A Brief Introduction to Inter-Universal Geometry I

§1. The Inter-Universal Geometry of Categories
§2. A Survey of Absolute Anabelian Geometry

§1. The IU Geometry of Categories

§1.1. Motivation:

ABC Conjecture

⇝

scheme theory / $\mathbb{Z}$ insufficient;
need 'geometry / $\mathbb{F}_1$'

⇝

need 'global Hodge theory' / no. flds.
(cf. Hodge-Arakelov theory)

⇝ (technical obstacle)

must solve 'membership equation' $a \in a$

impossible in conventional set theory, by the axiom of foundation

⇒ must extend conventional set theory!

§1.2. Resolution via IU geometry: in a word, use labels obtained by extending the universe in question

cf. $a \in a$ ⇝ loop

⇝

+1 acts.

$a_4 \in \ldots$
$\parallel$
$a_3 \in \{a_3, b_3\}$
$\parallel$
$a_2 \in \{a_2, b_2\}$
$\parallel$
$a_1 \in \{a_1, b_1\}$

then identify
the a_i ⇝ a
the b_i ⇝ b
by forming a quotient

(a sort of 'limit')

$\boxed{\text{IU geom.}}$ $\boxed{\text{usual set th.}}$

a sort of analysis — algebra (approximations of the limit)

$\mathbb{F}_p[t^{1/p^\infty}]$ (perfection) ... $\mathbb{F}_p[t^{1/p^n}]$ ...

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§1.3. The Fundamental Theorem of the IU Geom. of Categories

> Thm: One may construct a geometry as above by considering categories / equiv.

Why categories?

... because $\boxed{\text{they are 1-dim.}}$
" 'most primitive units'

... by contrast, $\boxed{\text{rings are 2-dim.}}$
$\boxplus$, $\boxtimes$
addition multiplication

... The ABC conj. concerns the relationship betw. these 2 dims.

Ex.: M comm. monoid $\ni 1$

$\rightsquigarrow \mathcal{C}_M$: obj: $*$
mor: $\mathrm{End}(*) = M$

.... similarly, a ring may be represented as a 2-cat. of 1-cats.

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§2. A Survey of Absolute Anabelian Geometry:

Let. K: char. 0 fld.
X_K: hyperbolic curve $/K$
↳ (compact genus g) − (r pts.) s.t. $2g-2+r>0$

⇓

$$\boxed{1 \to \underset{\underset{\mathrm{Ker}(\ )}{\shortparallel}}{\Delta_X} \to \underset{\underset{\pi_1(X_K)}{\shortparallel}}{\Pi_{X_K}} \to G_K = 1}$$

('anabelian geometry'
" representing schemes via Galois cats.
" a special case of IU geom.)

relative theory: '$/G_K$'

Ex: $\mathrm{Aut}_{G_K}(\Pi_{X_K}) \overset{?}{\cong} \mathrm{Aut}_K(X_K)$

absolute theory: 'not nec. $/G_K$'

§2.1. The Case of No. Flds.: '$K/\mathbb{Q} < \infty$' ; Neukirch-Uchida Thm. $\Rightarrow$ $\boxed{\text{rel.} \Leftrightarrow \text{abs.}}$

Thm. (NU) $\mathrm{Isom}(\overline{K}/K, \overline{K}'/K') \xrightarrow{\sim} \mathrm{Isom}(G_K, G_{K'})$

Cor (NU + Tamagawa, M) $\mathrm{Isom}_{\mathbb{Q}}(X_K, X'_{K'}) \xrightarrow{\sim} \mathrm{OutIsom}(\Pi_{X_K}, \Pi_{X'_{K'}})$

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§2.2. The Case of p-adic Local Flds.: $\boxed{K/\mathbb{Q}_p < \infty}$

- analogue of NU: false
- analogue of 'Cor.': unknown (but expected to be false!)
  $\parallel$
  'abs pGC'

Rmk: By a lemma of Tamagawa, the quotient $\Pi_{X_K} \twoheadrightarrow G_K$ is characteristic.

