Fundamental groups of log configuration spaces and the cuspidalization problem

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Contents

1 Introduction

In this paper, we consider *the cuspidalization problem* of the fundamental group of a curve. Let X be a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K whose (not necessarily positive) characteristic we denote by p.

Problem 1.1. *Let $U \hookrightarrow X$ be an open subscheme of X. Then can one reconstruct the* (*arithmetic*) *fundamental group*

$$\pi_1(U)$$

of U from the (*arithmetic*) *fundamental group $\pi_1(X)$ of X?*

More "generally",

Problem 1.2. *Let r be a natural number. Then can one reconstruct the* (*arithmetic*) *fundamental group*

$$\pi_1(U_{(r)})$$

of the r-th configuration space $U_{(r)}$ of X (*i.e., the open subscheme of the r-th product of X* [*over K*] *whose complement consists of the diagonals "$D_{(r)\{i,j\}} = \{(x_1, \cdots, x_r) \mid x_i = x_j\}$" $(i \neq j)$*) *from the* (*arithmetic*) *fundamental group $\pi_1(X)$ of X?*

In this paper, we study Problem 1.2 by means of the log geometry of the *the log configuration scheme of X*, which is a natural compactification of $U_{(r)}$.

Let $\overline{\mathcal{M}}^{\log}_{g,r}$ be the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ of r-pointed stable curves of genus g whose r sections are equipped with an ordering with the log structure associated to the divisor with normal crossings which parametrizes singular curves. Then, for a natural number r, we define the (n-th) log configuration scheme $X^{\log}_{(r)}$ as the fiber product

$$\operatorname{Spec} K \times_{\overline{\mathcal{M}}^{\log}_{g,0}} \overline{\mathcal{M}}^{\log}_{g,r},$$

where the (1-) morphism $\operatorname{Spec} K \to \overline{\mathcal{M}}^{\log}_{g,0}$ is the classifying (1-)morphism determined by the curve $X \to \operatorname{Spec} K$, and the (1-)morphism $\overline{\mathcal{M}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,0}$ is the (1-)morphism obtained by forgetting the sections. Note that the interior of $X^{\log}_{(r)}$ (i.e., the largest open subset of the underlying scheme of $X^{\log}_{(r)}$ on which the log structure is trivial) is *the usual (r-th) configuration space* $U_{(r)}$ of X, and that the natural inclusion $U_{(r)} \hookrightarrow X^{\log}_{(r)}$ induces an isomorphism of *the geometric pro-prime to p quotient* of $\pi_1(U_{(r)})$ (i.e., the quotient of $\pi_1(U_{(r)})$ by the kernel of the natural surjection from the geometric fundamental group of $U_{(r)}$ to its maximal pro-prime to p quotient) with *the geometric pro-prime to p quotient* of $\pi_1(X^{\log}_{(r)})$.

Let Σ be a (non-empty) set of prime numbers. We shall denote by $\Pi^{\log}_{X_{(r)}}$ the geometric pro-Σ quotient of $\pi_1(X^{\log}_{(r)})$, and by $\Pi^{\log}_{\mathbb{P}_K}$ the geometric pro-Σ quotient of the log fundamental group of the log scheme $\mathbb{P}^{\log}_K$ obtained by equipping the projective line $\mathbb{P}^1_K$ with the log structure associated to the divisor $\{0, 1, \infty\} \subseteq \mathbb{P}^1_K$. Then the first main result of this paper is as follows (cf. Theorem 7.4):

Theorem 1.3. *Let $r \geq 3$ be an integer. Then there exist extensions*

$$\Pi_1,\ \Pi_3$$

of $\Pi^{\log}_{X_{(r-1)}}$ *by* $\hat{\mathbb{Z}}^{(\Sigma)}(1)$, *an extension*

$$\Pi_2$$

of $\Pi^{\log}_{X_{(r-2)}} \times_{G_K} \Pi^{\log}_{\mathbb{P}^1_K}$ *by* $\hat{\mathbb{Z}}^{(\Sigma)}(1)$ *and continuous homomorphisms*

$$\Pi_i \longrightarrow \Pi^{\log}_{X_{(r)}} \ \ (1 \leq i \leq 3)$$

such that the morphism

$$\Pi^{\mathcal{G}}_{X_{(r)}} \stackrel{\text{def}}{=} \varinjlim(\Pi_1 \leftarrow \{1\} \rightarrow \Pi_2 \leftarrow \{1\} \rightarrow \Pi_3) \longrightarrow \Pi^{\log}_{X_{(r)}}$$

induced by the morphisms $\Pi_i \rightarrow \Pi^{\log}_{X_{(r)}}$ *is surjective, where the inductive limit is taken in the category of profinite groups.*

Note that Theorem 1.3 can be regarded as a logarithmic analogue of [13], Remark 1.2.

We shall denote by $p^{\log}_{X_{(r)}i} : X^{\log}_{(r+1)} \rightarrow X^{\log}_{(r)}$ the morphism obtained by "forgetting" the i-th section. Then the second main result of this paper is as follows (cf. Theorem 7.15):

Theorem 1.4. *Let $r \geq 2$ be an integer. Moreover, we assume that*

$$\Sigma = \begin{cases} \text{the set of all prime numbers or } \{l\} & \text{if } p = 0 \\ \{l\} & \text{if } p \geq 2\,. \end{cases}$$

If the collection of data consisting of the profinite groups $\Pi^{\log}_{X_{(k)}}$ $(0 \leq k \leq r)$, *the profinite group* $\Pi^{\log}_{\mathbb{P}}$, *the surjections* $\Pi^{\log}_{X_{(k)}} \rightarrow \Pi^{\log}_{X_{(k-1)}}$ $(1 \leq k \leq r)$ *induced by the* $p^{\log}_{X_{(k-1)}i}$*'s* $(1 \leq k \leq r,\ 1 \leq i \leq k)$ *and the structure morphism of* X, *the morphism* $\Pi^{\log}_{\mathbb{P}} \rightarrow G_K$ *induced by the structure morphism of* $\mathbb{P}^{\log}_K$ *and some data concerning the log fundamental groups of the irreducible components of the divisor at infinity (i.e., the divisor with normal crossings which defines the log structure) of* $X^{\log}_{(r)}$ *is given, then we can "reconstruct" the profinite group*

$$\Pi^{\mathcal{G}}_{X_{(r+1)}}$$

defined in Theorem 1.3 and morphisms

$$q_{X_{(r)}i} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \longrightarrow \Pi^{\log}_{X_{(r)}} \ \ (1 \leq i \leq r+1)$$

such that $q_{X_{(r)i}}$ factors as the composite

$$\Pi^{\mathcal{G}}_{X_{(r+1)}} \longrightarrow \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\text{via } p^{\log}_{X_{(r)^i}}} \Pi^{\log}_{X_{(r)}}$$

where the first morphism is the morphism obtained in Theorem 1.3.

In Theorem 1.4, we use the terminology "reconstruct" as a sort of "abbreviation" for the somewhat lengthy but mathematically precise formulation given in the statement of Theorem 7.15.

By Theorem 1.3 and Theorem 1.4, if one can also reconstruct group-theoretically *the kernel of the surjection* $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (which appears in the above composite), then, by taking the quotient by this kernel, one can reconstruct the profinite group $\Pi^{\log}_{X_{(r+1)}}$. However, unfortunately, reconstruction of this kernel is not performed in this paper. Moreover, it seems to the author that if such a reconstruction should prove to be possible, it is likely that the method of reconstruction of this kernel should depend on the "arithmetic" of K in an essential way.

This paper is organized as follows:

In Section 2, we prove the existence of *a logarithmic version of the Stein factorization* under some hypotheses (cf. Definition 2.11, Theorem 2.9, also Remark 2.13). In [7], Exposé X, Corollaire 1.4, the exactness of the homotopy sequence associated to a proper, separable morphism is proven. In this proof, the existence of the Stein factorization plays an essential role. Therefore, to prove a logarithmic analogue of the exactness of the homotopy sequence, we consider the existence of a logarithmic analogue of the Stein factorization.

In Section 3, we prove a logarithmic analogue of [7], Exposé X, Corollaire 1.4, i.e., the exactness of *the log homotopy sequence* by means of the existence of the log Stein factorization (cf. Theorem 3.3). Moreover, a logarithmic analogue of the fact that the fundamental group of the scheme obtained by taking the product of schemes is naturally isomorphic to the product of the fundamental groups of these schemes (cf. [7], Exposé X, Corollaire 1.7) is proven (cf. Proposition 3.4). These results are used in Section 5 and 7.

In Section 4, we define the notion of a log structure on a formal scheme and establish a theory of *algebraizations* of log formal schemes. One can develop a theory of algebraizations of log formal schemes (cf. Theorem 4.5) in a similar fashion to the classical theory of algebraizations of formal schemes (for example, the theory considered in [4], §5). However, in the case of algebraizations of log formal schemes, it is insufficient only to assume a "compactness condition" of the sort that is required in the classical algebraization theory of formal schemes; in addition to such a "compactness condition", a certain reducedness hypothesis is necessary (cf. Remark 4.6, 4.7). This algebraization theory of formal log schemes implies a logarithmic analogue of the fact that the fundamental group of a proper smooth scheme over a "complete base" is naturally isomorphic to the fundamental group of the closed fiber (cf. [7], Exposé X, Théorème 2.1, also [22], Théorème 2.2, (a)) (cf. Corollary 4.8). This result is used in the next section.

In Section 5, we define the notion of a *morphism of type* $\mathbb{N}^{\oplus n}$ and consider fundamental properties of such a morphism. Roughly speaking, a morphism of log schemes is of type $\mathbb{N}^{\oplus n}$ if the relative characteristic is locally constant with stalk isomorphic to $\mathbb{N}^{\oplus n}$. The main result of this section is the fact that at the level of anabelioids (i.e., Galois categories) (determined by ket coverings), certain morphisms of type $\mathbb{N}^{\oplus n}$ can be regarded as "$\mathbb{G}_m^{\times n}$-fibrations" (cf. Theorem 5.18). Moreover, following [15], Lemma 4.4, we give a sufficient condition for the homomorphism from the log fundamental group of the fiber of the "$\mathbb{G}_m^{\times n}$-fibration" determined by such a morphism of type $\mathbb{N}^{\oplus n}$ to the log fundamental group of total space of the "$\mathbb{G}_m^{\times n}$-fibration" to be injective (cf. Proposition 5.23).

In Section 6, we consider the scheme-theoretic and log scheme-theoretic properties of log configuration schemes. Moreover, we study the geometry of the divisor at infinity of $X_{(r)}^{\log}$ in more detail.

In Section 7, we consider *the reconstruction of the fundamental groups of higher dimensional log configuration schemes* by means of the results obtained in previous sections.

Finally, in the Appendix, we prove the well-known fact that the category of ket coverings of a connected locally noetherian fs log scheme is a *Galois category*; this implies, in particular, the existence of log fundamental groups (cf. Theorem A.1, also Theorem A.2). The log fundamental group has already been constructed by several people (e.g., [3], [8], 4.6, [20], 3.3, [22], 1.2). Since, however, at the time of writing, a proof of this fact was not available in published form, and, moreover, various facts used in the proof of this fact are necessary elsewhere in this paper, we decided to give a proof of this fact. Moreover, although other authors approach the problem of showing that the category of ket coverings of a log scheme is a Galois category by considering the category of locally constant sheaves on the Kummer log étale site, we take a more direct approach to this problem which allows us to avoid the use of locally constant sheaves on the Kummer log étale site.

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Notation

Symbols

We shall denote by $\mathbb{Z}$ the set of rational integers, by $\mathbb{N}$ the set of rational integers $n \geq 0$, by $\mathbb{Q}$ the set of rational numbers and by $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$.

Subscripts

For a ring A (respectively, a scheme X), we shall denote by A_{red} (respectively, X_{red}) the quotient ring by the ideal of all nilpotent elements of A (respectively, the reduced closed subscheme of X associated to X). For a ring A, we shall denote by A^* the group of unity of A. For a field k, we shall use the notation k^{sep} to denote a separable closure of k. For a monoid P, (respectively, a sheaf of monoids $\mathcal{P}$) we shall denote by P^{gp} the group associated to P (respectively, $\mathcal{P}^{\mathrm{gp}}$ the sheaf of groups associated to $\mathcal{P}$). For a group G, we shall denote by $G^{\mathrm{a}b}$ the abelianization of G.

