Inter-Universal Hodge-Arakelov Theory

§1. Scheme-Theoretic Hodge-Arakelov Theory
§2. Inter-Universal Geometry
§3. Differential Poles.

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§1. Scheme-theoretic Hodge-Arakelov Theory

original motivation: ABC Conjecture ('absolute derivative of $\mathbb{Z}/\mathbb{F}_1$')

$\leadsto$ Hodge-Arakelov Theory: F: fld. of char. 0, E/F: elliptic curve.

(cf. 'Survey')

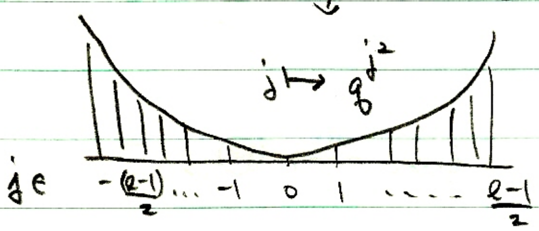
$\left\{ \begin{matrix} \text{space of theta fns.} \\ \text{and (certain) derivatives} \end{matrix} \right\} \xrightarrow{\text{evaluate}} \left\{ \begin{matrix} \text{functions} \\ \text{on } \ell\text{-tors. pts.} \end{matrix} \right\}$

$\cup$ Hodge filtration $\qquad (\ell: \text{prime}) \qquad \curvearrowleft$ Gal.

.... easier to understand for Tate curves: $\mathbb{G}_m/q^{\mathbb{Z}}$, $\Theta = \sum_{n \in \mathbb{Z}} q^{n^2} U^n$

$$\Theta \Big|_{\mu_\ell \subseteq \mathbb{G}_m} = \sum_{j \in \mathbb{Z}/\ell\mathbb{Z}} (U|_{\mu_\ell})^j \sum_{n \in j} q^{n^2}$$

$\Big\}$ up to highest order



'GP' = Gaussian poles ($\leadsto$ integral structure on $\mathbb{Z}[[q]]$)

Gal $\curvearrowright$ Hodge filtration $\leadsto$ 'arithmetic Kodaira-Spencer'

'geometric portion' of KS: For $(GL_2(\mathbb{F}_\ell) \supseteq) \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, closely related to usual KS at p over ℓ, mod p^2 (i.e., $q \mapsto q^\ell \cdot \zeta$, $\zeta^\ell = 1$) (p of a number field)

Key: $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ preserves GP 'int. str.'

(geom. portion) : insufficient for application to number fields

$\rightsquigarrow$ **Problem:** for number fields, need $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

which permutes GP

(i.e., does not preserve)

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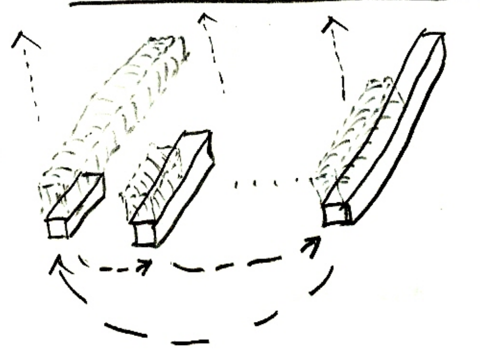
§2. Inter-Universal Geometry:

Basic idea: think of piling blocks:

(infinite piles)



} **Problem:** How to realize this situation.

.... think of formal infinite products:

$\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix}\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix}\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix} \cdots\cdots$ $\overset{\text{(exponentiate)}}{\rightsquigarrow}$ $\left(\left(\left(q\right)^{\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix}}\right)^{\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix}}\right)^{\begin{pmatrix} 1^2 \\ 2^2 \\ 3^2 \\ \vdots \end{pmatrix}} \cdots$

(e.g., p!)

i.e., think of q-parameter 'q' $\in \mathcal{O}_{F_v}$, for some v of a number field F:

$$q \overset{\text{form}}{\rightsquigarrow} \Theta =: q_{new} \overset{\text{form}}{\rightsquigarrow} \Theta_{new} =: q_{newer} \overset{\text{form}}{\rightsquigarrow} \Theta_{newer} \rightsquigarrow \cdots$$

fundamentally impossible in scheme-theory: since $\mathbb{Z}$, $\mathbb{Z}_p$, 'p' are absolute in scheme-theory!

