Anabelian Geometry from an Inter-universal Point of View I

§1. Not 'what' but 'why'
§2. The Membership Equation
§3. The Inter-universal Geometry of Categories
§4. Analogy with (Uniformizing) $\mathcal{MF}^{\nabla}$-objects

§1. Not 'what' but 'why':

What is anab. geom.?

connected scheme $X \rightsquigarrow \pi_1(X)$ (étale fund. gp.: SGA1)

... To what extent / for which X ('anabelian'), can one $\pi_1(X) \rightsquigarrow X$ (reconstruct)?

Why does one care? ... not so clear, but influences fundamental direction of research

... i.e., without an answer to 'why?') $\rightsquigarrow$ tends to degenerate into a meaningless contest of stoicness (e.g., smaller portions of π_1, more general X)

Original Apparent Motivation (Grothendieck? Deligne?)

Diophantine Geometry: e.g., Mordell Conjecture

X: smooth, proper curve, genus ≥ 2 / number field F; $G_F := \mathrm{Gal}(\overline{F}/F)$; $\Pi_X := \pi_1(X)$

$X(F) \ni x_1, \dots, x_n, \dots \rightsquigarrow s_1, \dots, s_n, \dots \quad \begin{matrix} \Pi_X \\ \downarrow \\ G_F \end{matrix}$ ... $\exists$ subsequence converges $\to s_\infty: G_F \to \Pi_X$

note: theme (analysis / global fields)

$\Rightarrow$ if 'Section Conjecture' holds, then s_∞ arises from $\exists x_\infty$

$\Rightarrow$ Contradiction ???

Speaker's conclusion: This approach is of historical interest, but is not really the 'correct approach'.

...Here, we consider new, different approach.

.... cf. one mathematician's remark after work of Tamagawa, Mochizuki on Grothendieck's Anabelian Conjecture (GC):

'In some sense, it's a shame that GC was proven'

..., i.e., if GC was false, then $\exists$ interesting new non-scheme-theoretic $\in \mathrm{Aut}(\pi_1(X))$ $\rightsquigarrow$ i.e., interesting new geometry (lying outside scheme theory)

.... In other words, one should be interested in phenomena that lie on the BOUNDARY between 'anabelian' / 'non-anabelian'

... in a word, our answer is that = 'absolute p-adic anabelian geom. of hyperbolic curves'

§2. The Membership Equation:

motivation: $\boxed{\text{ABC Conjecture}}$ $\rightsquigarrow$ $\boxed{\text{Need geometry (e.g., derivative) } / \pi_1}$ $\rightsquigarrow$ $\boxed{\text{Need 'global Hodge Theory' cf. Hodge-Arakelov Theory: close, but still scheme-theoretic}}$ $\rightsquigarrow$ $\boxed{\text{Solve '}a \in a\text{'!}}$ 'membership equation'

$$\begin{array}{l} a_4 \in \cdots \\ \parallel \\ a_3 \in \{a_3, b_3\} \\ \parallel \\ a_2 \in \{a_2, b_2\} \\ \parallel \\ a_1 \in \{a_1, b_1\} \end{array} \quad \left\} \begin{array}{l} \text{form quotient by identifying} \\ a_i\text{'s} \longrightarrow a \\ b_i\text{'s} \longrightarrow b \end{array} \right\} \rightsquigarrow \text{Solve '}a \in a\text{'!}$$

(contradicts 'axiom of foundation'!)

relation to Hodge Theory; cf. indigenous bundles on a hyperbolic Riemann surface X:

$$\pi_1(X) \circlearrowright \tilde{X} \xrightarrow{\sim} \mathfrak{h} \rightsquigarrow \pi_1(X) \circlearrowright \mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}} \rightsquigarrow \begin{tikzcd} \pi_1(X) \circlearrowright \tilde{X} \times \mathbb{P}^1 \arrow[r] \arrow[d] & P \arrow[d] \\ \pi_1(X) \circlearrowright \tilde{X} \arrow[u, bend left, "\tilde{s}"] \arrow[r] & X \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} P \arrow[d] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$

IB

Hodge section

$$\text{—i.e., } \begin{tikzcd} \tilde{X} \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 & \text{a fiber} \arrow[l] \\ \tilde{X} = \mathfrak{h} \arrow[u, bend left, "\tilde{s}"] \arrow[ur] & \text{moduli of fibers} \arrow[l] \end{tikzcd}$$

Hodge section = identification of $\boxed{\text{a fiber}}$ with $\boxed{\{\text{all fibers}\}}$

"pathology / distortion" arising from this identification (= Hodge section = '$a \in a$')