Terminologies

We shall assume that the underlying topological space of a *connected* scheme is not empty. In particular, if a morphism is geometrically connected, then it is surjective.

Let Σ be a set of prime numbers, and n an integer. Then we shall say that n is a Σ-*integer* if the prime divisors of n are in Σ. Let Γ be a profinite group. We shall refer to the quotient

$$\varprojlim \Gamma/H$$

(where the projective limit is over all open normoal subgroups $H \subseteq \Gamma$ whose orders are Σ-integers) as the pro-Σ quotient of Γ. We shall denote by $\Gamma^{(\Sigma)}$ the pro-Σ quotient of Γ.

We shall refer to the largest open subset (possibly empty) of the underlying scheme of an fs log scheme on which the log structure is trivial as the *interior* of the fs log scheme. We shall refer to a Kummer log étale (respectively, finite Kummer log étale) morphism of fs log schemes as a *ket* morphism (respectively, a *ket covering*).

Log schemes

For a log scheme $X^{\log}$, we shall denote by $\mathcal{M}_X$ the sheaf of monoids that defines the log structure of $X^{\log}$.

Let $\mathcal{P}$ be a property of schemes [for example, "quasi-compact", "connected", "normal", "regular"] (respectively, morphisms of schemes [for example, "proper", "finite", "étale", "smooth"]). Then we shall say that a log scheme (respectively, a morphism of log schemes) satisfies $\mathcal{P}$ if the underlying scheme (respectively, the underlying morphism of schemes) satisfies $\mathcal{P}$.

For a log scheme $X^{\log}$ (respectively, a morphism $f^{\log}$ of log schemes), we shall denote by X the underlying scheme (respectively, by f the underlying morphism of schemes). For fs log schemes $X^{\log}$, $Y^{\log}$ and $Z^{\log}$, we shall denote by $X^{\log} \times_{Y^{\log}} Z^{\log}$ the fiber product of $X^{\log}$ and $Z^{\log}$ over $Y^{\log}$ in the category of fs log schemes. In general, the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is not $X \times_Y Z$. However, since strictness (a morphism $f^{\log} : X^{\log} \to Y^{\log}$ is called *strict* if the induced morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ on X is an isomorphism) is stable under base-change in the category of *arbitrary* log schemes, if $X^{\log} \to Y^{\log}$ is strict, then the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is $X \times_Y Z$. Note that since the natural morphism from the saturation of a fine log scheme to the original fine log scheme is finite, properness and finiteness are stable under fs base-change.

If there exist both schemes and log schemes in a commutative diagram, then we regard each scheme in the diagram as the log scheme obtained by equipping the scheme with the trivial log structure.

2 The log Stein factorization

Definition 2.1. Let $X^{\log}$ be an fs log scheme, and $\overline{x} \to X$ a geometric point.

(i) We shall refer to the strict morphism $\overline{x}^{\log} \to X^{\log}$ whose underlying morphism of schemes is $\overline{x} \to X$ as the *strict geometric point* over $\overline{x} \to X$.

(ii) We shall refer to $\overline{x}_1^{\log} \to X^{\log}$ as a *reduced covering point* over the strict geometric point $\overline{x}^{\log} \to X^{\log}$ or, alternatively, over the geometric point $\overline{x} \to X$, if it is obtained as a composite

$$\overline{x}_1^{\log} \longrightarrow \overline{x'}_1^{\log} \longrightarrow \overline{x}^{\log} \longrightarrow X^{\log},$$

where $\overline{x}^{\log} \to X^{\log}$ is the strict geometric point over $\overline{x} \to X$, $\overline{x'}_1^{\log} \to \overline{x}^{\log}$ is a connected ket covering, and $\overline{x}_1^{\log} \to \overline{x'}_1^{\log}$ is a strict morphism of fs log schemes for which the underlying morphism of schemes determines an isomorphism $\overline{x}_1 \simeq \overline{x'}_{1,\mathrm{red}}$. Note that, in general, $\overline{x}_1^{\log} \to \overline{x}^{\log}$ is *not* a ket covering. (See Remark 2.2 below.)

Remark 2.2. The underlying scheme of the domain of a strict geometric point $\overline{x}^{\log} \to X^{\log}$ is the spectrum of a separably closed field. However, in general, the underlying scheme of the domain of a connected ket covering

$\overline{x'}_1^{\log} \to \overline{x}^{\log}$ is not the spectrum of a separably closed field. On the other hand, if we denote by $\overline{x}_1^{\log}$ the log scheme obtained by equipping $\overline{x'}_{1,\mathrm{red}}$ with the log structure induced by the log structure of $\overline{x'}_1^{\log}$ (i.e., the natural morphism $\overline{x}_1^{\log} \to X^{\log}$ is a reduced covering point over $\overline{x}^{\log} \to X^{\log}$), then the following hold:

(i) The underlying scheme of $\overline{x}_1^{\log}$ is the spectrum of a separably closed field (by Proposition A.4).

(ii) There is a natural equivalence between the category of ket coverings of $\overline{x}_1^{\log}$ and the category of ket coverings of $\overline{x'}_1^{\log}$ (by Proposition A.8). In particular, $\pi_1(\overline{x'}_1^{\log}) \simeq \pi_1(\overline{x}_1^{\log})$. (Concerning the log fundamental group, see Theorem A.1.)

(iii) The natural morphism $\overline{x}_1^{\log} \to \overline{x'}_1^{\log}$ is a homeomorphism on the underlying topological spaces and remains so after any base-change in the category of *fs log schemes* over $\overline{x'}_1^{\log}$. Indeed, this follows from the fact that this morphism is strict, together with the fact that the underlying morphism of schemes is a universal homeomorphism.

Definition 2.3. Let $X^{\log}$ be an fs log scheme, $\overline{x} \to X$ a geometric point of X, $U \to X$ an étale neighborhood of $\overline{x} \to X$, and $P \to \mathcal{O}_U$ an fs chart at $\overline{x} \to X$. Then we shall say that the chart $P \to \mathcal{O}_U$ is clean at $\overline{x} \to X$ if the composite $P \to \mathcal{M}_{X,\overline{x}} \to (\mathcal{M}_X/\mathcal{O}_X^*)_{\overline{x}}$ is an isomorphism. A clean chart of $X^{\log}$ always exists over an étale neighborhood of any given geometric point of X. (See the following discussion of [14], Definition 1.3.)

The following technical lemma follows immediately from Proposition A.8.

Lemma 2.4. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring. Then for a strict geometric point $\overline{x}^{\log} \to X^{\log}$ for which the image of the underlying morphism of schemes is the closed point of X, and any reduced covering point $\overline{x}_1^{\log} \to X^{\log}$ over $\overline{x}^{\log} \to X^{\log}$, there exists a ket covering $Y^{\log} \to X^{\log}$ and a strict geometric point $\overline{y}^{\log} \to Y^{\log}$ such that $\overline{y}^{\log} \to Y^{\log} \to X^{\log}$ factors as a composite $\overline{y}^{\log} \to \overline{x}_1^{\log} \to X^{\log}$, where the morphism $\overline{y}^{\log} \to \overline{x}_1^{\log}$ is a reduced covering point over the strict geometric point $\overline{x}_1^{\log} \to \overline{x}_1^{\log}$ given by the identity morphism of $\overline{x}_1^{\log}$.*

In the following discussion, we will show the existence of a logarithmic version of the Stein factorization.

Lemma 2.5. *Let $X^{\log}$ be a quasi-compact fs log scheme equipped with the trivial log structure, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper* *log smooth morphism. Then the morphism $X' \to X$ that appears in the Stein factorization $Y \to X' \to X$ of f is finite étale.*

Proof. By [7], Exposé X, Proposition 1.2, it is enough to show that f is proper and separable. The properness of f is assumed in the statement of Lemma 2.5. Since the log structure of $X^{\log}$ is trivial, $f^{\log}$ is integral ([10], Proposition 4.1). Since an integral log smooth morphism is flat ([10], Theorem 4.5), f is flat. For the rest of the proof of the separability of f, by base-changing, we may assume that $X = \operatorname{Spec} k$, where k is a field whose characteristic we denote by p. Then étale locally on Y, there exist an fs monoid P whose associated group P^{gp} is p-torsion-free if p is not zero and an étale morphism $Y \to \operatorname{Spec} k[P]$ over k ([10], Theorem 3.5). On the other hand, $k[P] \otimes_k K \subseteq k[P^{\mathrm{gp}}] \otimes_k K$, and $k[P^{\mathrm{gp}}] \otimes_k K = K[P^{\mathrm{gp}}]$ is reduced for any extension field K of k by the assumption on P^{gp}; thus, $k[P] \otimes_k K$, hence also Y is reduced. Therefore, f is separable. □

Lemma 2.6. *Let $X^{\log}$ be a log regular, quasi-compact fs log scheme, $U_X \subseteq X$ the interior of $X^{\log}$ $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. If we denote by $Y \times_X U_X \to V \to U_X$ the Stein factorization of $f\,|_{Y \times_X U_X}$, then the following hold:*

(*i*) *$V \to U_X$ is finite étale.*

(*ii*) *The normalization of X in V is tamely ramified over the generic points of $D_X = X \setminus U_X$.*

Proof. Since log smoothness and properness are stable under base-change, (i) follows from Lemma 2.5. For (ii), since normalization and the operation of taking Stein factorization commute with étale localization, we may assume that X is the spectrum of a strictly henselian discrete valuation ring R whose field of fractions we denote by K, and whose residue field we denote by k. Then the log regularity of $X^{\log}$ implies that the log structure of $X^{\log}$ is trival, or is defined by the closed point of X ([11], Theorem 11.6). If the log structure of $X^{\log}$ is trivial, then (ii) follows from (i). Thus, we may assume that the log structure of $X^{\log}$ is not trivial. Moreover, for (ii), we may assume that V is connected. Then, by (i), $\Gamma(V, \mathcal{O}_V)$ is a finite separable extension field of K. We denote this field by L.

Let us denote the integral closure of R in L by R_L. Thus, the normalization X' of X in V is $\operatorname{Spec} R_L$, $U_X = \operatorname{Spec} K$, and $V = \operatorname{Spec} L$. Therefore, we obtaine the following commutative diagram:

$$\begin{array}{ccccc} & & \operatorname{Spec} L & & \operatorname{Spec} K \\ & & \| & & \| \\ Y^{\log} \times_{X^{\log}} U_X & \longrightarrow & V & \longrightarrow & U_X \\ \downarrow & & \downarrow & & \downarrow \\ Y^{\log} & \longrightarrow & X' & \longrightarrow & X \\ & & \| & & \| \\ & & \operatorname{Spec} R_L & & \operatorname{Spec} R\,. \end{array}$$

Note that since $V \to U_X$ is finite étale, R_L is finite over R. Let $\overline{y} \to Y$ be a geometric point of Y over the closed point of X'.

Now, by [10], Theorem 3.5, there exists

- a connected étale neighborhood W of $\overline{y} \to Y$;
- an fs monoid chart $P \to \mathcal{O}_W$ of $Y^{\log}$; and
- a chart

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & P \\ \downarrow & & \downarrow \\ R & \longrightarrow & \mathcal{O}_W \end{array}$$

  of $Y^{\log} \to (\operatorname{Spec} R)^{\log}$ (where $\mathbb{N} \to R$ is a chart of $(\operatorname{Spec} R)^{\log}$ such that $1 \mapsto \pi_R$ [π_R is a prime element of R])

such that

(i) $\mathbb{N} \to P$ is injective, and if the image of 1 is $t \in P$, then the torsion part of $P^{\mathrm{gp}}/\langle t \rangle$ is a finite group of order invertible in R; and

(ii) the natural morphism $W \to \operatorname{Spec} R[P]/(\pi_R - t)$ is étale.

Thus, we have a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \operatorname{Spec} R[P]/(\pi_R - t) \\ \downarrow & & \downarrow \\ \operatorname{Spec} R_L & \longrightarrow & \operatorname{Spec} R\,. \end{array}$$

Therefore, it follows from the above conditions (i) and (ii) that if the image of π_R in R_L has valuation r, then r is invertible in R, hence in k.

