The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

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§1. Uniformization Theory as a Hodge Theory at Arithmetic Primes

§2. The Physical Aspect: Embedding by Automorphic Forms

§3. The Modular Aspect: Canonical Frobenius Actions

§1. Unif. Theory as a Hodge Theory at Arithmetic Primes

(A.) Uniformization as a Catalogue of Rational Points

Problem: For a variety $Z/\mathbf{C}$, list/ *catalogue* explicitly the set of rational rational points $Z(\mathbf{C})$, e.g.,

Complex Planar Varieties:

$$Z(\mathbf{C}) \stackrel{\text{def}}{=} V(f(X,Y))$$
$$= \{(x,y) \in \mathbf{C}^2 | \ \ f(x,y) = 0\}$$

1. Linear: $X = 0 \implies$

$$(0, ?) : \mathbf{C} \stackrel{\sim}{\to} Z(\mathbf{C})$$

2. Quadratic: $X \cdot Y = 1 \implies$

$$\exp : \mathbf{C} \;\to\; Z(\mathbf{C}) = \mathbf{C}^{\times}$$

3. Cubic: Elliptic Curve $E \implies$

$$\exp_E : \mathbf{C} \;\to\; E(\mathbf{C}) = \mathbf{C}/\Lambda$$

4. Higher Degree:

Hyperbolic Curve Z: smooth, proper connected genus g algebraic curve $-\ r$ points, s.t. $2g - 2 + r > 0$

$$\Downarrow \text{(Köbe)}$$

Upper half-plane $\mathcal{H} \to Z(\mathbf{C}) = \mathcal{H}/\Gamma$

(B.) "Intrinsic" Hodge Theories

Unif. = linear/geometric catalogue
of rational points

$\Downarrow$ an equivalence:

$$\begin{pmatrix} \text{alg.} \\ \text{geom.} \\ \text{(e.g.,} \\ \text{rat.} \\ \text{pts.)} \end{pmatrix} \iff \begin{pmatrix} \underline{\mathbf{C}}\text{: top. +} \\ \text{diff. geom.} \\ \underline{p\text{-adics}}\text{:} \\ \text{pro-}p\text{ étale} \\ \text{top. + Gal.} \\ \text{action} \end{pmatrix}$$

A "Hodge Theory": but not of cohoms.
(i.e., de Rham. $\iff$ sing./et. cohom.)

Uniformization = case where:

"alg. geom." = "variety itself," e.g.,

○ Its Rat. Points $\rightsquigarrow$ Physical Aspect
(uniformization of variety itself)

○ Its Moduli $\rightsquigarrow$ Modular Aspect
(uniformization of moduli space)

... "Intrinsic Hodge Theory" (IHT)

Note: For $\mathbf{G}_{\mathrm{m}} = V(XY - 1)$, elliptic curve E,

$$\text{IHT} = \text{HT of cohoms.}$$

since these are 1-motives!

(C.) Completion at Arith. Primes

To realize (I)HT, typically must complete at "arithmetic primes":

inf. primes ($\mathbf{C}$), p-adic primes ($\mathbf{Q}_p$), degenerate object (power series/$\mathbf{Z}$)

Guiding Principle: $\forall$ arithmetic prime,

$\exists$ canonical unif. theory at that prime.

(but, in general, theories at different primes are not compatible!!

Examples: (Phys./Mod.)

(1.) Abelian Varieties

(2.) Hyperbolic Curves

(1.) Abelian Varieties

C	*p*-adic	Deg. Obj.
exp. map. of AV/ Siegel upper half-pl.	Tate's thm./ Serre-Tate theory	Schottky unifs. of Tate/ Mumford

(2.) Hyperbolic Curves

C	*p*-adic	Deg. Obj.
Fuchsian unif./ Teich.-Bers Unif. Theory	§2 (anab. conj.)/ §3 (*p*-adic Teich. th.)	formal algebr. Schottky unif. of Mumford

§2. The Physical Aspect: Embedding by Aut. Forms

(A.) The Complex Case

alg. curve $X \iff SO(2)\backslash PSL_2(\mathbf{R})/\Gamma$

(physical/analytic obj.)

