma_0}^{1/2}(\mathbb{R}, L_q(\mathbb{R}_+^N)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\mathbb{R}_+^N)) & \text{if } \beta > 0, \end{cases}$$

problem (2) with $F = 0$ and time interval $\mathbb{R}$ admits a unique solution U such that

$$U \in W_{p,0,\gamma_0}^1(\mathbb{R}, L_q(\mathbb{R}_+^N)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^2(\mathbb{R}_+^N))$$

with the maximal L_p-L_q regularity;

$$\begin{aligned} &\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} \\ &\leq \begin{cases} C\|e^{-\gamma t}(\partial_t H, \Lambda_\gamma^{1/2}\nabla H, \nabla^2 H)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} & \text{if } \beta = 0 \\ C\|e^{-\gamma t}(\Lambda_\gamma^{1/2} H, \nabla H)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))} & \text{if } \beta > 0. \end{cases} \end{aligned}$$

for any $\gamma \geq \gamma_0$ with some constant $C = C_{N,p,q,\gamma_0,\alpha,\beta}$.

Appendix A Proof of Theorem 4.1; estimate for the whole space

Proof. Let $\mathbf{A} = \sqrt{\sum_{j=1}^N \xi_j^2}, \mathbf{B} := \sqrt{\lambda + \mathbf{A}^2}$. We have $u = \mathcal{F}^{-1}\mathbf{B}^{-2}\mathcal{F}f$. Functions $\lambda\mathbf{B}^{-2}$, $\lambda^{1/2}\mathbf{A}\mathbf{B}^{-2}$, $\mathbf{A}^2\mathbf{B}^{-2}$ correspond to the symbols of left-hand sides, which are holomorphic and bounded by Lemma 6.1. Therefore we can use Theorem 3.3. This proves the first estimate. For the non-stationary problem, the solution is $U = \mathcal{L}_\lambda^{-1}\mathcal{F}^{-1}\mathbf{B}^{-2}\mathcal{F}\mathcal{L}F = e^{\gamma t}\mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)}\mathbf{B}^{-2}\mathcal{F}_{(t,x)\to(\tau,\xi)}(e^{-\gamma t}F)$. The symbol $(\tau, \xi) \mapsto \lambda\mathbf{B}^{-2}$ is holomorphic and bounded. Therefore we have $\|e^{-\gamma t}\partial_t U\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \leq C\|e^{-\gamma t}F\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))}$ by [16, Proposition 4.2.1] and [18, Proposition 4.3.10]. The others are same. □

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