Moreover, by base-changing by $R \to k$ and taking "$(-)_{\mathrm{red}}$", we obtain a commutative diagram

$$\begin{array}{ccc}
(W \times_R k)_{\mathrm{red}} & \longrightarrow & \operatorname{Spec}(k[P]/(t))_{\mathrm{red}} \\
\downarrow & & \downarrow \\
W \times_R k & \longrightarrow & \operatorname{Spec} k[P]/(t) \\
\downarrow & & \downarrow \\
\operatorname{Spec} R_L/\pi_R R_L & \longrightarrow & \operatorname{Spec} k\,.
\end{array}$$

Since the middle horizontal arrow of the diagram is étale, it follows that the upper square is cartesian; thus, $(W \times_R k)_{\mathrm{red}} \to \operatorname{Spec}(k[P]/(t))_{\mathrm{red}}$ is also étale. Since $\operatorname{Spec}(k[P]/(t))_{\mathrm{red}}$ is geometrically reduced over k, it follows that $\operatorname{Spec}(k[P]/(t))_{\mathrm{red}}$, hence also, $(W \times_R k)_{\mathrm{red}}$ has a k-rational point. Therefore the residue field of R_L is k. $\square$

Definition 2.7. Let $X^{\log}$ and $Y^{\log}$ be fs log schemes. Then we shall say that a morphism $f^{\log} : Y^{\log} \to X^{\log}$ is *log geometrically connected* if for any reduced covering point $\overline{x}_1^{\log} \to \overline{x}^{\log}$ over a strict geometric point $\overline{x}^{\log} \to X^{\log}$, the fiber product $Y^{\log} \times_{X^{\log}} \overline{x}_1^{\log}$ is connected.

Note that it follows from Remark 2.2, (iii), that this condition is equivalent to the condition that for any connected ket covering $\overline{x'}^{\log} \to \overline{x}^{\log}$ of a strict geometric point $\overline{x}^{\log} \to X^{\log}$, $Y^{\log} \times_{X^{\log}} \overline{x'}^{\log}$ is connected.

Remark 2.8. In log geometry, there exists the notion of a *log geometric point*. In fact, one can regard a log geometric point as a *limit* of ket coverings over a strict geometric point. Thus, one natural way to define log geometric connectedness is by the condition that every base-change via a log geometric point is connected. However, in general, a log geometric point is not a fine log scheme. Hence we can not perform such a base-change in the category of fs log schemes.

Theorem 2.9. *Let $X^{\log}$ be a log regular, quasi-compact fs log scheme, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. If we denote by $Y \xrightarrow{f'} X' \xrightarrow{g} X$ the Stein factorization of f, then X' admits a log structure that satisfies the following properties:*

- *(i) There exists a ket covering $X'^{\log} \to X^{\log}$ whose underlying morphism of schemes is g.*
- *(ii) $Y^{\log} \to X'^{\log}$ is log geometrically connected.*

Proof. Let $U_X \subseteq X$ be the interior of $X^{\log}$. If we denote by $Y \times_X U_X \to V \to U_X$ the Stein factorization of $Y \times_X U_X \to U_X$, then, by Lemma 2.6, $V \to U_X$ is finite étale, and the normalization Z of X in V is tamely ramified over the generic points of $D_X = X \setminus U_X$. Hence Z admits a log structure that determines a ket covering $Z^{\log} \to X^{\log}$ by the log purity theorem in [14]. (Concerning *the log purity theorem*, see Remark 2.10 below.) Now $Y^{\log}$ is log regular, hence normal ([10], Theorem 4.1); thus, X' is normal. Therefore $X' \to X$ factors through Z. Since both $X' \times_X U_X$ and $Z \times_X U_X$ are naturally isomorphic to V, we have $X' \simeq Z$. This completes the proof of (i).

For (ii), since the operation of taking the Stein factorization commutes with étale base-change, by base-changing, we may assume that both X and X' are the spectra of strictly henselian local rings. Moreover, by Lemma 2.4, it is enough to show that for any connected ket covering $X_1^{\log} \to X^{\log}$ and any strict geometric point $\overline{x}^{\log} \to X'^{\log} \times_{X^{\log}} X_1^{\log}$ for which the image of the unerlying morphism of schemes is the closed point, $Y^{\log} \times_{X'^{\log}} \overline{x}^{\log}$ is connected.

Let us denote by $Y_1^{\log}$ the fiber product $Y^{\log} \times_{X^{\log}} X_1^{\log}$. Since log smoothness and properness are stable under base-change, $Y_1^{\log} \to X_1^{\log}$ is log smooth and proper. By (i), if we denote by $Y_1 \to X_1' \to X_1$ the Stein factorization of $Y_1 \to X_1$, then X_1' admits a log structure such that the resulting morphism $X_1'^{\log} \to X_1^{\log}$ is a ket covering. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc}
Y_1^{\log} & \longrightarrow & X_1'^{\log} & \longrightarrow & X_1^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
Y^{\log} & \longrightarrow & X'^{\log} & \longrightarrow & X^{\log} .
\end{array}$$

Now I claim that the right-hand square in the above commutative diagram is cartesian. Note that it follows formally from this claim that the left-hand square is also cartesian. In particular, it follows from this claim, together with the connectedness property of the Stein factorization, that $Y^{\log} \times_{X'^{\log}} \overline{x}^{\log} = Y_1^{\log} \times_{X_1'^{\log}} \overline{x}^{\log}$ is connected for any strict geometric point $\overline{x}^{\log} \to X_1'^{\log}$.

The claim of the preceding paragraph may be verified follows: If we base-change by $U_X \to X^{\log}$, then we obtain a commutative diagram

$$\begin{array}{ccccc}
Y_1^{\log} \times_{X^{\log}} U_X & \longrightarrow & X_1'^{\log} \times_{X^{\log}} U_X & \longrightarrow & X_1^{\log} \times_{X^{\log}} U_X \\
\downarrow & & \downarrow & & \downarrow \\
Y^{\log} \times_{X^{\log}} U_X & \longrightarrow & X'^{\log} \times_{X^{\log}} U_X & \longrightarrow & U_X .
\end{array}$$

Since $U_X \to X^{\log}$ is a strict morphism, and the log structures of U_X and $X_1^{\log} \times_{X^{\log}} U_X$ are trivial, the underlying scheme of $Y_1^{\log} \times_{X^{\log}} U_X$ [$= Y^{\log} \times_{X^{\log}}$

$X_1^{\log} \times_{X^{\log}} U_X = Y^{\log} \times_{X^{\log}} U_X \times_{U_X} (U_X \times_{X^{\log}} X_1^{\log})]$ is $Y_1 \times_X U_X$. Moreover, $X_1^{\log} \times_{X^{\log}} U_X \to U_X$ is finite étale, hence flat. Thus, the underlying morphism of schemes of $Y_1^{\log} \times_{X^{\log}} U_X \to (X'^{\log} \times_{X^{\log}} X_1^{\log}) \times_{X^{\log}} U_X \to X_1^{\log} \times_{X^{\log}} U_X$ is the Stein factorization of the underlying morphism of schemes of $Y_1^{\log} \times_{X^{\log}} U_X \to X_1^{\log} \times_{X^{\log}} U_X$; in particular, $X_1'^{\log} \times_{X^{\log}} U_X \simeq (X'^{\log} \times_{X^{\log}} X_1^{\log}) \times_{X^{\log}} U_X$. Therefore $X_1'^{\log} \simeq X'^{\log} \times_{X^{\log}} X_1^{\log}$ by Proposition A.10. □

Remark 2.10. In [14], Theorem 3.3, it is only stated that:

Let $X^{\log}$ be a log regular, quasi-compact fs log scheme and U_X the interior of $X^{\log}$. Let $V \to U_X$ be a finite étale morphism which is tamely ramified over the generic points of $X \setminus U_X$. Let Y be the normalization of X in V and $Y^{\log}$ the log scheme obtained by equipping Y with the log structure $\mathcal{O}_Y \cap (V \hookrightarrow Y)_* \mathcal{O}_V^* \to \mathcal{O}_Y$. Then the following hold:

- *$Y^{\log}$ is log regular.*
- *The finite étale morphism $V \to U_X$ extends uniquely to a log étale morphism $Y^{\log} \to X^{\log}$.*

However, in fact, $Y^{\log} \to X^{\log}$ is Kummer by the proof of the log purity theorem in *loc. cit.* (More precisely, in the notation of *loc.cit.*, the inclusions $P \subseteq P_Y \subseteq (1/n)P$ imply this fact.) Moreover, since $V \to U_X$ is finite étale, it follows that the normalization $Y \to X$ is finite.

Definition 2.11. In the notation of Theorem 2.9, we shall refer to $Y^{\log} \to X'^{\log} \to X^{\log}$ as the *log Stein factorization* of $f^{\log}$. This name is motivated by condition (ii) in the statement of Theorem 2.9.

Proposition 2.12. *The operation of taking log Stein factorization commutes with base-change by a morphism which satisfies the following condition $(*)$:*

$(*)$ *The domain is a log regular, quasi-compact fs log scheme, and the restriction of the morphism to the interior is flat.*

(*For example, a quasi-compact ket morphism satisfies* $(*)$.)

Proof. Let $X^{\log}$ be a log regular, quasi-compact fs log scheme, $f^{\log} : Y^{\log} \to X^{\log}$ a proper, log smooth morphism, and $g^{\log} : X_1^{\log} \to X^{\log}$ a morphism which satisfies the condition $(*)$ in the statement of Proposition 2.12. Let us denote by $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ the base-change of $f^{\log}$ by $g^{\log}$, and by $Y^{\log} \to X'^{\log} \to X^{\log}$ (respectively, $Y_1^{\log} \to X_1'^{\log} \to X_1^{\log}$) the log Stein factorization of $f^{\log}$ (respectively $f_1^{\log}$). Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccc} Y_1^{\log} & \longrightarrow & X_1'^{\log} & \longrightarrow & X_1^{\log} \\ \downarrow & & \downarrow & & \downarrow g^{\log} \\ Y^{\log} & \longrightarrow & X'^{\log} & \longrightarrow & X^{\log}. \end{array}$$

If we denote by $X_2^{\log}$ the fiber product $X_1^{\log} \times_{X^{\log}} X'^{\log}$, then the above commutative diagram determines a morphism $X_1'^{\log} \to X_2^{\log}$. Our claim is that this morphism is an isomorphism.

Let $U_1 \subseteq X_1$ be the interior of $X_1^{\log}$. Since $g^{\log}$ is Kummer, the morphism $U_1 \to X^{\log}$ factors through U; in particular, $U_1 \to X^{\log}$ is strict. Therefore the underlying scheme of $Y_1^{\log} \times_{X_1^{\log}} U_1$ is $Y \times_X U_1$, and the factorization induced on the underlying schemes by the factorization $Y_1^{\log} \times_{X_1^{\log}} U_1 \to X_1'^{\log} \times_{X_1^{\log}} U_1 \to U_1$ is the Stein factorization of the underlying morphism of $Y_1^{\log} \times_{X_1^{\log}} U_1 \to U_1$. On the other hand, it follows from the flatness of $U_1 \to X$ that the factorization induced on the underlying schemes by the factorization $Y_1^{\log} \times_{X_1^{\log}} U_1 \to X_2^{\log} \times_{X_1^{\log}} U_1 \to U_1$ is also the Stein factorization of the underlying morphism $Y_1^{\log} \times_{X_1^{\log}} U_1 \to U_1$. Thus, we obtain $X_1'^{\log} \times_{X_1^{\log}} U_1 \simeq X_2^{\log} \times_{X_1^{\log}} U_1$. Now $X_1'^{\log} \to X^{\log}$ and $X_2^{\log} \to X^{\log}$ are ket coverings; thus, by Proposition A.10, $X_1'^{\log} \simeq X_2^{\log}$. ☐

Remark 2.13. In this section, we only consider the log Stein factorization in the case where the base log scheme is log regular. However, if a morphism $f^{\log} : Y^{\log} \to X^{\log}$ of fs log schemes admits the following *cartesian diagram*:

$$\begin{array}{ccc} Y^{\log} & \xrightarrow{f^{\log}} & X^{\log} \\ \downarrow & & \downarrow \\ Y_1^{\log} & \xrightarrow{f_1^{\log}} & X_1^{\log}, \end{array}$$

where

- $X_1^{\log}$ is a log regular, quasi-compact, fs log scheme, and $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ is a proper, log smooth morphism from an fs log scheme,
- $X^{\log} \to X_1^{\log}$ is strict,

then the factorization $Y^{\log} \to X_1'^{\log} \times_{X_1^{\log}} X^{\log} \to X^{\log}$ obteined by base-changing the log Stein factorization $Y_1^{\log} \to X_1'^{\log} \to X_1^{\log}$ of $f_1^{\log}$ by $X^{\log} \to X_1^{\log}$ satisfies the following:

- $Y^{\log} \to X_1'^{\log} \times_{X_1^{\log}} X^{\log}$ is log geometrically connected.

