Categorical Representation of Arithmetic Log Schemes, with Applications to the Arithmetic of Elliptic Curves

§1. The Inter-Universal Geometry of Categories
§2. Basic Examples of IU Geometry
§3. Global Multiplicative Subspaces
§4. Distributed Versions

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§1. The Inter-Universal Geometry of Categories:

Motivation:

$\boxed{\text{ABC Conjecture}}$ ⇝ $\boxed{\text{scheme theory}/\mathbb{Z} \text{ insufficient; need geometry}/\mathbb{F}_1}$ ⇝ $\boxed{\text{need 'global Hodge theory' over number fields}}$ ⇝ $\boxed{\text{must solve 'membership equation' } a \in a}$

impossible in conventional set theory, by axiom of foundation

⇒ extend conventional set theory! ⇒ 'IU geometry': use labels obtained by extending the universe in question

$$\begin{array}{l} a_4 \in \dots \\ \| \\ a_3 \in \{a_3, b_3\} \\ \| \\ a_2 \in \{a_2, b_2\} \\ \| \\ a \in \{a_1, b_1\} \end{array}$$

identify $\left.\begin{array}{l} a_i \rightsquigarrow a \\ b_i \rightsquigarrow b \end{array}\right\}$ sort of 'limit' = 'analysis' or quotient

(vs. usual set theory = 'algebra' = 'approximations')

Thm: ('Fundamental Thm. of IU Geom. of Categories')
One may construct such a geometry by considering categories/equiv.

.... Why categories? Because $\underbrace{\text{they are 1-dim}}_{\text{'most primitive units'}}$

.... by contrast, rings are 2-dim. → ⊞ addition, ⊠ multiplication

.... The ABC Conj. concerns the relationship between these 2 dims.

Ex: M monoid ($\ni 1$)
⇝ $\mathcal{C}_M$: obj.: $*$
mor.: $\mathrm{End}(*) = M$

.... similarly, a ring may be naturally represented as a 2-cat. of 1-cats.

§2. Basic Examples of IU geom.

§2.1. Absolute Anabelian Geometry: X_K hyperbolic curve / char. 0 fld. K

hyperbolic curve → (compact genus g) − (r pts.) s.t. $2g-2+r>0$

$$\boxed{1 \to \Delta_X \to \Pi_{X_K} \to G_K \to 1}$$

$\Delta_X = \mathrm{Ker}(-)$, $\Pi_{X_K} = \pi_1(X_K)$

Then: anabelian geom. = 'to what extent can one recover the curve X_K from the profinite group G_K ⇔ the assoc'd Galois category?'

relative theory = 'over G_K' ; absolute theory = 'not nec'ly over G_K'

	relative	absolute			
over:	arbitrary curve / K	arbitrary curve	canonical curves (of p-adic Teich. theory)	log special fiber	theory of correspondences, e.g. arithmeticity (whether or not isogenous to Shimura curve via fin. ét. morphisms)
number fields	○	○	○	○	○
p-adic local fields	○	?, X?	○ (RIMS Preprint 1379)	○ (RIMS Preprint 1363)	○

→ first application of p-adic Teich. theory!

$\boxed{\text{'Curves satisfying abs. pGC are Zariski dense in moduli space } \mathcal{M}_{g,r}(\overline{\mathbb{Q}}_p)\text{'}}$

Motivating Analogy

Over p-adic local fields	Over $\mathbb{F}_p((t))$
treating Π_{X_K} absolutely	working with $X_{\mathbb{F}_p((t))}$ not over $\mathbb{F}_p((t))$, but over $\mathbb{F}_p$!!

⇒ one only sees properties invariant w.r.t. coordinate transformations $t \mapsto t + (\ )t^2 + (\ )t^3 + \dots$ (cf. the ABC conj.!)

Problem with anab. geometry (as a special case of IU geom.): only applies to very special schemes!

§2.2. Arbitrary arith. log schemes: X noetherian

$Sch(X) := \begin{cases} \text{obj: } Y \to X \text{ fin. type morphism} \\ \text{mor: } Y_1 \to Y_2 \text{ .... morphism of } X\text{-schemes} \end{cases}$

equivs. of cats. / isom

$\boxed{\text{Thm: } Isom(X, X') \xrightarrow{\sim} Isom(Sch(X), Sch(X'))}$ (RIMS Preprint 1364) } cf. 'Grothendieck conj.'

.... also $\exists$ log version for fine, saturated log schemes X^{log}

$\exists$ arith. version for X fin. type / $\mathbb{Z}$ or $\mathbb{Q}$: arch. structures: $H_X \subseteq X(\mathbb{C})$ compact, stable under complex conjugation.

$\exists$ log arith. version: $X(\mathbb{C}) \rightsquigarrow X^{log}(\mathbb{C})$

'spaces of Kato-Nakayama'

Problem: lacks rigidity of Gal. categories

Ex: for arith. line bundle take 'H_X' to be $|-| \leq 1$. ('the unit ball')

§2.3. Categories of multiplicative localizations:

$$F/\mathbb{Q} < \infty, \quad G = Gal(\bar{F}/F) \twoheadleftarrow G_F = Gal(\bar{F}/F)$$

$$\Downarrow$$

'pro-arith. log scheme' $\bar{S}^{log}_F$: (Spec $\mathcal{O}_F$ + log str. at all closed pts. + arch. str. = {all arch. primes})