$\parallel$

'Kodaira-Spencer'

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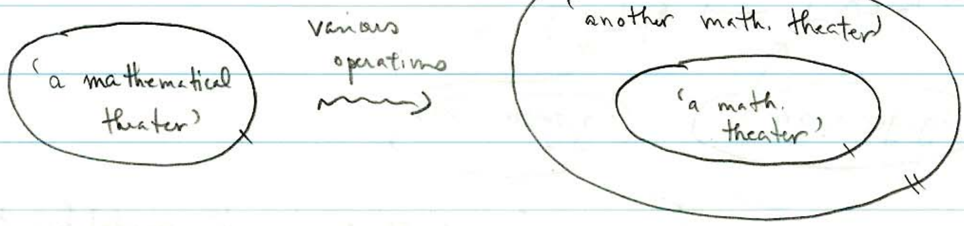
§3. The Inter-Universal Geometry of Categories:



Goal: to identify ◯ with ◯ ⇝ 'a loop' (cf. $a \in a$)

... if 'math. theater' is built up with usual set-theoretic objects ("schemes + metrics", etc.), then it typically has a very complicated '$\in$-structure'

$\Downarrow$ $\quad \in \cdot \in \cdot \in \cdot \in \cdots$

difficult to form a loop

$\Downarrow$

if take 'categories' as fund. geom. objects, then 'simple $\in$-structure'

$\Downarrow$

'IU Geom. of Categories': 'categories up to equivalence'

Also, note: • M monoid $\Rightarrow \mathcal{C}_M : \left\{ \begin{array}{l} \text{obj: } * \\ \text{mor: } \mathrm{End}(*) = M \end{array} \right\} \Rightarrow$ 'gp./monoid' is a special case of 1-cats.

• R ring $\Rightarrow \mathcal{D}_R : \left\{ \begin{array}{l} \mathcal{C}_{R_+} \\ \circlearrowleft \\ R \ni a \cdot \text{ : functor} \end{array} \right\}$ ... 2-cat. of 1-cats.

.... i.e., ring has '2 dims./levels': '+', '×'

ABC conj concerns the relation betw. these two levels.

⇒ need to be able to consider them 'indepedently'

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§4. Analogy with (Uniformizing) MF^{∇}-objects (~ 'filtered Frobenius crystals')

Observe:

$$\cdots \to \mathbb{F}_p[X] \xrightarrow{\text{Frob.}} \mathbb{F}_p[X] \to \cdots$$

alg. geom. over   alg. geom. over

the relation betw. these two 'math. theaters' is the model for §3.

⇓

'loop' of §3 ⟷ MF^{∇}-objects (i.e., 'come back to themselves rel. to Frob.)

.... cf. especially uniformizing MF^{∇}-objects of 'p-adic Teichmüller theory'

↳ 'give rise to p-adic unifs.'

'intrinsic Hodge theory' (cf. ICM '98)

of hyperbolic curves ⟷ Koebe unif. /$\mathbb{C}$

their moduli ⟷ Bers unif. /$\mathbb{C}$

.... note: for MF^{∇}-objects, $d(\text{Frob.}) = p \cdot (\ )$ ⇝ (preview) in the case of ABC, abs. p-adic anab. geom. ⇝ 'differential poles'

i.e. 'log different' term of ABC

'diff.' /$\mathbb{F}_1$'

.... note: perfection is a sort of 'limit'

⟷ IU geom. also a sort of 'limit' = 'analysis'.

(roughly) cf. 'Arakelov geom./scheme theory': take into account arch. primes (+ their analysis)

'IU geom./Arak. geom.': ..... generic primes (+ their analysis)

Anabelian Geometry from an Inter-universal Point of View II

§1. Anabelian Geometry as a Special Case of Inter-Universal Geometry
§2. Arithmetic Fields
§3. Hyperbolic Curves

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§1. Anabelian Geometry as a Special Case of Inter-universal Geometry

In a word, corresponds to the case of $\boxed{\text{Galois categories}}$

i.e.: Π profinite gp. $\rightsquigarrow$ $\mathcal{B}(\Pi)$: obj: finite sets with cont. Π-action
mor: Π-morphisms.

$\Downarrow$

(geom. via) $\boxed{\text{'cats, up to equiv.'}}$ $\longleftrightarrow$ (geom. via) $\boxed{\text{profinite gps (e.g., } \pi_1(\overset{\text{conn.}}{\text{scheme}})\text{) up to automorphism}}$

$\Downarrow$

then the discussion of $\boxed{\text{I}}$ applies to anab. geometry.