On the other hand, the rel. pGC

Thm (M): $\mathrm{Isom}_K(X_K, X'_K) \xrightarrow{\sim} \mathrm{OutIsom}_{G_K}(\Pi_{X_K}, \Pi_{X'_K})$

(Invent. Math. '99)

admits an absolute interpretation:

$$\mathrm{Loc}_K(X_K) := \left\{ \begin{array}{l} \text{obj: } Y \text{ s.t. } \exists \text{ fin. et. } Y \to X_K \\ \text{mor: } Y_1 \to Y_2 \text{ fin. et. (not nec'ly } /X_K\text{!)} \end{array} \right\} \text{ cat.}$$

Cor: $\forall\ \Pi_{X_K} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X'_{K'}}$ induces a functorial (in α) equiv. of cats.:

$$\boxed{\mathrm{Loc}_K(X_K) \xrightarrow{\sim} \mathrm{Loc}_{K'}(X'_{K'})}$$

In particular, the arithmeticity of X_K is completely determined by the isom. class of the profinite gp. Π_{X_K}.

('whether or not X_K is isogenous to a Shimura curve via fin. et. morphisms')

§2,3. One important analogy:

Over p-adic flds	Over $\mathbb{F}_p((t))$
treating Π_{X_K} absolutely	working with $X_{\mathbb{F}_p((t))}$ not over $\mathbb{F}_p((t))$, but over $\mathbb{F}_p$! $\Downarrow$ One only sees properties that are invariant w.r.t. coordinate transformations $t \mapsto t + (?)t^2 + \ldots$ (Cf. the ABC conj.!)

this suggests $\Longrightarrow$ One should be able to recover the following Π_{X_K}:

(i) the special fiber of X_K

(ii) X_K itself, whenever X_K is 'constant'

Indeed,

$$\boxed{\text{Thm (M via T+...)} \quad \forall \Pi_{X_K} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X'_{K'}} \xrightarrow{\text{functorially in } \alpha} \boxed{X_k^{\log} \xrightarrow{\sim} (X'_{k'})^{\log}}}$$

($X_K, X'_{K'}$: stable reduction) (an isom. betw. the log special fibers) (RIMS Preprint 1363)

Thm: (M) $\forall \Pi_{X_K} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X'_{K'}} \rightsquigarrow$ (i) X_K can. lift $\Leftrightarrow$ $X'_{K'}$ can. lift (in the sense of p-adic Teich. theory!)

(K, K': abs. unram.; $p>5$)

(ii) if can. lift, then the above isom. of log. special fibers lifts to a (unique) $\boxed{X_K \xrightarrow{\sim} X'_{K'}}$ (RIMS Preprint 1379)

Defn: X_K absolute:

$\forall X'_{K'}$ s.t. $\Pi_{X_K} \overset{\exists}{\xrightarrow{\sim}} \Pi_{X'_{K'}}$,

$\exists X_K \xrightarrow{\sim} X'_{K'}$

$$\boxed{\text{Cor: For } p>5, \text{ the 'abs. curve pts.' are Zariski dense in } \mathcal{M}_{g,r}(\overline{\mathbb{Q}}_p).}$$

Rmk: (i) This is the first genuine application of p-adic Teich. theory.

(ii) cf. 'can. lifts. of abel. vars. are CM': $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C} \circlearrowleft \mathrm{Aut}$. $\leftsquigarrow$ consider such Auts $\Updownarrow$ consider Π_{X_K} absolutely

A Brief Introduction to Inter-Universal Geometry II

§1. Categories of Arbitrary Arithmetic Log Schemes
§2. Categories of Multiplicative Localizations
§3. Global Multiplicative Subspaces
§4. Distributed Versions.

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§1. Categories of Arbitrary Arith. Log Schemes:

anabelian geom.: only applies to very special schemes
$\Downarrow$
want to do IU geom. with more general schemes

X noetherian scheme

$Sch(X) := \begin{cases} \text{obj: } Y \to X \text{ fin. type morphism} \\ \text{mor: } Y_1 \to Y_2 \text{ ... morphism of } X\text{-schemes} \end{cases}$

$\boxed{\text{Thm. (M) } Isom(X, X') \xrightarrow{\sim} Isom(Sch(X), Sch(X'))}$ (RIMS Preprint 1364) cf. "Grothendieck Conj."

eq. of cats./isom.

..... also log version for fine, saturated log schemes X^{log}

"arith. (log) schemes": R: $\mathbb{Z}$ or $\mathbb{Q}$
X: fin. type $/R$.

Defn: arch. str. on X: $H_X \subseteq X(\mathbb{C})$ s.t. H_X compact, stabilized by complex conj.

..... $\overline{X} = (X, H_X)$: "arith. sch."