$\iff \pi_1(\mathcal{X}) +$ action on $\mathcal{H}$

$\iff$ top. + arith. str. (geom.)

Autom. forms define first "$\iff$": i.e., to recover alg. str., constr. aut. forms analytically, then embed in proj. sp.

Point: *analytic* repr. of alg. diff. forms.

$\Longrightarrow$ do this *p-adically* using p-adic HT

$$\begin{array}{ccc} \text{Upp. half-pl. } \mathcal{H} & \rightarrow & \text{Proj. Sp.} \\ \downarrow & & \| \\ \text{Alg. Curve} & \hookrightarrow & \text{Proj. Sp.} \end{array}$$

The Case of $SL_2(\mathbf{Z})$

(B.) The Arith. Fund. Group

$K \overset{\text{def}}{=}$ char. 0 field, $\Gamma_K \overset{\text{def}}{=} \text{Gal}(\overline{K}/K)$

X: hyp. curve/K, $\overline{X} \overset{\text{def}}{=} X \times_K \overline{K}$.

$\Rightarrow \quad 1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \Gamma_K \to 1$

$\pi_1(\overline{X})$ (geom. π_1): indep. of moduli (but in char. p, may determine moduli! – A. Tamagawa)

Grothendieck's anabelian philosophy:
"Extension should determine moduli."

(C.) The Main Theorem

Theorem 1: $K \subseteq$ fin. gen. extn./$\mathbf{Q}_p$,
X: hyperbolic curve/K,
S : smooth variety/K.

$$\Rightarrow X(S)^{\mathrm{dom}} \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(S), \pi_1(X))$$

i.e., alg. curve $X \iff$
phys./an. obj. $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(-, \pi_1(X))$

Builds on work of: H. Nakamura, A. Tamagawa + G. Faltings, Bloch/Kato.

Proof: Consider p-adic analytic diff. forms on $(\mathbf{Z}_p[\ T\ ]^{\mathrm{tame}}_{(p)}\ )^\wedge$ (maps to X) – cf. mod. forms on $\mathcal{H}$.

Remark: Also pro-p, function field versions (cf. F. Pop).

(D.) Comparison with the Case of Abelian Varieties

Th.1 resembles Tate Conjecture, i.e.,

Hom(abelian varieties) $\Longleftrightarrow$ Hom(Tate modules)

But T. C. false over *local fields*!

New point of view:

Theorem 1 = p-adic version of physical aspect of of Fuchsian unif.

§3. The Modular Aspect: Canonical Frobenius Actions

(A.) The Complex Case

{hyp. curve X + proj. str.} $\rightsquigarrow \mathcal{S}_{g,r}$
= {squ. diffs.}-('Schwarz') torsor/$\mathcal{M}_{g,r}$

(*proj. str.* $\subseteq \mathcal{O}_X$, sects. differ loc. by *lin. fract. trans.*)

Over **C**: $\exists$ "open ball of quasi-conformal gps."

$$\begin{array}{ccc}
`\mathcal{S}_{g,r}^{\text{int}_\infty}\text{'} \curvearrowleft \text{Fr}_\infty & & \\
\| & & \\
\text{Rep}^{\text{QC}}(\pi_1(X), & & \\
PGL_2(\mathbf{C})) & \overset{\text{open}}{\hookrightarrow} & (\mathcal{S}_{g,r})_{\mathbf{C}} \\
\bigcup & & \downarrow \\
\text{Fr}_\infty\text{-invars.} & \overset{\sim}{\rightarrow} & (\mathcal{M}_{g,r})_{\mathbf{C}}
\end{array}$$

$\rightsquigarrow$ <u>can. real an.</u> $s : (\mathcal{M}_{g,r})_{\mathbf{C}} \to (\mathcal{S}_{g,r})_{\mathbf{C}}$

(= section assoc. to <u>Fuchs. unif.</u>)

s.t.: (1) $\overline{\partial}\ s =$ Weil-Petersson metric

(2) $\int$ WP = Bers coords.