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§2. Arithmetic Fields:

Number Fields: $F/\mathbb{Q} < \infty \rightsquigarrow G_F := \mathrm{Gal}(\overline{F}/F)$

Most basic result: 'no. flds. are anab.'

Theorem (Neukirch-Uchida): F_1, F_2 : no. flds. $\Rightarrow$

$$\mathrm{Isom}(F_1, F_2) \xrightarrow{\sim} \mathrm{OutIsom}(G_{F_1}, G_{F_2})$$

Also, if $G_F \twoheadrightarrow G_F^{\mathrm{sol}}$ denotes the 'maximal solvable quotient', then:

Theorem (Uchida): F_1, F_2 : no. flds $\Rightarrow$

$$\mathrm{Isom}(F_1, F_2) \xrightarrow{\sim} \mathrm{OutIsom}(G_{F_1}^{\mathrm{sol}}, G_{F_2}^{\mathrm{sol}})$$

p-adic local fields: $K/\mathbb{Q}_p < \infty$, $G_K := \mathrm{Gal}(\overline{K}/K)$

$\Rightarrow$ $\boxed{\text{NU Thm. is } \underline{\text{false}} \text{ for } G_K \,!}$ $\Rightarrow$ $\boxed{\text{possibility of interesting new geometry!}}$

(cf. I, §1; III)

But if consider higher ramification filtration

Theorem (M): K_1, K_2: p-adic local fields $\Rightarrow$

$$\mathrm{Isom}(K_1, K_2) \xrightarrow{\sim} \mathrm{OutIsom}^{\text{ram.}}(G_{K_1}, G_{K_2})$$

ram.: preserves higher ram. filt.

not quite absolute!

§3. Hyperbolic Curves:

K: fld.

X: hyperbolic curve $/K$: (smooth, proper, geom. conn. curve $/F$ of genus g) \ (reduced divisor of deg. r)

s.t. $2g-2+r>0$

$\Downarrow$

$$\boxed{1 \to \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \to 1}$$

$\Delta_X := \pi_1(X_{\overline{K}})$ ‖ $G_K := \mathrm{Gal}(\overline{K}/K)$

$\Pi_X := \pi_1^{t}(X)$ — t: tame ramification at cusps (vacuous if $\mathrm{char}(K)=0$)

note: if $\mathrm{char}(K)=0$, then Δ_X det'd up to isom, by (g,r), center-free.
if $\mathrm{char}(K)=p$, then — not — (cf. below), but is center-free

note: if K sub-p-adic (i.e., $\exists \hookrightarrow$ ($\exists$ fin. gen. extn. of $\mathbb{Q}_p$): e.g., no. flds, p-adic local flds,)
then G_K: center-free (false for, e.g., finite fields.)

characteristic $p>0$:

Theorem (Tamagawa) For $i=1,2$, $k_i/\mathbb{F}_p < \infty$; X_i: affine hyperbolic curve$/k_i$
$\tilde{X}_i \to X_i$: profinite universal covering

cf. 'not center-free'

$\Downarrow$

absolute

$$\mathrm{Isom}(\tilde{X}_1/X_1, \tilde{X}_2/X_1) \xrightarrow{\sim} \mathrm{Isom}(\Pi_{X_1}, \Pi_{X_2})$$

Remark: $\exists$ unpublished extn. to proper case (M)

Theorem (Tamagawa) X_i: hyperbolic curve$/\overline{\mathbb{F}}_p$ (for $i=1,2$) $\Rightarrow$

(i) $\exists$ only finitely many X_2 s.t. $\Pi_{X_1} \simeq \Pi_{X_2}$

(ii) X_i: genus 0 $\Rightarrow$

$$\left\{ \left(\Pi_{X_1} \overset{\exists}{\xrightarrow{\sim}} \Pi_{X_2} \right) \Leftrightarrow \left(X_1 \overset{\exists}{\xrightarrow{\sim}} X_2 \right) \right\}$$

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characteristic 0:

S 'smooth pro-variety$/K$': $\quad S = \varprojlim S_i$, where $\ldots \to S_i \to S_j \to \ldots$

smooth, geom. conn.$/k$

open immersion

if $\emptyset \neq \Sigma$ is a set of primes, write:

$$\Delta_S^\Sigma := \text{maximal pro-}\Sigma \text{ quotient of } \Delta_S$$

$$\Pi_S^\Sigma := \Pi_S / \mathrm{Ker}(\Delta_S \to \Delta_S^\Sigma)$$

relative (not absolute)