$Loc_G(\bar{S}^{log}_F)$: $\begin{cases} \cdot\ \bar{F} \supseteq L/F < \infty:\ \bar{S}^{log}_L \rightsquigarrow \text{(global objs.)} \\ \cdot \text{ Zariski localizations of gl. objs. at nonarch., arch. primes} \rightsquigarrow \text{'local objs'} \\ \cdot\ \mathcal{O}_F\text{-morphisms} \end{cases}$

$Loc^{\times}_G(\bar{S}^{log}_F)$: $\begin{cases} \cdot \text{ arith. l.b. } V \text{ (cf. Ex.) over objs. } T \text{ of } Loc_G(\bar{S}^{log}_F) \\ \cdot\ \mathcal{O}_F\text{-morphisms (of arith. log schemes)} \Rightarrow \text{morphisms of the form '} T \mapsto c \cdot T^n \text{'} \end{cases}$

... + other inessential details. (T: geom. l.b. coord.; $n \geq 1$ integer; c integral)

Thm: $\deg_{ar}$ (global objs.), $\deg_{ar}$ (change in integral structure of local objs.) $\in \mathbb{R}$ } category-theoretic! (for $Loc^{\times}$)

Merits of $Loc^{\times}$:

- has Gal. category portion: G
- has arch. portion $\Rightarrow$ global degs, heights!

§3. Global Mult. s/spaces: S' : Riemann surface of fin. type $(= \text{compact} \setminus \text{fin. set})$

$$\begin{matrix} E \\ \downarrow \\ S \end{matrix} \Big\}\ \text{family of ell. curves} \rightsquigarrow \begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}_{\mathbb{C}} \cong \mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}} \arrow[d] \\ S \arrow[r] & \mathcal{M}_{\mathbb{C}} \end{tikzcd} = \{\text{moduli stack of ell. curves}/\mathbb{C}\}$$

$\mathcal{E}$: local system $/S'$, fiber $\cong \mathbb{Z}^2$

want an arith. analogue

$$\boxed{\text{Fundamental Thm. of Hodge Theory}/\mathbb{C}:\ \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_S \supseteq \omega_E}$$

rank 1 subbundle

We want an arith. analogue of this rank 1 sub.

$\Downarrow$

the closest well-known arith. analogue occurs in the case of Tate curves

$\Downarrow$

$\mathcal{E}|_{\widetilde{S}} \cong \mathbb{Z}^2 \Rightarrow$ rank 1 sub. is a varying rank 1 subspace of $\mathbb{C}^2$

$\Rightarrow$ classifying morphism may be thought of as the resulting:

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}_{\mathbb{C}} \cong \mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}} \arrow[d] \\ S \arrow[r] & \mathcal{M}_{\mathbb{C}} \end{tikzcd}$$

$$\mathbb{C}^\times \to E = \mathbb{C}^\times/q^{\mathbb{Z}} \rightsquigarrow \boxed{0 \to \mu_n \to E[n] \to \mathbb{Z}/n\mathbb{Z} \to 0}$$

exists over $\mathbb{Z}[[q]]$, $\therefore$ at bad mult. primes for ell. curves/no. flds.

$\Rightarrow$ we wish to globalize this rank 1 sub.

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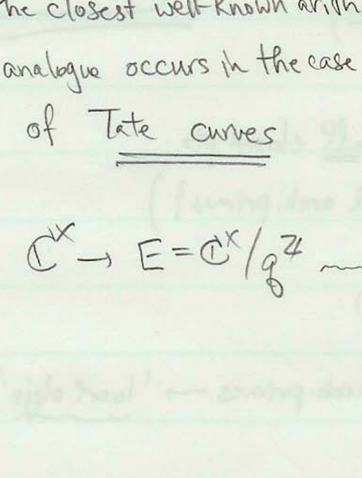
§4. Distributed Versions: $F/\mathbb{Q} < \infty$, E : ell. curve$/F$, $K := F(E[l])$; assume $\mathrm{Gal}(K/F) = GL_2(\mathbb{Z}/l\mathbb{Z})$

K



F

extract only the bad primes where some chosen rank 1 sub. is mult.

$\Downarrow$

distributed versions of $\mathrm{Loc}^\times$

here, $\exists$ tautological global mult. s/sp.!

$\mathrm{Loc}[\underline{\Delta}^+]$ } primes of K over a single prime of F treated independently

'U'

$\mathrm{Loc}^\times[\underline{\Delta}]$ } linked

$\downarrow \wr$

$\mathrm{Loc}^\times$ } the usual category of §2, 3, for F

(*)

(*) : cf. FU geom! ‘$a \in a$’

$\cdot = \vdots \; \xrightarrow{\sim} \; \cdot$ } ‘fundamental loop’

(on F) (on K)

Moreover: (i) Over $\mathrm{LOC}^{\times}[\underline{\Delta}^{\dagger}]$, one has: · $\deg_{ar}$(global objs.) ⇝ one can do ‘Arakelov theory’

· Galois ⟶ one can do ‘Gal. theory’

[· also ∃ ‘Frobenius’]

(ii) $\mathrm{LOC}^{\times}[\underline{\Delta}^{\dagger}]$ related to $\mathrm{Loc}^{\times}$ via $\mathrm{Loc}^{\times}[\underline{\Delta}]$ (cf. ↓2)

$\boxed{\text{Thus, over the ‘geom. obj.’ (cf. FU geom.) } \mathrm{LOC}^{\times}[\underline{\Delta}^{\dagger}] \text{, we have constructed a ‘classifying morphism}_{\mathbb{F}_1}\text{’ ‘}\mathrm{Loc}^{\times}[\underline{\Delta}^{\dagger}] \to \mathcal{M}_{\mathbb{F}_1}\text{’}}$

..., to verify ABC, remains to compute its derivative (work in progress)!