$= \mathrm{pr}_{\mathcal{M}_{g,r}} \circ (\mathrm{Fr}_{\infty}|_{\mathrm{Fiber}})$

"Bers unif. <u>is</u> a Frobenius action!"

(B.) Teich. Theory in Char. p

Proj. str. s.t. p-curv. (= "[Frob., ∇]") squ. nilp. (cf. Shimura curves) $\Longrightarrow$

$$\begin{array}{ccc} (\mathcal{S}_{g,r})_{\mathbf{F}_p} & = & (\mathcal{S}_{g,r})_{\mathbf{F}_p} \\ \bigcup & & \downarrow \\ \mathcal{N}_{g,r} & \to & (\mathcal{M}_{g,r})_{\mathbf{F}_p} \end{array}$$

Theorem 2: $\mathcal{N}_{g,r} \to (\mathcal{M}_{g,r})_{\mathbf{F}_p}$: fin., flat, loc. compl. int., deg. $= p^{3g-3+r}$, i.e.,

$\mathcal{N}_{g,r}$ "almost" a section of Sch. torsor!

Remarks: (1) $\mathcal{N}_{g,r} \rightsquigarrow$ 'p-adic Teich. th.'

(2) $\mathcal{N}_{g,r} \Rightarrow$ new prf. that $\mathcal{M}_{g,r}$ conn.! (cf. Teich. th./$\mathbf{C}$; ab. vars. (Oort)!)

(C.) Canonical p-adic Liftings

$$\mathcal{N}_{g,r} \supseteq (\mathcal{N}^{\mathrm{ord}}_{g,r})_{\mathbf{F}_p} \stackrel{\mathrm{def}}{=} \text{ét. loc.}/(\mathcal{M}_{g,r})_{\mathbf{F}_p}$$

$$\begin{array}{ccccc} \Longrightarrow & (\mathcal{S}_{g,r})_{\mathbf{Z}_p} & = & (\mathcal{S}_{g,r})_{\mathbf{Z}_p} \\ & \cup & & \downarrow \\ \Phi_{\mathcal{N}} \curvearrowright & (\mathcal{N}^{\mathrm{ord}}_{g,r})_{\mathbf{Z}_p} & \stackrel{p\text{-adic et}}{\longrightarrow} & (\mathcal{M}_{g,r})_{\mathbf{Z}_p} \\ & \cdot||\cdot & & \\ & \mathrm{Rep}^{\mathrm{Crys}}(\pi_1(X_{\mathbf{Q}_p}), PGL_2(\mathbf{Z}_p)) & & \end{array}$$

Thm. 3: $\{\Phi_N, \cup\}$ are ! Frob.-inv. pair.

$\Longrightarrow s_{\mathcal{N}} =$ can. sect. of Sch. torsor!

Remarks: (1) $(1/p) \cdot \mathrm{d}\Phi_{\mathcal{N}}$ is isom., i.e., $\Phi_{\mathcal{N}}$ is ordinary Frobenius lifting

(2) ‘$\int$ ord. F.L.’ $\Longrightarrow$

<u>canonical mult. parameters</u>

(cf. <u>real analytic Kähler metrics</u>)

(3) $\Phi_{\mathcal{N}} \longleftrightarrow$ Weil-Petersson metric

‘$\int \Phi_{\mathcal{N}}$’ $\longleftrightarrow$ Bers coordinates

(4) $\exists$ ord. F.L. $\Phi_{\mathcal{A}} \Longrightarrow$

Serre-Tate Theory for ord. AV's

(can. mult. pars. $\longleftrightarrow$ "S.-T. pars.")

$\Rightarrow$ Th.3 = Serre-Tate

theory for hyp. curves!

(5) $\Phi_{\mathcal{N}}$, $\Phi_{\mathcal{A}}$, "ord's" <u>not compatible</u>!

$\longleftrightarrow \quad \mathcal{M}_g \to \mathcal{A}_g$ <u>not isometric</u>/$\mathbf{C}$

(for WP metric,

Siegel upper half-plane metric)