Theorem (M): K: sub-p-adic; S: smooth pro-variety$/K$; X: hyperbolic curve$/K$; $p \in \Sigma \Rightarrow$

$$\mathrm{Hom}_K^{\text{dominant}}(S, X) \xrightarrow{\sim} \mathrm{OutHom}_{G_K}^{\text{open}}(\Pi_S^\Sigma, \Pi_X^\Sigma)$$

K generalized sub-p-adic : $K \overset{\exists}{\hookrightarrow} (\exists \text{ fin. gen. extn. of } \mathbb{Q}(W(\overline{\mathbb{F}}_p)))$

($\Rightarrow G_K$ center-free)

$$\boxed{\begin{array}{l}\textbf{Theorem (M)}: K \text{ generalized sub-p-adic; for } i=1,2,\ X_i : \text{hyperbolic curve}/K,\ p \in \Sigma \Rightarrow \\ \qquad \mathrm{Isom}_K(X_1, X_2) \xrightarrow{\sim} \mathrm{OutIsom}_{G_K}(\Pi^{\Sigma}_{X_1}, \Pi^{\Sigma}_{X_2})\end{array}}$$

relative (not absolute)

Remark: cf. Tamagawa/$\overline{\mathbb{F}}_p$

Absolute Results / Number Fields :

$$\boxed{\begin{array}{l}\text{Corollary}: F_i/\mathbb{Q} < \infty;\ X_i : \text{hyperbolic curve}/F_i;\ \Sigma \text{ arbitrary} \Rightarrow \\ \qquad \mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{OutIsom}(\Pi^{\Sigma}_{X_1}, \Pi^{\Sigma}_{X_2})\end{array}}$$

'X: Σ-solvable.'

If $|\Sigma| = 1$ (cardinality), and, for $p \in \Sigma$, p-torsion points of Jacobian of X (hyp. curve/K) are defined over K, then

$$\begin{array}{ccc} G_K & \longrightarrow & \mathrm{Out}(\Delta^{\Sigma}_X) \\ \searrow & & \nearrow \exists! \\ & G_K^{\mathrm{sol}} & \end{array}$$

maximal solvable quotient

$$\boxed{\begin{array}{l}\text{Corollary}: F_i/\mathbb{Q} < \infty;\ X_i : \text{hyperbolic curve}/F_i,\ \Sigma\text{-solvable} \Rightarrow \\ \qquad \mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{OutIsom}(\Pi^{\Sigma}_{X_1}, \Pi^{\Sigma}_{X_2})\end{array}}$$

Remark: Since 'NU for p-adic local fields' is false, unclear (UNKNOWN) if abs. result holds for p-adic local fields, = '(abs pGC)'

... my expectation: $\boxed{\text{false}}$ (corollary of work on ABC ???)

Anabelian Geometry from an Inter-universal Point of View III

§1. p-adic Absolute Anabelian Geometry
§2. The Logarithmic Special Fiber
§3. Canonical Curves
§4. Absolute Relative Anabelian Geometry

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§1. p-adic Absolute Anabelian Geometry

Expectation: abs pGC false!

Motivation: 'ANALOGY'

p-adic local fields	$\mathbb{F}_p((t))$
treating Π_X absolutely	treating X not as an object over $\mathbb{F}_p((t))$, but rather as an object over $\mathbb{F}_p$
$\uparrow$	$\Downarrow$
cf. differentiation over '$\mathbb{F}_1$'!	should only be able to recover properties preserved by coordinate transformations $t \mapsto t + (?)t^2 + \ldots$

$\Downarrow$

one expects that one should be able to recover

(i) the (log) special fiber

(ii) the generic fiber if X is a constant curve

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§2. The Logarithmic Special Fiber

Theorem (M via Tamagawa) For $i = 1, 2$, $k_i / \mathbb{Q}_{p_i} < \infty$, X_i : hyperbolic curve$/k_i$ with log special fiber $X^{\log}_{k_i}$

$$\forall\ \Pi_{X_1} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X_2} \ \overset{\text{functorially in } \alpha}{\rightsquigarrow} \ X^{\log}_{k_1} \xrightarrow{\sim} X^{\log}_{k_2}$$

§3. Canonical Curves

	hyperbolic curves	abelian varieties
$\mathbb{C}$	Bers embedding (Teich. theory)	hyperbolic geometry of Siegel upper half plane
p-adics	$\exists$ (p-adic Teich. theory (cf. [I], §4))	Serre – Tate theory.

$\Downarrow$ canonical liftings $\wr$ canonical liftings $\Downarrow$

not compatible!!