- $X_1'^{\log} \times_{X_1^{\log}} X^{\log} \to X^{\log}$ is a ket covering.

3 The log homotopy exact sequence

Proposition 3.1. *Let $X^{\log}$ be a log regular, connected, quasi-compact fs log scheme, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. Then the following conditions are equivalent:*

(i) $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.

(ii) If we denote the Stein factorization of f by $Y \to X' \to X$, then the morphism $X' \to X$ is an isomorphism (i.e., f is geometrically connected).

(iii) If we denote the log Stein factorization of $f^{\log}$ by $Y^{\log} \to X'^{\log} \to X^{\log}$, then the morphism $X'^{\log} \to X^{\log}$ is an isomorphism (i.e., $f^{\log}$ is log geometrically connected).

(iv) Y is connected, and $f^{\log}$ induces a surjection $\pi_1(Y^{\log}) \to \pi_1(X^{\log})$.

Moreover, the above four conditions imply the following condition:

(v) Y is connected, and f induces a surjection $\pi_1(Y) \to \pi_1(X)$.

Proof. The equivalence of the first three conditions is immediate from the constructions of the Stein and log Stein factorizations.

Now we assume the first three conditions. Then since f is surjective (by condition (i)), proper, and geometrically connected (by condition (ii)), it follows that Y is connected. Now let $X_1^{\log} \to X^{\log}$ be a connected ket covering, and $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ the base-change $Y^{\log} \times_{X^{\log}} X_1^{\log} \to X_1^{\log}$. Then f_1 is also sujective and proper. Moreover, it follows from Proposition 2.12 that f_1 is geometrically connected. Thus, Y_1 is connected. This completes the proof that the first three conditions imply (iv).

Next, we will show that (iv) implies (iii). Assume that $f^{\log}$ induces a surjection $\pi_1(Y^{\log}) \to \pi_1(X^{\log})$. If we denote by $Y^{\log} \to X'^{\log} \to X^{\log}$ the log Stein factorization of $f^{\log}$, then since Y is connected and $Y \to X'$ is surjective, X' is connected. Moreover, it follows from Theorem 2.9, (i), that $X'^{\log} \to X^{\log}$ is a ket covering. By the assumption (iv), $Y^{\log} \times_{X^{\log}} X'^{\log} \to Y^{\log}$ is also a connected ket covering. However, this covering has a section, hence $Y^{\log} \times_{X^{\log}} X'^{\log} \simeq Y^{\log}$. Thus, by applying the general theory of Galois categories to $\mathrm{K\acute{e}t}(X'^{\log})$ and $\mathrm{K\acute{e}t}(Y^{\log})$, we obtain $X'^{\log} \simeq X^{\log}$ (concerning $\mathrm{K\acute{e}t}(X^{\log})$, see Theorem A.1).

Finally, we will show that (iv) implies (v). It is immediate that the morphism $X^{\log} \to X$ determined by the morphism of sheaves of monoids $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ induces a surjection $\pi_1(X^{\log}) \to \pi_1(X)$. Thus, it follows from condition (iv), the fact that $\pi_1(X^{\log}) \to \pi_1(X)$ is surjective, and the existence of the commutative diagram

$$\begin{array}{ccc} \pi_1(Y^{\log}) & \longrightarrow & \pi_1(X^{\log}) \\ \big\downarrow & & \big\downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X)\,, \end{array}$$

that $\pi_1(Y) \to \pi_1(X)$ is surjective. $\square$

Remark 3.2. In the statement of Proposition 3.1, condition (v) does not imply condition (iv). Indeed, let R be a strictly henselian discrete valuation ring, K the field of fractions of R, L a tamely ramified extension of K, and R_L the integral closure of R in L. If we denote by $(\operatorname{Spec} R)^{\log}$ (respectively, $(\operatorname{Spec} R_L)^{\log}$) the log scheme obtained by equipping $\operatorname{Spec} R$ (respectively, $\operatorname{Spec} R_L$) with the log structure defined by the closed point, then the natural morphism $(\operatorname{Spec} R_L)^{\log} \to (\operatorname{Spec} R)^{\log}$ satisfies (v) (since $\pi_1(\operatorname{Spec} R) = 1$), but $\pi_1((\operatorname{Spec} R_L)^{\log}) \to \pi_1((\operatorname{Spec} R)^{\log})$ is not surjective unless $K = L$ (since $(\operatorname{Spec} R_L)^{\log} \to (\operatorname{Spec} R)^{\log}$ is a connected ket covering).

Next, we will show the exactness of the *log homotopy sequence.*

Theorem 3.3. *Let $X^{\log}$ be a log regular, connected, quasi-compact fs log scheme, $Y^{\log}$ a connected fs log scheme and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. Moreover, we assume one of conditions (i), (ii), (iii) and (iv) in Proposition 3.1. Then for any strict geometric point $\overline{x}^{\log} \to X^{\log}$, the following sequence:*

$$\varprojlim \pi_1(Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \xrightarrow{s} \pi_1(Y^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1$$

is exact, where the projective limit is over all reduced covering points $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$, and s is induced by the natural projections $Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to Y^{\log}$.

Proof. Note that, by Proposition 3.1, (iii), and the connectedness property of the log Stein factorization, $Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ is connected for any reduced covering point $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$ over $\overline{x}^{\log}$.

Next, observe that the surjectivity of $\pi_1(f^{\log})$ follows from Proposition 3.1, (iv). Moreover, it is immediate that $\pi_1(f^{\log}) \circ s = 1$. Hence it is sufficient to show that the kernel of $\pi_1(f^{\log})$ is generated by the image of s. By the general theory of profinite groups, it is enough to show that for an open subgroup G of $\pi_1(Y^{\log})$, if G contains the image of s, then G contains the kernel of $\pi_1(f^{\log})$. Let $Y_1^{\log} \to Y^{\log}$ be the connected ket covering corresponding to G. Then since G contains the image of s, there exists a reduced covering point $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$ such that $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ has a (ket) section. Since $Y_1^{\log} \to Y^{\log}$ is finite and log étale, it follows that $Y_1^{\log} \to X^{\log}$ is proper and log smooth. Let $Y_1^{\log} \to X_1^{\log} \to X^{\log}$ be the log Stein factorization of this morphism and $Y_2^{\log}$ the fiber product $Y^{\log} \times_{X^{\log}} X_1^{\log}$. Thus, we have a commutative diagram

$$\begin{tikzcd}
Y_1^{\log} \arrow[r] \arrow[d, equal] & Y_2^{\log} \arrow[r] \arrow[d] & X_1^{\log} \arrow[d] \\
Y_1^{\log} \arrow[r] & Y^{\log} \arrow[r, "f^{\log}"] & X^{\log}
\end{tikzcd}$$

(where the right-hand sequare is cartesian). Now I claim that $Y_1^{\log} \to Y_2^{\log}$ is an isomorphism. To prove this claim, it is enough to show the following:

(i) $Y_2^{\log}$ is connected.

(ii) $Y_1^{\log} \to Y_2^{\log}$ is a ket covering.

(iii) $Y_1^{\log} \to Y_2^{\log}$ has rank one at some point. (We shall say that a ket covering $Y^{\log} \to X^{\log}$ of locally noetherian fs log scheme has *rank one at some point*, if there exists a log geometric point of $X^{\log}$ such that, for the fiber functor F of $\mathrm{K\acute{e}t}(X^{\log})$ defined by the log geometric point [cf. Theorem A.1], the cardinality of $F(Y^{\log})$ is one.)

The first assertion follows from Proposition 3.1, (iv), and the second assertion follows from the fact that $Y_1^{\log} \to Y^{\log}$ and $Y_2^{\log} \to Y^{\log}$ are ket coverings and Proposition A.5. Hence, in the rest of the proof, we will show the third assertion.

Replacing the reduced covering point $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$ by the composite $\overline{x}_{\lambda'}^{\log} \to \overline{x}_\lambda^{\log} \to \overline{x}^{\log}$, where $\overline{x}_{\lambda'}^{\log} \to \overline{x}_\lambda^{\log}$ is a reduced covering point, if necessary, we may assume that $X_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ splits as a disjoint union of copies of $\overline{x}_\lambda^{\log}$. If we base-change the above commutative diagram by $\overline{x}_\lambda^{\log} \to X^{\log}$, then we obtain the following commutative diagram:

$$\begin{tikzcd}
Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \arrow[r] \arrow[d, equal] & \overbrace{(Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \sqcup \ldots \sqcup (Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log})}^{n} \arrow[r] \arrow[d] & \overbrace{\overline{x}_\lambda^{\log} \sqcup \ldots \sqcup \overline{x}_\lambda^{\log}}^{n} \arrow[d] \\
Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \arrow[r] & Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \arrow[r] & \overline{x}_\lambda^{\log}
\end{tikzcd}$$

(where the right-hand sequare is cartesian). By the general theory of Galois categories, it is enough to show that

$$Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \longrightarrow Y_2^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} (= \overbrace{(Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \sqcup \ldots \sqcup (Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log})}^{n})$$

has rank one at some point.

Now $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ has a (ket) section; thus, one of the connected components of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ is isomorphic to $Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$. Since $Y_1^{\log} \to Y_2^{\log}$ is a surjective ket covering,

$$Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \longrightarrow \overbrace{(Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \sqcup \ldots \sqcup (Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log})}^{n}$$

is surjective ([18], Proposition 2.2.2). On the other hand, the number of connected components of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ is n by the connectedness property of the log Stein factorization $Y_1^{\log} \to X_1^{\log} \to X^{\log}$. Thus, $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to Y_2^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ induces a bijection between the set of connected components of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ and that of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$. Since one of the connected components of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ is isomorphic to $Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$, $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to Y_2^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ is an isomorphism on the connected component of $Y_1^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$ which isomorphic to $Y^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}$. $\square$

Proposition 3.4. *Let k be a field. Let $X^{\log}$ be a log smooth, proper, log geometrically conncted fs log scheme over k, and $Y^{\log}$ a connected, quasi-compact, log regular fs log scheme over k. Moreover, we assume that there exists a finite separable extension k' of k such that $Y^{\log} \to s \stackrel{\text{def}}{=} \operatorname{Spec} k$ admits a morphism $\operatorname{Spec} k' \to Y^{\log}$ over s. Let $p_1^{\log} : X^{\log} \times_s Y^{\log} \to X^{\log}$ (respectively, $p_2^{\log} : X^{\log} \times_s Y^{\log} \to Y^{\log}$) be the 1-st (respectively, 2-nd) projection. Then the following hold:*

(*i*) *$X^{\log} \times_s Y^{\log}$ is connected.*

(*ii*) *The natural morphism*

$$\pi_1(X^{\log} \times_s Y^{\log}) \longrightarrow \pi_1(X^{\log}) \times_{\mathrm{Gal}(k^{\mathrm{sep}}/k)} \pi_1(Y^{\log})$$

determined by $p_1^{\log}$ and $p_2^{\log}$ is an isomorphism.

Proof. First, we prove (i). Since $X^{\log} \to s$ is proper, $p_2^{\log} : X^{\log} \times_s Y^{\log} \to Y^{\log}$ is proper. Thus, to verify that $X^{\log} \times_s Y^{\log}$ is connected, it is enough to show that each fiber of p_2 at any geometric point of Y is connected. On the other hand, since $X^{\log} \to s$ is log geometrically connected, each fiber of p_2 at any geometric point of Y is connected. Therefore, $X^{\log} \times_s Y^{\log}$ is connected.

