Absolute Anabelian Cuspidalizations

§1. Maximal Cuspidally Abelian Extensions

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§1. Maximal Cuspidally Abelian Extensions

$p^{\dagger} := p = \mathrm{char}(k)$, $d_k = 1$ $\qquad$ $p^{\dagger} := p = \mathrm{res.char.}(k)$, $d_k = 2$

X: proper hyperbolic curve / field k (finite or nonarch. local)

$\cup\!\mid$

U: open subscheme

$X \setminus S$, $S \subseteq X(k)$ $\nwarrow$ for simplicity $\rightsquigarrow$

$$\begin{tikzcd}
 & & \pi_1(U) \arrow[d, equal] & \mathrm{Gal}(\overline{k}/k) \arrow[d, equal] & \\
1 \arrow[r] & \Delta_U \arrow[r] \arrow[d, two heads] & \Pi_U \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_X \arrow[r] & \Pi_X \arrow[r] \arrow[d, equal] & G_k \arrow[r, equal] & 1 \\
 & & \pi_1(X) & &
\end{tikzcd}$$

$$\boxed{\text{Fundamental Question: To what extent can one reconstruct } \Pi_U \text{ from } \Pi_X\text{?}}$$

(Note: 'converse' $\Pi_U \rightsquigarrow \Pi_X$ well-known over finite, nonarch. loc. fields)

First approximation: $I_U := \mathrm{Ker}(\underbrace{\Pi_U \twoheadrightarrow \Pi_X}_{\text{'cuspidal subgroup'}}) \subseteq \Pi_U \twoheadrightarrow \underbrace{\Pi_U^{c\text{-}cn}}_{} \twoheadrightarrow \Pi_X$

'maximal cuspidally (geometrically) central quotient'

— i.e., '$I_U^{ab} / \circlearrowleft \Delta_X$'

fin.: $\hat{\mathbb{Z}} \twoheadrightarrow \hat{\mathbb{Z}}^{\dagger} \xrightarrow{\sim} \hat{\mathbb{Z}}'$

nonarch.loc.: $\hat{\mathbb{Z}} \xrightarrow{\sim} \hat{\mathbb{Z}}^{\dagger} \twoheadrightarrow \hat{\mathbb{Z}}'$

$\Downarrow$

$$\boxed{1 \to \bigoplus_{x \in S} \hat{\mathbb{Z}}^{\dagger}(1) \to \Pi_U^{c\text{-}cn.} \to \Pi_X \to 1}$$

$\Updownarrow$

$\forall x \in S,\quad c_1(x) \in H^2(\Pi_X, \hat{\mathbb{Z}}^{\dagger}(1))$

$\qquad\qquad \|$

$\left(c_1(\mathrm{diag.}) \in H^2(\Pi_{X \times X}, \hat{\mathbb{Z}}^{\dagger}(1))\right)\Big|_{\underbrace{D_x}_{\text{decomp. gp. of } x} \times_{G_k} \Pi_X}$

('group-theoretic' / fin. flds. by Tamagawa)

Note: $c_1(\mathrm{diag})$: 'absolutely gp.-th.' by Poincaré duality: dual of

$H^{2+d_k}(\Pi_{X \times X}, \hat{\mathbb{Z}}^{\dagger}(d_k))$

$\xrightarrow{(\mathrm{diag})^*} H^{2+d_k}(\Pi_X, \hat{\mathbb{Z}}^{\dagger}(d_k))$

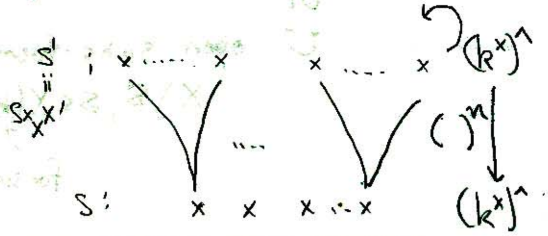
'maximal cuspidally abelian quotient' ; (I_U^{ab})