> Theorem (M) For $i=1,2$, $K_i/\mathbb{Q}_p < \infty$ abs. unram., $p>5$; X_i : hyperbolic curve$/K_i$,
>
> $\Pi_{X_1} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X_2} \Rightarrow$ (i) X_1 can. lift $\Leftrightarrow$ X_2 can. lift
>
> (ii) if can. lift, then log sp. fiber isom. lifts uniquely to $X_1 \xrightarrow{\sim} X_2$

Defn: X absolute : $\forall X'$ s.t. $\Pi_X \overset{\exists}{\xrightarrow{\sim}} \Pi_{X'}$, $\exists\, X \xrightarrow{\sim} X'$

> Corollary : The points determined by absolute curves are Zariski dense in $\mathcal{M}_{g,r}(\overline{\mathbb{Q}}_p)$, $p>5$.

Remarks: (i) So far, these are the only known absolute curves.

(ii) Thus, 'can. lifts are like constant curves' (cf. abel. vars)

(iii) 'Cor' is first application of p-adic Teich. theory (!)

(iv) cf. 'can. lifts. of abel. vars. are CM' : $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C} \circlearrowleft \mathrm{Aut}$

cf. 'abs. Π_X'

§4. Absolute Relative Anabelian Geometry

X: hyp. curve / fld. K.

$\mathrm{Loc}_K(X)$: obj: Y s.t. $\exists$ fin. et. $Y \to X$

mor: $Y_1 \to Y_2$ fin. et., $/K$ (not nec. $/X$!)

$\mathrm{DLoc}_K(X)$: obj: $Y \hookrightarrow Z$ s.t. $\exists$ fin. et. $Y \to X$; Z hyperbolic curve;

$\underbrace{Y \hookrightarrow Z \hookrightarrow \overline{Y}}$

usual compactification embedding

mor:
$$\begin{array}{ccc} Y_1 & & Y_2 \\ \cap & & \cap \\ Z_1 & \to & Z_2 \end{array}$$
dominant, $/K$.

Thus, $\exists$ natural faithful $\mathrm{Loc}_K(X) \to \mathrm{DLoc}_K(X)$

$\Downarrow$

rel pGC implies the following absolute results:

Theorem (M) For $i = 1, 2$, $K_i / \mathbb{Q}_p < \infty$; X_i: hyp. curve / K_i, $\alpha: \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ $\Rightarrow$

$\exists$ $\mathrm{Loc}_{K_1}(X_1) \xrightarrow{\sim} \mathrm{Loc}_{K_2}(X_2)$; $\mathrm{DLoc}_{K_1}(X_1) \xrightarrow{\sim} \mathrm{DLoc}_{K_2}(X_2)$

.... functorial in α

Corollary: notation as in Theorem.

(i) Suppose $(g_i, r_i) = (1, 1)$. Then α preserves decomp. gps. of torsion points.

(ii) Suppose X_i isogenous (:= admits a common fin. et. covering) to a genus 0 hyp. curve def'd. over a number field. Then α preserves decomp. gps. of all closed points.

Remark: 'Cor' may be interpreted as a sort of 'weak Section Conjecture',
– cf. $\boxed{\text{I}}$, §1, the importance of 'why': $\begin{array}{c}\Pi_{X_K} \\ \downarrow \uparrow s \\ G_K\end{array}$ 'i.e., when do not obviously geom. s occur?'

Suppose $x \in \bar{X} \setminus X$: 'cusp' $\Rightarrow$

$$\boxed{\begin{array}{c} 1 \to I_x \to D_x \to G_K \to 1 \\ \quad\ \ \wr\!\wr \\ \quad\ \ \hat{\mathbb{Z}}(1) \end{array}} \rightsquigarrow \text{'torsor of splittings' over } H^1(G_K, I_x) \cong (K^\times)^\wedge$$

For arb. (resp. stable reduction) X, local (resp. local integral) coord. t at x,

$$\{t^{1/N}\} \rightsquigarrow \text{reduction of structure gp } (K^\times)^\wedge \rightsquigarrow K^\times$$
$$(\text{resp. } (K^\times)^\wedge \rightsquigarrow \mathcal{O}_K^\times)$$

> **Corollary**: notation as in Theorem. Suppose X_i isogenous to genus 0 hyp. curve.
> (i) α preserves red. of str. gp. $(K^\times)^\wedge \rightsquigarrow K^\times$
> (ii) in stable red. case, α preserves red. of str. gp. $(K^\times)^\wedge \rightsquigarrow \mathcal{O}_K^\times$

Remarks: (i) These results extend immediately to André's tempered fund. gp.
(ii) 'Cor' (& 'Thm') 's related via 'étale theta function' (= sort of p-adic analytic version of theta fn.) to Hodge-Arakelov theory, ABC (cf. functional equation of classical theta function)