Next, we prove (ii). By the existence of a morphism $\operatorname{Spec} k' \to Y^{\log}$, we obtain the following cartesian diagram:

$$\begin{array}{ccc} X^{\log} \times_k k' & \longrightarrow & \operatorname{Spec} k' \\ \big\downarrow & & \big\downarrow \\ X^{\log} \times_s Y^{\log} & \xrightarrow{p_2^{\log}} & Y^{\log} . \end{array}$$

Thus, by Theorem 3.3, we obtain the following exact sequence:

$$\pi_1(X^{\log} \times_k k^{\mathrm{sep}}) \longrightarrow \pi_1(X^{\log} \times_s Y^{\log}) \xrightarrow{\pi_1(p_2^{\log})} \pi_1(Y^{\log}) \longrightarrow 1 .$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & \pi_1(X^{\log} \times_k k^{\mathrm{sep}}) & \longrightarrow & \pi_1(X^{\log} \times_s Y^{\log}) & \xrightarrow{\pi_1(p_2^{\log})} & \pi_1(Y^{\log}) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \pi_1(X^{\log} \times_k k^{\mathrm{sep}}) & \longrightarrow & \pi_1(X^{\log}) \times_{\mathrm{Gal}(k^{\mathrm{sep}}/k)} \pi_1(Y^{\log}) & \longrightarrow & \pi_1(Y^{\log}) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \pi_1(X^{\log} \times_k k^{\mathrm{sep}}) & \longrightarrow & \pi_1(X^{\log}) & \longrightarrow & \mathrm{Gal}(k^{\mathrm{sep}}/k) & \longrightarrow & 1 , \end{array}$$

where all horizontal sequences are exact. Then it follows from the injectivity of the left-hand bottom horizontal arrow $\pi_1(X^{\log} \times_k k^{\mathrm{sep}}) \to \pi_1(X^{\log})$ that the left-hand top horizontal arrow $\pi_1(X^{\log} \times_k k^{\mathrm{sep}}) \to \pi_1(X^{\log} \times_s Y^{\log})$ is injective. Thus, assertion (ii) follows from the five lemma. $\square$

4 Log formal schemes and the algebraization

In this section, we define the notion of a log structure on a formal scheme and establish a theory of *algebraizations* of log formal schemes.

First, we define the notion of a log structure on a locally noetherian formal scheme.

Definition 4.1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally noetherian formal schemes.

(i) Let $\mathcal{M}_{\mathfrak{X}}$ be a sheaf of monoids on the étale site of $\mathfrak{X}$ (concerning the étale site of a locally noetherian formal scheme, see [6], 6.1). We shall refer to a homomorphism of sheaves of monoids $\mathcal{M}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (where we regard $\mathcal{O}_{\mathfrak{X}}$ as a sheaf of monoids via the monoid structure determined by the multiplicative structure on the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$) as a *pre-log structure* on $\mathfrak{X}$.

A morphism $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}})$ of locally noetherian formal schemes equipped with pre-log structures is defined to be a pair $(\mathfrak{f}, h)$ of a morphism of locally noetherian formal schemes $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ and a homomorphism $h : \mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} \to \mathcal{M}_{\mathfrak{X}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} & \longrightarrow & \mathcal{M}_{\mathfrak{X}} \\ \downarrow & & \downarrow \\ \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} & \longrightarrow & \mathcal{O}_{\mathfrak{X}} . \end{array}$$

(ii) We shall refer to a pre-log structure $\alpha : \mathcal{M}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}$ as a *log structure* on $\mathfrak{X}$ if the homomorphism α induces an isomorphism $\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^{*}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}^{*}$.

We shall refer to a locally noetherian formal scheme equipped with a log structure as a *locally noetherian log formal scheme*. A morphism of locally noetherian log formal schemes is defined as a morphism of locally noetherian formal schemes equipped with pre-log structures.

For simplicity, we shall use the notation $\mathfrak{X}^{\log}$ to denote a locally noetherian log formal scheme whose underlying formal scheme is $\mathfrak{X}$. Then we shall denote by $\mathcal{M}_{\mathfrak{X}}$ the sheaf of monoids that determines the log structure of $\mathfrak{X}^{\log}$. Note that by a similar way to the way in which we regard the category of locally noetherian schemes as a full subcategory of the category of locally noetherian formal schemes (by regarding a scheme S as the formal scheme obtained by the completion of S along the closed subset S of S), we regard the category of locally noetherian log schemes as a full subcategory of the category of locally noetherian log formal schemes.

(iii) Let $\alpha : \mathcal{M}'_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ be a pre-log structure on $\mathfrak{X}$. We shall refer to the log structure determined by the push-out in the category of sheaves of monoids on the étale site of $\mathfrak{X}$ of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^{*}) & \xrightarrow{\text{via } \alpha} & \mathcal{O}_{\mathfrak{X}}^{*} \\ \downarrow & & \\ \mathcal{M}'_{\mathfrak{X}} & & \end{array}$$

as the *log structure associated to the pre-log structure* $\alpha : \mathcal{M}'_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$.

(iv) Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes, and $\mathcal{M}_{\mathfrak{Y}}$ a log structure on $\mathfrak{Y}$. We shall refer to the log structure associated to the pre-log structure $\mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} \to \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{X}}$ as the *pull-back of the log structure* $\mathcal{M}_{\mathfrak{Y}}$, or, alternatively, the *log structure on* $\mathfrak{X}$ *induced by* $\mathfrak{f}$.

Let $X^{\log}$ be a log scheme, and $F \subseteq X$ a closed subspace of the underlying topological space of X. Then we shall refer to the log formal scheme $\hat{X}^{\log}$ obtained by equipping the completion $\hat{X}$ of X along F with the pull-back of the log structure of $X^{\log}$ as the *log completion of* $X^{\log}$ *along* F.

(v) Let $\mathfrak{X}^{\log}$ be a locally noetherian log formal scheme. Then we shall say that $\mathfrak{X}^{\log}$ is a *locally noetherian fs log formal scheme* if étale locally on $\mathfrak{X}$, there exists an fs monoid P and a homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (where $P_{\mathfrak{X}}$ is the constant sheaf on the étale site of $\mathfrak{X}$ determined by P) such that the log structure of $\mathfrak{X}^{\log}$ is isomorphic to the log structure associated to the homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$.

(vi) Let $\mathfrak{X}^{\log}$ be a locally noetherian fs log formal scheme, P is a monoid (respectively, an fs monoid), and $P_{\mathfrak{X}}$ the constant sheaf on the étale site of $\mathfrak{X}$ determined by P. We shall refer to a homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ such that the log structure of $\mathfrak{X}^{\log}$ is isomorphic to the log structure associated to the homomorphism as a *chart* (respectively, an *fs chart*) of $\mathfrak{X}^{\log}$. By the definition of a locally noetherian fs log formal scheme, an fs chart always exists étale locally on $\mathfrak{X}^{\log}$.

Let $\overline{x} \to \mathfrak{X}$ be a geometric point of $\mathfrak{X}$ (i.e., $\overline{x} = \operatorname{Spec} k$ for some separably closed field k). We shall say that an fs chart $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ is *clean* at $\overline{x} \to \mathfrak{X}$ if the composite $P \to \mathcal{M}_{\mathfrak{X},\overline{x}} \to (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}^*_{\mathfrak{X}})_{\overline{x}}$ is an isomorphism. It follows immediately from a similar argument to the argument used to prove the existence of a clean chart for an fs log scheme that a clean chart of $\mathfrak{X}^{\log}$ always exists over an étale neighborhood of any given geometric point of $\mathfrak{X}$.

(vii) Let $\mathfrak{X}^{\log}$ and $\mathfrak{Y}^{\log}$ be locally noetherian fs log formal schemes, and $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ a morphism of log formal schemes. We shall refer to a collection of data consisting of

- a chart $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ of $\mathfrak{X}^{\log}$,
- a chart $Q_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}$ of $\mathfrak{Y}^{\log}$, and
- a morphism $Q \to P$ of monoids such that the following diagram

commutes

$$\begin{array}{ccc} Q_{\mathfrak{X}} & \longrightarrow & P_{\mathfrak{X}} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} & \longrightarrow & \mathcal{O}_{\mathfrak{X}} \end{array}$$

as a *chart of the morphism* $\mathfrak{f}^{\log}$. It follows from a similar argument to the argument used to prove the existence of a chart of a morphism of fs log schemes that given a chart $Q_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}$ of $\mathfrak{Y}^{\log}$, there exist an étale morphism $\mathfrak{U} \to \mathfrak{X}$, an fs chart $P_{\mathfrak{U}} \to \mathcal{O}_{\mathfrak{U}}$ of the log structure of $\mathfrak{U}^{\log}$ induced by the log structure of $\mathfrak{X}^{\log}$, and a morphism $P \to Q$ of monoids such that these data form a chart of the morphism $\mathfrak{f}^{\log}$.

Lemma 4.2. *Let A be an adic noetherian ring, I an ideal of definition of A, and $f : X \to \operatorname{Spec} A$ a proper morphism. If a subspace F of the underlying topological space of X contains the underlying topological space of $X \times_A (A/I)$, and is stable under generization, then F coincides with the underlying topological space of X.*

Proof. Assume that F does not coincide with the underlying topological space of X (and that X is non-empty). Then there exists an element x of $X \setminus F$. Since F is stable under generization, for any element a of F, there exists an open neighborhood U_a of a in X such that x does not belong to U_a. Thus, the open set $U \overset{\text{def}}{=} \bigcup_{a \in F} U_a$ of the underlying topological space of X contains the underlying topological space of $X \times_A (A/I)$, and x does not belong to U. It thus follows from the properness of f that $f(X \setminus U)$ is a non-empty closed subset of the underlying topological space of $\operatorname{Spec} A$, and does not contain the underlying topological space of $\operatorname{Spec}(A/I)$. However, since A is an adic noetherian ring, $\operatorname{Spec}(A/I)$ contains all closed points of $\operatorname{Spec} A$. Thus, there exists no such a set; hence we obtain a contradiction. ☐

Lemma 4.3. *Let R be a strictly henselian excellent reduced local ring, $\hat{R}$ the completion of R with respect to the maximal ideal $\mathfrak{m}$ of R, and $R \to \hat{R}$ the natural morphism. If the following diagram commutes*

$$\begin{array}{ccccc} P & \xrightarrow{\alpha_P} & \mathfrak{m} & \xrightarrow{\text{inclusion}} & R \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Q & \xrightarrow{\alpha_Q} & \hat{\mathfrak{m}} & \xrightarrow{\text{inclusion}} & \hat{R}, \end{array}$$

where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{R}$, P and Q are clean monoids, and the left-hand vertical arrow $P \to Q$ is Kummer, then the morphism $\alpha_Q : Q \to \hat{R}$ factors through $\mathfrak{m}$.

Proof. Let q be an element of Q. Our claim is that the image $\alpha_Q(q)$ of q via α_Q is in R. Let $\mathfrak{p}_1, \cdots \mathfrak{p}_r \subseteq R$ be the associated primes of R. Then, by the fact that R is reduced, the natural morphism $R \to R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_r$ is injective. We denote by K_i the field of fractions of $R/\mathfrak{p}_i$. Now since R is excellent, $R/\mathfrak{p}_i$ is excellent. Therefore, by [5], Corollaire 18.9.2, the completion $(\widehat{R/\mathfrak{p}_i})(\simeq R/\mathfrak{p}_i \otimes_R \hat{R})$ of $R/\mathfrak{p}_i$ with respect to the maximal ideal is an integral domain. We denote by $\hat{K}_i$ the field of fractions of $(\widehat{R/\mathfrak{p}_i})$. Thus, we obtain the following diagram:

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_r & \longrightarrow & K_1 \oplus \cdots \oplus K_r \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \hat{R} & \longrightarrow & (\widehat{R/\mathfrak{p}_1}) \oplus \cdots \oplus (\widehat{R/\mathfrak{p}_r}) & \longrightarrow & \hat{K}_1 \oplus \cdots \oplus \hat{K}_r \end{array}$$

(where all morphisms are injective).