Note that
$$\begin{tikzcd} \Pi_U \arrow[r, two heads] & \Pi_U^{c\text{-}ab} \arrow[r, two heads] \arrow[d, "\wr"] & \Pi_X \\ & \varprojlim_{\substack{X' \to X \\ \text{fin. et. Gal.}}} \left(\Pi_{U'}^{c\text{-}cn} \overset{\mathrm{out}}{\rtimes} \mathrm{Gal}(X'/X) \right) & \end{tikzcd}$$
(where $U' := U \times_X X'$)

$\Rightarrow$ $\boxed{\text{reconstruct } \Pi_U^{c\text{-}ab}.}$

Moreover, unlike $\Pi_U^{c\text{-}cn} \circlearrowright H^1(X, \hat{\mathbb{Z}}^{\dagger}(1)) \simeq (k^{\times})^{\wedge}$,

$\Pi_U^{c\text{-}ab}$ has remarkable rigidity ; :)



$n = \deg(X'_{\bar{k}} / X_{\bar{k}})$

For instance : $\alpha \in \mathrm{Aut}(\Pi_U^{c\text{-}ab})$

s.t. (i) α preserves each $\hat{\mathbb{Z}}^{\dagger}(1)$ of $\bigoplus_S \hat{\mathbb{Z}}^{\dagger}(1)$

(ii) α preserves, induces id. on $\Pi_U^{c\text{-}ab} \twoheadrightarrow \Pi_X$

$\Rightarrow$ α cuspidally inner

Once one has $\Pi_U^{c\text{-}ab}$, one can reconstruct.

$$\boxed{\Gamma(U, \mathcal{O}_U^{\times}) \cdot (k^{\times})^{\wedge}} \underset{\text{Kummer}}{\hookrightarrow} H^1_{et}(U, \hat{\mathbb{Z}}^{\dagger}(1)) \simeq H^1(\Pi_U, \hat{\mathbb{Z}}^{\dagger}(1))$$
$$\simeq H^1(\Pi_U^{c\text{-}ab}, \hat{\mathbb{Z}}^{\dagger}(1))$$
$$\simeq H^1(\Pi_U^{c\text{-}cn}, \hat{\mathbb{Z}}^{\dagger}(1)),$$

Hence, by enlarging S, taking $\varinjlim$, can reconstruct.

$$\boxed{K \cdot (k^{\times})^{\times}}$$

function field of X.

§2. Green's trivializations:

$$\underbrace{\Pi_{U_{X\times X}}}_{\substack{\shortparallel \\ X\times X\setminus(\text{diag})}} \twoheadrightarrow \Pi^{\text{c-cn.}}_{U_{X\times X}} \twoheadrightarrow \Pi_{X\times X}; \quad 1 \to \hat{\mathbb{Z}}^{\dagger}(1) \to \overset{\overset{\mathfrak{D}}{\shortparallel}}{\Pi^{\text{c-cn.}}_{U_{X\times X}}} \to \Pi_{X\times X} \to 1.$$

$$\rightsquigarrow c_1(\text{diag}) \in H^2(X\times X, \hat{\mathbb{Z}}^{\dagger}(1))$$

$$\Downarrow$$

$$\forall x,y \in X(k) \text{ s.t. } x \neq y, \quad G_k \xrightarrow{\sim} D_x \times_{G_k} D_y \hookrightarrow \Pi_{U_{X\times X}} \twoheadrightarrow \mathfrak{D} \to G_k.$$

$$\rightsquigarrow \begin{tikzcd} \mathfrak{D}_{x,y} \arrow[d, two heads] \\ G_k \arrow[u, bend left, dashed, "\gamma_{x,y}"] \end{tikzcd} := \mathfrak{D}\Big|_{D_x \times_{G_k} D_y \subseteq \Pi_{X\times X}}$$

Symmetric in x,y $\Big\{$ **Green's trivialization** (cf. Green's fns. of Arak. theory)