cf. 'H-A' in [scheme vs. IU], 'char. p Picard gp' in [varieties vs. schemes]

$\Rightarrow$ [enter Inter-Universal Geometry]

.... consider species: 'type of mathematical object' } 'set-theoretic realization of objects of a category, functors'

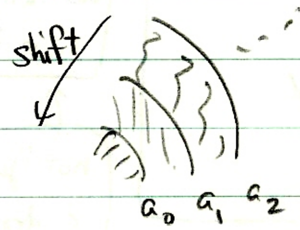
mutation: $A \rightsquigarrow B$ (A, B: species.)

then consider loops of mutations ('simulate $a \in a$'):

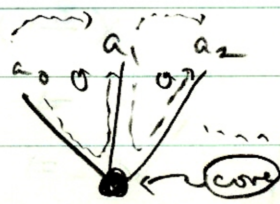
$$A \rightsquigarrow B \rightsquigarrow C \rightsquigarrow D \rightsquigarrow A \ldots$$

e.g.: anabelian geometry (certain schemes) $\overset{\pi_1}{\rightsquigarrow}$ (certain profinite gps) $\overset{?}{\rightsquigarrow}$ (certain schemes)

.... when considering formal composites of operations:



a priori, can't change order (e.g., move a_2 to between a_0, a_1)



} 'commute'

but can commute if operations really only depend on a core.

.... need to (shift & commute) to form infinite product a^∞ s.t.

$$\boxed{a \cdot a^\infty = a^\infty.}$$

i.e. $a_1(a_0(a_{-1}(a_{-2}(\cdots))))$
$= a_0(a_{-1}(a_{-2}(\cdots)))$

.... fundamental dichotomy:

et	Fr'd
gp	monoid $\mathbb{Z}_{\geq 0}$
order-indep.	order-conscious

note: need 'et core'
⇓
use arith. π_1 to represent ell. curve, loc. fld., gl. fld.
⇓
enter anabelian geometry

Ex: et site of char. p scheme X under Frob:

$$X \overset{Fr}{\rightsquigarrow} X^{(p)} \overset{Fr}{\rightsquigarrow} X^{(p^2)} \overset{Fr}{\rightsquigarrow} \ldots$$

Et(X)

$\begin{pmatrix}1^2\\2^2\\3^2\\ \vdots\\ n^2\end{pmatrix}\begin{pmatrix}1^2\\2^2\\3^2\\ \vdots\\ n^2\end{pmatrix}\dots$ } may be thought of as a 'homotopy' from $1^\infty = 1$ to n^∞ = Frobenius, i.e., a Frob. lifting! } i.e., '(H) is a F.L.!'

$\Downarrow$

ABC argument analogous to:

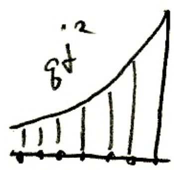
> Ex: X smooth proper curve / $W(\mathbb{F}_p)$
>
> $X \xrightarrow{\Phi} X$ Frob. lift.
>
> $\Rightarrow \frac{1}{p}\, d\Phi : \Phi^*\Omega_X \hookrightarrow \Omega_X$
>
> $\Rightarrow$ $\boxed{g \not\geq 2}$

i.e., derivative of a FL

↑ not 'just a deriv.' as for geom. ABC!

§3. Differential poles:

(Deriv. of (H)) ⟷



'differentiating' = 'distinguishing' pts. $\in (\mathbb{Z}/\ell\mathbb{Z})/\{\pm 1\}$. 'labels'

$\Downarrow$

Values at various copies of $\mathcal{O}_K$ ($K/\mathbb{Q}_p < \infty$):

$\mathcal{O}_K \otimes \dots \otimes \mathcal{O}_K \quad \subseteq \quad \bigoplus \mathcal{O}_K.$ } j copies for j^{th} value

($\mathcal{O}_K \otimes \dots \otimes \mathcal{O}_K$): can't take $\otimes_{\mathcal{O}_K}$ since this requires crushing labels

($\subseteq$): ↖ difference in integral structure (easy exercise) $\sim j \cdot \log \mathrm{diff}(\mathcal{O}_K)$ ... 'Differential Poles'

$\Downarrow$

using $\boxed{a \cdot a^\infty = a^\infty}$ to differentiate the F.L. '(H)' } $\Rightarrow$ $0 \le -GP + DP$, i.e. $GP \le DP$ $\iff$ ABC inequality (by basic computation of H-A Theory)