Now the Kummerness of $P \to Q$ implies that $\alpha_Q(q)^n \in \mathfrak{m}$. Therefore, the image of $\alpha_Q(q)^n$ in $\hat{K}_i$ is in K_i. On the other hand, by the excellentness of $R/\mathfrak{p}_i$ and [5], Corollaire 18.9.3, K_i is algebraically closed in $\hat{K}_i$; it thus follows that the image of $\alpha_Q(q)$ in $\hat{K}_i$ is in K_i. Moreover, the image of $\alpha_Q(q)$ in $\hat{K}_i$ is in $R/\mathfrak{p}_i$. Indeed, for the fractional ideal $I \stackrel{\text{def}}{=} t_i R/\mathfrak{p}_i + R/\mathfrak{p}_i$ (where $t_i \in K_i$ is the image of $\alpha_Q(q)$ in $\hat{K}_i$), the fact that $(I/(R/\mathfrak{p}_i)) \otimes_{R/\mathfrak{p}_i} \widehat{R/\mathfrak{p}_i} = \hat{I}/(\widehat{R/\mathfrak{p}_i}) = 0$ (since t_i is in $\widehat{R/\mathfrak{p}_i}$) implies that $I/(R/\mathfrak{p}_i) = 0$ (since $\widehat{R/\mathfrak{p}_i}$ is faithfully flat over $R/\mathfrak{p}_i$). Thus, the image of $\alpha_Q(q)$ in $(\widehat{R/\mathfrak{p}_1}) \oplus \cdots \oplus (\widehat{R/\mathfrak{p}_r})$ is in $R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_r$. Moreover, it follows from a similar argument of the argument used in the proof of that the image of $\alpha_Q(q)$ in $\hat{K}_i$ is in $R/\mathfrak{p}_i$ that $\alpha_Q(q) \in R$. $\square$

Definition 4.4. Let $\mathfrak{X}^{\log}$ and $\mathfrak{Y}^{\log}$ be locally noetherian fs log formal schemes. We shall refer to a morphism $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ as a Kummer morphism if for any geometric point $\overline{x} \to \mathfrak{X}$ of $\mathfrak{X}$, the morphism of monoids $(\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}^*_{\mathfrak{Y}})_{\mathfrak{f}(\overline{x})} \to (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}^*_{\mathfrak{X}})_{\overline{x}}$ induced by $\mathfrak{f}^{\log}$ is Kummer (where the geometric point $\mathfrak{f}(\overline{x}) \to \mathfrak{Y}$ is the geometric point determined by the composite $\overline{x} \to \mathfrak{X} \to \mathfrak{Y}$).

The main result in this section is the following theorem.

Theorem 4.5. *Let A be an adic noetherian ring, and I an ideal of definition of A. Let $S^{\log}$ be a fs log scheme whose underlying scheme S is the spectrum of A, $X^{\log}$ a noetherian excellent fs log scheme, $X^{\log} \to S^{\log}$ a morphism that is separated and of finite type, and $\hat{X}^{\log}$ (respectively, $\hat{S}^{\log}$) the log completion of $X^{\log}$ (respectively, $S^{\log}$) along $X/I \stackrel{\text{def}}{=} X \times_A (A/I)$ (respectively* Spec (A/I)*).*

Then the functor determined by the operation of taking the log completion along the fiber of $S/I \stackrel{\text{def}}{=}$ Spec (A/I) *induces a natural equivalence between the* *category $\mathcal{C}_{X^{\log}}$ of reduced fs log schemes that are finite and Kummer over $X^{\log}$ and proper over $S^{\log}$ and the category $\mathcal{C}_{\hat{X}^{\log}}$ of reduced fs log formal schemes that are finite and Kummer over $\hat{X}^{\log}$ and proper over $\hat{S}^{\log}$.*

Proof. Note that if $Y^{\log} \to X^{\log}$ is an object of the category $\mathcal{C}_{X^{\log}}$, then the excellentness of X implies that the completion $\hat{Y}$ along $Y \times_A A/I$ is reduced. Therefore, the functor is well-defined.

First, we prove that the functor is fully faithful. Let $Y_1^{\log} \to X^{\log}$ and $Y_2^{\log} \to X^{\log}$ be objects of the category $\mathcal{C}_{X^{\log}}$.

Let $f^{\log}$, $g^{\log} : Y_1^{\log} \to Y_2^{\log}$ be morphisms in the category $\mathcal{C}_{X^{\log}}$ such that $\hat{f}^{\log} = \hat{g}^{\log}$, where $\hat{f}^{\log}$, $\hat{g}^{\log} : \hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ are the morphisms induced by $f^{\log}$ and $g^{\log}$, respectively. Then since $\hat{f}^{\log} = \hat{g}^{\log}$, we obtain $\hat{f} = \hat{g}$. Thus, by [4], Théorème 5.4.1, we obtain $f = g$. To see that $f^{\log} = g^{\log}$, we take a geometric point $\overline{y}_1 \to Y_1$ of Y_1 whose image lies on $Y_1/I \stackrel{\text{def}}{=} Y_1 \times_A (A/I)$. Then it follows from the assumption that $\hat{f}^{\log} = \hat{g}^{\log}$ and a similar argument to the argument used in the proof of Proposition A.11 (note that $\mathcal{O}_{Y_1,\overline{y}_1} \to \mathcal{O}^{\wedge}_{Y_1,\overline{y}_1}$ is faithfully flat) that the homomorphism $\mathcal{M}_{Y_2,\overline{y}_2} \to \mathcal{M}_{Y_1,\overline{y}_1}$ induced by $f^{\log}$ (where we denote by $\overline{y}_2 \to Y_2$ the geometric point determined by the composite $\overline{y}_1 \to Y_1 \xrightarrow{f=g} Y_2$) coincides with the homomorphism $\mathcal{M}_{Y_2,\overline{y}_2} \to \mathcal{M}_{Y_1,\overline{y}_1}$ induced by $g^{\log}$. Therefore, $f^{\log}$ coincides with $g^{\log}$ on an étale neighborhood of the geometric point $\overline{y}_1 \to Y_1$. Moreover, by Lemma 4.2, this implies that $f^{\log}$ coincides with $g^{\log}$ on $Y_1^{\log}$. This completes the proof that the functor in question is faithful.

Next, let $\mathfrak{f}^{\log} : \hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ be a morphism in the category $\mathcal{C}_{\hat{X}^{\log}}$. By [4], Théorème 5.4.1, there exists a unique morphism $f : Y_1 \to Y_2$ such that $\hat{f}$ coincides with the underlying morphism $\mathfrak{f}$ of formal schemes of $\mathfrak{f}^{\log}$. Now if there exists an extension of the morphism f to a morphism of log schemes $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ such that the morphism $\hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ induced by $f^{\log}$ coincides with $\mathfrak{f}^{\log}$, then it is unique (by the proof of the faithfulness of the functor in question); therefore, it is enough to show that such an extension of f exists étale locally on $Y_1^{\log}$. Moreover, by Lemma 4.2, it is enough to show that for any geometric point of Y_1 whose image lies on Y_1/I, there exists such an extension of f on an étale neighborhood of the geometric point. To see this, let $\overline{y}_1 \to Y_1$ be a geometric point whose image lies on Y_1/I, and denote by $\overline{y}_2 \to Y_2$ the geometric point determined by the composite $\overline{y}_1 \to Y_1 \xrightarrow{f} Y_2$. If we denote by $P_2 \to \mathcal{O}_{Y_2,\overline{y}_2}$ a clean chart at $\overline{y}_2 \to Y_2$ of the log structure of $Y_2^{\log}$, then there exists a chart $P_1 \to \mathcal{O}^{\wedge}_{Y_1,\overline{y}_1}$ (where $\mathcal{O}^{\wedge}_{Y_1,\overline{y}_1}$ is the completion of $\mathcal{O}_{Y_1,\overline{y}_1}$ with respect to $I\mathcal{O}_{Y_1,\overline{y}_1}$) of the log structure of $\operatorname{Spf} \mathcal{O}^{\wedge}_{Y_1,\overline{y}_1}$ which is induced by the log structure of $\hat{Y}_1^{\log}$, and a diagram

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_2,\overline{y}_2} & \longrightarrow (\mathcal{O}_{Y_1,\overline{y}_1} \longrightarrow) & \hat{\mathcal{O}}_{Y_1,\overline{y}_1} \end{array}$$

such that the above diagram is a chart of the natural morphism $(\mathrm{Spf}\,\hat{\mathcal{O}}_{Y_1,\overline{y}_1})^{\log} \to Y_2^{\log}$. Note that the cleanness of the chart $P_2 \to \mathcal{O}_{Y_2,\overline{y}_2}$ and the Kummerness of $\mathfrak{f}^{\log}$ imply that the chart $P_1 \to \hat{\mathcal{O}}_{Y_1,\overline{y}_1}$ is a clean chart at the geometric point $\overline{y}_1 \to \mathrm{Spf}\,\hat{\mathcal{O}}_{Y_1,\overline{y}_1}$; thus, the top horizontal arrow $P_2 \to P_1$ is a Kummer morphism. In particular, the image of $P_1 \to \hat{\mathcal{O}}_{Y_1,\overline{y}_1}$ and the image of $P_2 \to \mathcal{O}_{Y_2,\overline{y}_2}$ are contained in the maximal ideals, respectively. Thus, by Lemma 4.3 (by considering the composite $P_1 \to \hat{\mathcal{O}}_{Y_1,\overline{y}_1} \to \tilde{\mathcal{O}}_{Y_1,\overline{y}_1}$, where $\tilde{\mathcal{O}}_{Y_1,\overline{y}_1}$ is the completion of $\mathcal{O}_{Y_1,\overline{y}_1}$ with respect to the maximal ideal of $\mathcal{O}_{Y_1,\overline{y}_1}$), the morphism $P_1 \to \hat{\mathcal{O}}_{Y_1,\overline{y}_1}$ factors through $\mathcal{O}_{Y_1,\overline{y}_1}$; moreover, the resulting morphism $P_1 \to \mathcal{O}_{Y_1,\overline{y}_1}$ is a clean chart at $\overline{y}_1 \to Y_1$ of the log structure of $Y_1^{\log}$. In particular, the diagram

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_2,\overline{y}_2} & \longrightarrow & \mathcal{O}_{Y_1,\overline{y}_1} \end{array}$$

is a chart of a morphism from an étale neighborhood of $\overline{y}_1 \to Y_1$ to Y_2 for which the morphism $\hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ determined by this morphism coincides with $\mathfrak{f}^{\log}$. This completes the proof that the functor in question is full.

Finally, we prove that the functor is essentially surjective. Let $\mathfrak{Y}^{\log} \to \hat{X}^{\log}$ be an object of $\mathcal{C}_{\hat{X}^{\log}}$. By [4], Théorème 5.4.1 and Proposition 5.4.4, there exists a unique noetherian scheme Y that is finite over X, and proper over S such that the completion $\hat{Y}$ of Y along $Y/I \stackrel{\mathrm{def}}{=} Y \times_A (A/I)$ is naturally isomorphic to $\mathfrak{Y}$. (Note that then the reducedness of $\mathfrak{Y}$ implies that Y is reduced.) If there exists an fs log structure of Y such that the pull-back of the log structure to $\hat{Y}$ is isomorphic to $\mathcal{M}_{\hat{Y}}$, then it is unique (note that by the proof of the fully faithfulness of the functor in question); therefore, it is enough to show that such an fs log structure exists étale locally on Y. Moreover, by Lemma 4.2, it is enough to show that for any geometric point of Y for which the image lies on Y/I, there exists such an fs log structure on an étale neighborhood of the geometric point.