$\rightsquigarrow$ by multiplying, $\forall$ divisor D on X,

$$\begin{tikzcd} \mathfrak{D}_{x,D} \arrow[d, two heads] \\ G_k \arrow[u, bend left, "\gamma_{x,D}"] \end{tikzcd}$$

Tautology: To evaluate $f \in K^{\times}$ at x, nec. & suff. to know $\gamma_{x,\text{div}(f)}$

($\because$ pull-back via $\text{Supp}(\text{div}(f)) \overset{\shortparallel}{} S$

$$\begin{tikzcd} \mathfrak{D}_{x,S} := \prod\limits_{y\in S} \mathfrak{D}_{x,y} \arrow[d, two heads] \arrow[r, phantom, "\simeq"] & \Pi^{\text{c-cn.}}_{U_S := X\setminus S}\Big|_{D_x} \arrow[d, two heads] \\ \mathfrak{D}_{x,\text{div}(f)} \arrow[r, two heads] & G_k \arrow[l, bend left, dashed, "\gamma_{x,\text{div}(f)}"] \end{tikzcd}$$

(over G_k))

fin. flds: Tamagawa
Belyi type/nonarch. loc. flds; RIMS Preprint 1473
(uses rel. p-adic GC. !!)

Corollary: $\Pi_X \overset{\alpha}{\xrightarrow{\sim}} \Pi_Y$ point-theoretic (= preserves decomp. gps.)
('Groth. Conj.') & Green-compatible (= preserves Green's trivs.)

$\uparrow$ trivial in affine case

$$\Downarrow$$

α arises geometrically

X, Y hyp. curves/fin. or nonarch. loc. fld.

$\exists$ only ctbly many examples! – cf. canonical curves of p-adic Teich. theory!

Example: U of Belyi type: defined/no. flds. & isogenous to genus 0 hyp. curve.

(= '$\exists$ common. fin. et. covering')

(nonarch. loc. case!) $\Rightarrow$ $\forall\, \Pi_U \xrightarrow{\sim} \Pi_V$ arises geometrically ('absolute p-adic G.C.' !!)

§3. Maximally cuspidally Pro-ℓ Extensions over Finite Fields

denote by

$$\begin{tikzcd} \Pi_U^{\leq\infty} \arrow[r, two heads] & \Pi_X \\ \Pi_U \supseteq I_U \arrow[u, two heads] & \end{tikzcd}$$

Ker = max. pro-ℓ quotient of I_U

(k = fin. fld.)
$\ell \neq \mathrm{char}(k)$

Thm: $U := X \setminus \{x\}$, $x \in X(k)$ $\Rightarrow$ can reconstruct $\Pi_U^{\leq\infty}$ from Π_X !

Idea of pf: (pro-ℓ gp) $\hookrightarrow$ ($\mathbb{Q}_\ell$-Lie alg) $\circlearrowleft$ Frob.

⇝ Lie. alg. splits

⇝ can reconstruct Lie alg.

⇒ remains to reconstruct integral str. = Im(pro-ℓ gp)

.... via similar technique to '$(k^\times)^\wedge \xrightarrow{(\ )^n} (k^\times)^\wedge$'

— i.e., 'obstruction to integrality from large covering'

$I(n\cdot) = 0$!

------ 'small' ...

Cor: Every $\Pi_X \xrightarrow{\sim} \Pi_Y$ is point-theoretic (Tamagawa) & Green-compatible.

Pf: $\underset{X \setminus \{x\}}{\underset{\parallel}{\Pi_U^{\leq\infty}}} \simeq \underset{Y \setminus \{y\}}{\underset{\parallel}{\Pi_V^{\leq\infty}}}$ point-theoretic (Tamagawa), but decomp. gps. of $\Pi_U^{\leq\infty}$, $\Pi_V^{\leq\infty}$ determine ℓ-portion of Green's trivs. !!

Cor: GC for proper hyp. curves / fin. flds. holds.