$\overline{Sch}(\overline{X}) = \begin{cases} \text{obj: } \overline{Y} = (Y, H_Y) \to \overline{X}: Y \to X \text{ fin. type, } H_Y \to H_X \\ \text{mor: } \overline{Y}_1 \to \overline{Y}_2 : Y_1 \to Y_2 \text{ fin. type}/X,\ H_{Y_1} \to H_{Y_2} \end{cases}$ (cat.)

$\boxed{\text{Thm: (M) } Isom(\overline{X}, \overline{X}') \xrightarrow{\sim} Isom(\overline{Sch}(\overline{X}), \overline{Sch}(\overline{X}'))}$

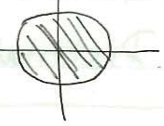
Rmk: (i) Thus, can treat arch. primes in IU geom. (unlike anab. geom.!)

(ii) $\exists$ log version via spaces of Kato-Nakayama "$\mathbb{N} \rightsquigarrow \mathbb{S}^1$"

Ex: $F/\mathbb{Q} < \infty$, $\overline{\mathcal{L}}$: arith. l.b. / Spec $(\mathcal{O}_F)$ (i.e., equipped with Herm. metrics at arch. primes)

$\Downarrow$

$V \to S = \mathrm{Spec}(\mathcal{O}_F)$ (geom. l.b.), $V^{\log}$: log str. from zero section

$\overline{V}^{\log}$: arch. str. from Herm. metric $|-| \leq 1$ $\longrightarrow$ 

$\Downarrow$

$$\boxed{\mathrm{Isom}(\overline{V}^{\log}, (\overline{V}')^{\log}) \xrightarrow{\sim} \mathrm{Isom}(\overline{\mathrm{Sch}}^{\log}(\overline{V}^{\log}), \overline{\mathrm{Sch}}^{\log}((\overline{V}')^{\log}))}$$

('the isom. class of $\overline{\mathcal{L}}$ has been represented cat.-theoretically')

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§2. Categories of Mult. Locs.: $F/\mathbb{Q} < \infty$, $G = \mathrm{Gal}(\widetilde{F}/F) \twoheadleftarrow G_F = \mathrm{Gal}(\overline{F}/F)$

$\Downarrow$

'pro-arith. log. sch.' $\overline{S}_F^{\log}$: (Spec $\mathcal{O}_F$ + log. str. at all closed pts. + arch. str. = {all arch. primes})

$\mathrm{Loc}_G(\overline{S}_F^{\log})$: $\begin{cases} \cdot\ \widetilde{F} \supseteq L/F < \infty : \overline{S}_L^{\log} \rightsquigarrow \text{'global objs.'} \\ \cdot\ \text{Zariski localizations of gl. objs. at nonarch., arch. primes} \\ \cdot\ \mathcal{O}_F\text{-morphisms} \qquad \rightsquigarrow \text{'local objs.'} \end{cases}$

$\mathrm{Loc}_G^{\times}(\overline{S}_F^{\log})$: $\begin{cases} \cdot\ \text{arith. l.b. } V \text{ (cf. Ex) over objs. } T \text{ of } \mathrm{Loc}_G(\overline{S}_F^{\log}) \\ \cdot\ \mathcal{O}_F\text{-morphisms (of arith. log schemes)} \Rightarrow \text{morphisms of the form} \end{cases}$

....+ other inessential details $\qquad$ ('$T \to c\, T^n$')

(T: geom. l.b. coord.)

$\Downarrow$

By considering 'morphisms of Frob. type'

i.e., 'c is invertible' $\begin{cases} V \longrightarrow W \\ T^n \longleftarrow T \end{cases}$ $\rightsquigarrow$ $\boxed{\begin{array}{c} \mathrm{Loc}^{\times} \xrightarrow{\overline{\Phi}} \mathrm{Loc}^{\times} \\ V \longmapsto W \\ \text{'Frobenius functor'} \end{array}}$

$\varinjlim$ (mors. of Frob. type) $\rightsquigarrow$ perfection $\mathrm{Loc}^{\times \otimes \mathbb{Q}}$ } cf. '$\mathbb{Q}$-, $\mathbb{R}$-divisors'
then 'completing w.r.t. valuation' $\rightsquigarrow$ realification $\mathrm{Loc}^{\times \mathbb{R}\mathbb{R}}$

(also intermediate forms $\mathrm{Loc}^{\times \mathbb{Z}\mathbb{Q}}$, $\mathrm{Loc}^{\times \mathbb{Z}\mathbb{R}}$, etc.: perfect or realify only global objs)

Then Frob. functor $\rightsquigarrow$ $\xrightarrow{\sim}$ on perfections, realifs.