By replacing $X^{\log}$ by the log scheme obtained by equipping Y with the log structure induced by the log structure of $X^{\log}$ via the morphism $Y \to X$, we may assume that the morphism $Y \to X$ is the identity morphism of X; thus, we may assume that the underlying morphism of formal schemes of $\hat{Y}^{\log} \to \hat{X}^{\log}$ is the identity morphism of $\hat{X}$. Let $\overline{x} \to X$ be a geometric point of X whose image lies on X/I. Then we obtain the following diagram

$$\begin{array}{ccc} \operatorname{Spf} \mathcal{O}^{\wedge}_{X,\overline{x}} & \longrightarrow & \operatorname{Spec} \mathcal{O}_{X,\overline{x}} \\ \downarrow & & \downarrow \\ \hat{X} & \longrightarrow & X\,, \end{array}$$

where $\mathcal{O}^{\wedge}_{X,\overline{x}}$ is the completion of $\mathcal{O}_{X,\overline{x}}$ with respect to $I\mathcal{O}_{X,\overline{x}}$. Now we obtain a chart of the morphism $(\operatorname{Spf} \mathcal{O}^{\wedge}_{X,\overline{x}})^{\log} \to X^{\log}$ (where the log structure of $(\operatorname{Spf} \mathcal{O}^{\wedge}_{X,\overline{x}})^{\log}$ is induced by the log structure of $\hat{Y}^{\log}$)

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,\overline{x}} & \longrightarrow & \mathcal{O}^{\wedge}_{X,\overline{x}}\,, \end{array}$$

where the left-hand vertical arrow $P \to \mathcal{O}_{X,\overline{x}}$ is a clean chart at $\overline{x}$ of $X^{\log}$, and the right-hand vertical arrow $Q \to \mathcal{O}^{\wedge}_{X,\overline{x}}$ is a chart of $(\operatorname{Spf} \mathcal{O}^{\wedge}_{X,\overline{x}})^{\log}$. Note that the cleanness of the chart $P \to \mathcal{O}_{X,\overline{x}}$ and the Kummerness of $\hat{Y}^{\log} \to \hat{X}^{\log}$ imply that the chart $Q \to \mathcal{O}^{\wedge}_{X,\overline{x}}$ is clean at the geometric point $\overline{x} \to \operatorname{Spf} \mathcal{O}^{\wedge}_{X,\overline{x}}$; thus, $P \to Q$ is a Kummer morphism. In particular, the image of $P \to \mathcal{O}_{X,\overline{x}}$ and the image of $Q \to \mathcal{O}^{\wedge}_{X,\overline{x}}$ are contained in the maximal ideals, respectively. Thus, by Lemma 4.3 (by considering the composite $Q \to \mathcal{O}^{\wedge}_{X,\overline{x}} \to \mathcal{O}^{\sim}_{X,\overline{x}}$, where $\mathcal{O}^{\sim}_{X,\overline{x}}$ is the completion of $\mathcal{O}_{X,\overline{x}}$ with respect to the maximal ideal of $\mathcal{O}_{X,\overline{x}}$), the chart $Q \to \mathcal{O}^{\wedge}_{X,\overline{x}}$ factors through $\mathcal{O}_{X,\overline{x}}$. It thus follows that the log structure of $\hat{Y}^{\log}$ can be descended to an étale neighborhood of the geometric point $\overline{x} \to X$. $\square$

Remark 4.6. If, in Theorem 4.5, one drops the reducedness hypothesis, the conclusion no longer holds in general. A counter-example is as follows:

Let k be a field whose characteristic we denote by p (≥ 2), $A = k[[t]][\epsilon]/(\epsilon^2)$, $X = \mathbb{P}^1_A$, $U_0 = X \setminus \{0_A\}$, $U_\infty = X \setminus \{\infty_A\}$, and $\mathfrak{X}$ (respectively, $\mathfrak{U}_0$; respectively, $\mathfrak{U}_\infty$) the t-adic completion of X (respectively, U_0; respectively, U_∞). We denote by $\mathcal{N} \to \mathcal{O}_{\mathfrak{X}}$ the log structure on $\mathfrak{X}$

$$\begin{array}{ccc} \mathbb{N}_{\mathfrak{X}} \oplus (\mathcal{O}_{\mathfrak{X}} \cap (\mathfrak{U}_0 \hookrightarrow \mathfrak{X})_* \mathcal{O}^*_{\mathfrak{U}_0}) & \longrightarrow & \mathcal{O}_{\mathfrak{X}} \\ (n, f) & \mapsto & \overline{\epsilon}^n \cdot f\,, \end{array}$$

where $\overline{\epsilon} = \epsilon \bmod (\epsilon^2)$. Thus, we have an isomorphism $\mathcal{N}/\mathcal{O}^*_{\mathfrak{X}} \simeq \mathbb{N}_{\mathfrak{X}} \oplus \mathbb{N}_{\{0_A\}}$. Let $\mathcal{P}$ be the subsheaf of monoids of $\mathcal{N}/\mathcal{O}^*_{\mathfrak{X}}$ generated by the global sections $(1,1)$ and $(1,0) \in \mathbb{N} \oplus \mathbb{N} \simeq (\mathcal{N}/\mathcal{O}^*_{\mathfrak{X}})(\mathfrak{X})$ and $\mathcal{N}' \to \mathcal{O}_{\mathfrak{X}}$ the log structure on $\mathfrak{X}$ determined by the composite $\mathcal{N} \times_{\mathcal{N}/\mathcal{O}^*_{\mathfrak{X}}} \mathcal{P} \hookrightarrow \mathcal{N} \to \mathcal{O}_{\mathfrak{X}}$ (i.e., $\mathcal{N}' \to \mathcal{O}_{\mathfrak{X}}$ is a log structure on $\mathfrak{X}$ whose characteristic $\mathcal{N}'/\mathcal{O}^*_{\mathfrak{X}}$ is isomorphic to $\mathcal{P}$).

We shall denote by

$$\mathfrak{D}$$

the divisor on $\mathfrak{X}$ determined by the t-completion of the (reduced) closed subscheme $\{0_A\} \subseteq X$, by

$$\mathcal{G}(m\mathfrak{D}) \ (m \in \mathbb{Z})$$

the $\mathbb{G}_m$-torsor sheaf on $\mathfrak{X}$ which corresponds to the invertible sheaf $\mathcal{O}_{\mathfrak{X}}(m\mathfrak{D})$, by

$$\iota_{m \to m'} : \mathcal{G}(-m\mathfrak{D}) \mid_{\mathfrak{U}_0} \xrightarrow{\sim} \mathcal{G}(-m'\mathfrak{D}) \mid_{\mathfrak{U}_0} \ \ (m \geq m')$$

the isomorphism induced by the natural inclusion $\mathcal{O}_{\mathfrak{X}}(-m\mathfrak{D}) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(-m'\mathfrak{D})$, and by

$$\mathcal{N}'_{n,m} \ (n \geq m)$$

the $\mathbb{G}_m$-torsor sheaf on $\mathfrak{X}$ obtained as the fiber product of

$$\begin{array}{ccc} & & \{(n,m)\} \\ & & \downarrow \\ \mathcal{N}' & \longrightarrow & \mathcal{P}\,, \end{array}$$

where $\{(n,m)\}$ is the sheaf of sets on X generated by the global section $(n,m) \in \mathcal{P}(\mathfrak{X})$ of $\mathcal{P}$, and the vertical arrow $\{(n,m)\} \to \mathcal{P}$ is the natural inclusion.

Then, by the definition of the log structure $\mathcal{N}' \to \mathcal{O}_{\mathfrak{X}}$, the following assertions hold:

(i) $\mathcal{N}'$ is generated by the $\mathcal{N}'_{n,m}$'s $(n \geq m)$.

(ii) The $\mathbb{G}_m$-torsor sheaf $\mathcal{N}'_{n,m}$ is naturally isomorphic to $\mathcal{G}(-m\mathfrak{D})$. We shall denote this isomorphism by

$$\phi_{n,m} : \mathcal{N}'_{n,m} \xrightarrow{\sim} \mathcal{G}(-m\mathfrak{D})\ .$$

(iii) The monoid structure on $\mathcal{N}'$ is determined by the composites

$$\begin{array}{ccc} \mathcal{N}'_{n,m} \times \mathcal{N}'_{n',m'} & & \mathcal{N}'_{n+n',m+m'} \\ {\scriptstyle \phi_{n,m} \times \phi_{n,m}} \downarrow & & \uparrow {\scriptstyle \phi^{-1}_{n+n',m+m'}} \\ \mathcal{G}(-m\mathfrak{D}) \times \mathcal{G}(-m'\mathfrak{D}) & \longrightarrow & \mathcal{G}(-(m+m')\mathfrak{D}) \\ (f,f') & \mapsto & f \cdot f'\,. \end{array}$$

(iv) The restriction of $\mathcal{N}' \to \mathcal{O}_{\mathfrak{X}}$ to $\mathcal{N}'$ coincides with the composite

$$\begin{array}{ccccc} \mathcal{N}'_{n,m} & \overset{\phi_{n,m}}{\xrightarrow{\sim}} & \mathcal{G}(-m\mathfrak{D}) & \longrightarrow & \mathcal{O}_{\mathfrak{X}} \\ & & f & \mapsto & \overline{\epsilon}^n \cdot \iota_{m\to 0}(f)\,. \end{array}$$

(v) Let $n \geq m \geq m'$ be natural numbers. Then the "glueing isomorphism" $\mathcal{N}'_{n,m}\mid_{\mathfrak{U}_0} \xrightarrow{\sim} \mathcal{N}'_{n,m'}\mid_{\mathfrak{U}_0}$ (note that by the definition of $\mathcal{P}$, the restrictions of the global sections $(0,m) \in \mathcal{P}(\mathfrak{X})$ ($m \in \mathbb{Z}$) to $\mathfrak{U}_0$ are 0, i.e., $(0,m)\mid_{\mathfrak{U}_0} = 0$; this means that "the restrictions of the $\mathbb{G}_m$-torsor sheaves $\mathcal{N}'_{n,m}$ ($m \in \mathbb{Z}$) to $\mathfrak{U}_0$ determine the same subsheaf of $\mathcal{N}'\mid_{\mathfrak{U}_0}$ ") is defined by the composite

$$\mathcal{N}'_{n,m}\mid_{\mathfrak{U}_0} \overset{\phi_{n,m}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{G}(-m\mathfrak{D})\mid_{\mathfrak{U}_0} \overset{\iota_{m\to m'}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{G}(-m'\mathfrak{D})\mid_{\mathfrak{U}_0} \overset{\phi^{-1}_{n,m'}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{N}'_{n,m'}\mid_{\mathfrak{U}_0}\ .$$

Let $\mathfrak{f} \in \Gamma(\mathfrak{U}_0, \mathcal{O}_{\mathfrak{U}_0})$ be a section such that $1 + \overline{\epsilon} \cdot \mathfrak{f}$ is not in the image of the natural morphism $\Gamma(U_0, \mathcal{O}_{U_0}) \to \Gamma(\mathfrak{U}_0, \mathcal{O}_{\mathfrak{U}_0})$ (for example, $\mathfrak{f} = \Sigma_{i=1}^{\infty} t^i (1/x)^i$, where $1/x \in \Gamma(U_0, \mathcal{O}_{U_0}) \xrightarrow{\sim} A[1/x]$).

Now we define the log structure $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$ as follows:

(I) Let $n \geq m$ be natural numbers. We shall denote by $\mathcal{M}_{n,m}$ a copy of $\mathcal{G}(-m\mathfrak{D})$, and by

$$\psi_{n,m} : \mathcal{M}_{n,m} \xrightarrow{\sim} \mathcal{G}(-m\mathfrak{D})$$

the "identity isomorphism".

(II) Let $n \geq m \geq m'$ be natural numbers. Then we define the isomorphism $\mathcal{M}_{n,m}\mid_{\mathfrak{U}_0} \xrightarrow{\sim} \mathcal{M}_{n,m'}\mid_{\mathfrak{U}_0}$ by the composite

$$\begin{array}{ccccc} \mathcal{M}_{n,m}\mid_{\mathfrak{U}_0} & \overset{\psi_{n,m}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} & \mathcal{G}(-m\mathfrak{D})\mid_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{G}(-m\mathfrak{D})\mid_{\mathfrak{U}_0} \\ & & f & \mapsto & f \cdot (1 + \overline{\epsilon} \cdot \mathfrak{f})^{m-m'} \end{array}$$

$$\overset{\iota_{m\to m'}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{G}(-m'\mathfrak{D})\mid_{\mathfrak{U}_0} \overset{\psi^{-1}_{n,m'}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{M}_{n,m'}\mid_{\mathfrak{U}_0}\ .$$

Note that, by the definition, for $n \geq m \geq m' \geq m''$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{n,m}\mid_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{M}_{n,m'}\mid_{\mathfrak{U}_0} \\ \| & & \downarrow \\ \mathcal{M}_{n,m}\mid_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{M}_{n,m''}\mid_{\mathfrak{U}_0}\,, \end{array}$$

where all morphisms are the isomorphisms defined as above.