Main Thm: $\deg_{ar}$(gl. objs.) , $\deg_{ar}$(change in integral str. of local objs.) $\in \mathbb{R}$ } category-theoretic! (for $\mathrm{Loc}^{\times}$)

$$\boxed{\deg_{ar}(\Phi(-)) = n \cdot \deg_{ar}(-)}$$ (Frob. functor assoc'd to 'n').

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§3. Global Mult. S/spaces: S: Riemann surface of fin. type (= compact \ fin. set)

$$\begin{array}{c} E \\ \downarrow \\ S \end{array} \Big\} \text{ family of ell. curves} \rightsquigarrow \begin{array}{ccc} \tilde{S} & \to & \tilde{\mathcal{M}}_{\mathbb{C}} \cong \mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{M}_{\mathbb{C}} \end{array} = \{\text{moduli stack of ell. curves}/\mathbb{C}\}$$

$\rightsquigarrow$

$\mathcal{E}$: local system/S, fiber $\cong \mathbb{Z}^2$

want an arith. analogue

$$\boxed{\text{'Fundamental Thm. of Hodge Theory}/\mathbb{C}\text{': } \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_S \supseteq \omega_E}$$

rank 1 subbundle

we want an arith. analogue of this rank 1 sub.

$\Downarrow$

the closest well-known arith. analogue occurs in the case of Tate curves

$\mathcal{E}|_{\tilde{S}} \cong \mathbb{Z}^2 \Rightarrow$ rank 1 sub. is a varying rank 1 subspace of $\mathbb{C}^2$

$\Rightarrow$ classifying morphism may be thought of as the resulting:

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & \tilde{\mathcal{M}}_{\mathbb{C}} \cong \mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{M}_{\mathbb{C}} \end{array}$$

$\mathbb{C}^\times \to E = \mathbb{C}^\times / q^{\mathbb{Z}} \rightsquigarrow$ $\boxed{0 \to \mu_n \to E[n] \to \mathbb{Z}/n\mathbb{Z} \to 0}$

exists over $\mathbb{Z}[[q]]$, i.e. at bad mult. primes for ell. curves/no. flds. $\Rightarrow$ we wish to globalize this rank 1 sub.

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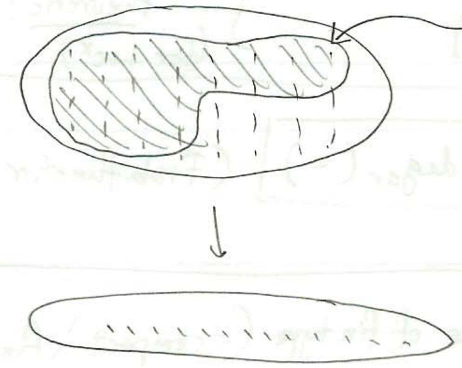
§4. Distributed Versions: $F/\mathbb{Q} < \infty$

E : ell. curve/F, $K := F(E[d])$

K



F

extract only the bad primes where some chosen rank 1 sub. is mult.

$\Downarrow$

'distributed versions of $\mathrm{Loc}^\times$'

$\mathrm{LOC}[\underline{\Delta}^+]$ } primes of K over a single prime of F treated independently

'UI'

$\mathrm{Loc}^\times[\underline{\Delta}]$ } linked

$\downarrow ?$

$\mathrm{Loc}^\times$ } the usual category of §2, for F

cf. 'IU geom.'

Here, a global mult. sub. exists tautologically!!

Moreover,

cf. B_{crys}!

(i) Over $\mathrm{LOC}^\times[\underline{\Delta}^+]$, one has:
- $\deg_{ar}$(global obj's.) $\rightsquigarrow$ one can do 'Arakelov theory'
- Frobenius functor Φ
- Galois $\rightsquigarrow$ one can do 'Gal. theory'

(ii) $\mathrm{LOC}^\times[\underline{\Delta}^+]$ is related to $\mathrm{Loc}^\times$ via $\mathrm{Loc}^\times[\underline{\Delta}]$

$\boxed{\text{Thus, over the 'geom. obj.' (cf. IU geom.) } \mathrm{LOC}^\times[\underline{\Delta}^+]\text{, we have constructed a 'classifying morphism}/\mathbb{F}_1\text{' '}\mathrm{LOC}^\times[\underline{\Delta}^+] \to \mathcal{M}_{\mathbb{F}_1}\text{'}}$