"By glueing by means of these isomorphisms", we obtain a sheaf of sets $\mathcal{M}$ on $\mathfrak{X}$. (See the note in (v). More precisely, by taking a quotient by means of these isomorphisms, we obtain $\mathcal{M}$.) Moreover, by the definition of $\mathcal{M}$, there is a natural inclusion $\mathcal{O}_{\mathfrak{X}}^* \overset{\psi_{0,0}^{-1}}{\underset{\sim}{\to}} \mathcal{M}_{0,0} \hookrightarrow \mathcal{M}$.

(III) By the definition of the glueing isomorphism defined in (II), for $n_i \geq m_i \geq m'_i$ ($i = 1, 2$), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{n_1,m_1}|_{\mathfrak{U}_0} \times \mathcal{M}_{n_2,m_2}|_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{M}_{n_1,m'_1}|_{\mathfrak{U}_0} \times \mathcal{M}_{n_2,m'_2}|_{\mathfrak{U}_0} \\ \downarrow & & \downarrow \\ \mathcal{M}_{n_1+n_2,m_1+m_2}|_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{M}_{n_1+n_2,m'_1+m'_2}|_{\mathfrak{U}_0}, \end{array}$$

where the horizontal arrows are the glueing isomorphisms defined in (II) and the vertical arrows are the composites

$$\begin{array}{ccc} \mathcal{M}_{n_1,l_1}|_{\mathfrak{U}_0} \times \mathcal{M}_{n_2,l_2}|_{\mathfrak{U}_0} & & \mathcal{M}_{n_1+n_2,l_1+l_2}|_{\mathfrak{U}_0} \\ {\scriptstyle \psi_{n_1,m_1}|_{\mathfrak{U}_0} \times \psi_{n_2,m_2}|_{\mathfrak{U}_0}} \downarrow & & \uparrow {\scriptstyle \psi^{-1}_{n_1+n_2,m_1+m_2}|_{\mathfrak{U}_0}} \\ \mathcal{G}(-l_1\mathfrak{D})|_{\mathfrak{U}_0} \times \mathcal{G}(-l_2\mathfrak{D})|_{\mathfrak{U}_0} & \longrightarrow & \mathcal{G}(-(l_1+l_2)\mathfrak{D})|_{\mathfrak{U}_0} \\ (f, f') & \mapsto & f \cdot f' \end{array}$$

$$(l_1 = m_1, m'_1;\ l_2 = m_2, m'_2).$$

Thus, we define the monoid structure on $\mathcal{M}$ by the composites (cf. (iii))

$$\begin{array}{ccc} \mathcal{M}_{n_1,m_1} \times \mathcal{M}_{n_2,m_2} & & \mathcal{M}_{n_1+n_2,m_1+m_2} \\ {\scriptstyle \psi_{n_1,m_1} \times \psi_{n_2,m_2}} \downarrow & & \uparrow {\scriptstyle \psi^{-1}_{n_1+n_2,m_1+m_2}} \\ \mathcal{G}(-m_1\mathfrak{D}) \times \mathcal{G}(-m_2\mathfrak{D}) & \longrightarrow & \mathcal{G}(-(m_1+m_2)\mathfrak{D}) \\ (f, f') & \mapsto & f \cdot f'. \end{array}$$

Moreover, by the definition of this monoid structure on $\mathcal{M}$, the inclusion $\mathcal{O}_{\mathfrak{X}}^* \hookrightarrow \mathcal{M}$ obtained in (II) is a morphism of sheaves of monoids, and the quotient $\mathcal{M}/\mathcal{O}_{\mathfrak{X}}^*$ is naturally isomorphic to $\mathcal{P}$.

(IV) By the definition of the glueing isomorphism defined in (II), for $n \geq m \geq m'$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{n,m}|_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{M}_{n,m'}|_{\mathfrak{U}_0} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathfrak{U}_0} & = & \mathcal{O}_{\mathfrak{U}_0}, \end{array}$$

where the top horizontal arrow is the glueing isomorphism defined in (II), and the vertical arrows are the composite

$$\begin{array}{ccccc} \mathcal{M}_{n,l} \mid_{\mathfrak{U}_0} & \overset{\psi_{n,l}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} & \mathcal{G}(-l\mathfrak{D}) \mid_{\mathfrak{U}_0} & \longrightarrow & \mathcal{O}_{\mathfrak{U}_0} \\ & & f & \mapsto & \overline{\epsilon}^n \cdot \iota_{l\to 0}(f)\,. \end{array} \qquad (l = m, m')$$

(Indeed, the image of $f \in \mathcal{G}(-m\mathfrak{D}) \mid_{\mathfrak{U}_0} \overset{\psi^{-1}_{n,m}\mid_{\mathfrak{U}_0}}{\xrightarrow{\sim}} \mathcal{M}_{n,m} \mid_{\mathfrak{U}_0}$ via the composite $\mathcal{M}_{n,m} \mid_{\mathfrak{U}_0} \xrightarrow{\sim} \mathcal{M}_{n,m'} \mid_{\mathfrak{U}_0} \to \mathcal{O}_{\mathfrak{U}_0}$ [respectively, the morphism $\mathcal{M}_{n,m} \mid_{\mathfrak{U}_0} \to \mathcal{O}_{\mathfrak{U}_0}$] is

$$\overline{\epsilon}^n \cdot (1+\overline{\epsilon}\cdot\mathfrak{f})^{m-m'} \cdot \iota_{m\to 0}(f) = \overline{\epsilon}^n \cdot \iota_{m\to 0}(f) + (m-m')\cdot\overline{\epsilon}^{n+1}\cdot\mathfrak{f}\cdot\iota_{m\to 0}(f)$$

$$[\text{respectively, } \overline{\epsilon}^n \cdot \iota_{m\to 0}(f)]\,.$$

Thus, the commutativity of the above diagram follows from the fact that $n \geq m \geq m'$ and $\overline{\epsilon}^2 = 0$.)

Thus, we define the morphism $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$ by glueing the morphisms (cf. (iv))

$$\begin{array}{ccccc} \mathcal{M}_{n,m} & \overset{\psi_{n,m}}{\xrightarrow{\sim}} & \mathcal{G}(-m\mathfrak{D}) & \longrightarrow & \mathcal{O}_{\mathfrak{X}} \\ & & f & \mapsto & \overline{\epsilon}^n \cdot \iota_{m\to 0}(f)\,. \end{array}$$

Then, by construction, the morphism $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$ is a log structure on $\mathfrak{X}$.

Now we prove that the log structure $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$ is not algebraizable, i.e., there is no log structure on X whose log completion is isomorphic to $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$.

Assume that there is a log structure $\mathcal{M}^{\mathrm{alg}} \to \mathcal{O}_X$ such that the log completion $\hat{\mathcal{M}}^{\mathrm{alg}} \to \mathcal{O}_{\mathfrak{X}}$ of $\mathcal{M}^{\mathrm{alg}} \to \mathcal{O}_X$ is isomorphic to $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$. We shall denote by

$$\rho : \hat{\mathcal{M}}^{\mathrm{alg}} \xrightarrow{\sim} \mathcal{M}$$

the isomorphism, by

$$\hat{\mathcal{M}}^{\mathrm{alg}}_{n,m}$$

the $\mathbb{G}_m$-torsor sheaf on $\mathfrak{X}$ (cf. the definition of $\mathcal{N}'_{n,m}$) obtained as the fiber product of

$$\begin{array}{ccccc} & & & & \{(n,m)\} \\ & & & & \downarrow \\ \hat{\mathcal{M}}^{\mathrm{alg}} & \longrightarrow & \hat{\mathcal{M}}^{\mathrm{alg}}/\mathcal{O}^*_{\mathfrak{X}} & \xrightarrow{\sim} & \mathcal{P}\,, \end{array}$$

and by

$$\rho_{n,m} : \mathcal{M}_{n,m} \xrightarrow{\sim} \hat{\mathcal{M}}^{\mathrm{alg}}_{n,m}$$

the isomorphism induced by the isomorphism $\rho : \hat{\mathcal{M}}^{\mathrm{alg}} \xrightarrow{\sim} \mathcal{M}$. Then the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0} & \xrightarrow[\sim]{\psi^{-1}_{1,1}\mid_{\mathfrak{U}_0}} & \mathcal{M}_{1,1} \mid_{\mathfrak{U}_0} & \xrightarrow[\sim]{\rho_{1,1}\mid_{\mathfrak{U}_0}} & \hat{\mathcal{M}}^{\mathrm{alg}}_{1,1} \mid_{\mathfrak{U}_0} \\
 & & \downarrow & & \downarrow \\
\mathcal{G} \mid_{\mathfrak{U}_0} & \xrightarrow[\sim]{\psi^{-1}_{1,0}\mid_{\mathfrak{U}_0}} & \mathcal{M}_{1,0} \mid_{\mathfrak{U}_0} & \xrightarrow[\sim]{\rho_{1,0}\mid_{\mathfrak{U}_0}} & \hat{\mathcal{M}}^{\mathrm{alg}}_{1,0} \mid_{\mathfrak{U}_0} ,
\end{array}$$

where the vertical arrows are the glueing morphisms. Now, by (II), the composite

$$\mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\psi^{-1}_{1,1}\mid_{\mathfrak{U}_0}} \mathcal{M}_{1,1} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\text{glueing}} \mathcal{M}_{1,0} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\psi_{1,0}\mid_{\mathfrak{U}_0}} \mathcal{G} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\iota^{-1}_{1\to 0}\mid_{\mathfrak{U}_0}} \mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0}$$

coincides with

$$\begin{array}{ccc}
\mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0} & \xrightarrow{\sim} & \mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0} \\
f & \mapsto & f \cdot (1 + \overline{\epsilon} \cdot \mathfrak{f}) ,
\end{array}$$

i.e., by the assumption on $\mathfrak{f}$, it is not algebraizable. On the other hand, the composite

$$\mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\psi^{\mathrm{alg}\,-1}_{1,1}\mid_{\mathfrak{U}_0}} \hat{\mathcal{M}}^{\mathrm{alg}}_{1,1} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\text{glueing}} \hat{\mathcal{M}}^{\mathrm{alg}}_{1,0} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\psi^{\mathrm{alg}}_{1,0}\mid_{\mathfrak{U}_0}} \mathcal{G} \mid_{\mathfrak{U}_0} \xrightarrow[\sim]{\iota^{-1}_{1\to 0}\mid_{\mathfrak{U}_0}} \mathcal{G}(-\mathfrak{D}) \mid_{\mathfrak{U}_0}$$

(where $\hat{\psi}^{\mathrm{alg}}_{n,m} = \psi_{n,m} \circ \rho^{-1}_{n,m}$) is algebraizable. (Indeed, this follows from the fact that the properness of X implies that the isomorphism $\hat{\psi}^{\mathrm{alg}}_{n,m}$ is algebraizable, together with the fact that the glueing isomorphism $\hat{\mathcal{M}}^{\mathrm{alg}}_{1,1} \mid_{\mathfrak{U}_0} \xrightarrow{\sim} \hat{\mathcal{M}}^{\mathrm{alg}}_{1,0} \mid_{\mathfrak{U}_0}$ is defined on U_0.) Therefore, we obtain a contradiction. This completes the proof that $\mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$ is not algebraizable.

Moreover, if we denote by $\mathcal{Q}$ the subsheaf of monoids of $\mathcal{P}$ generated by the global sections (p, p) and $(p, 0) \in \mathcal{P}_{\mathfrak{X}}$ and by $\widetilde{\mathcal{M}} \to \mathcal{O}_{\mathfrak{X}}$ the log structure on $\mathfrak{X}$ determined by the composite $\mathcal{M} \times_{\mathcal{P}} \mathcal{Q} \hookrightarrow \mathcal{M} \to \mathcal{O}_{\mathfrak{X}}$, then the inclusion $\widetilde{\mathcal{M}} \hookrightarrow \mathcal{M}$ induces a natural morphism of log formal schemes

$$(\mathfrak{X}, \mathcal{M} \to \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{X}, \widetilde{\mathcal{M}} \to \mathcal{O}_{\mathfrak{X}})$$

which is finite and Kummer. On the other hand, the log formal scheme $(\mathfrak{X}, \widetilde{\mathcal{M}} \to \mathcal{O}_{\mathfrak{X}})$ is algebraizable. (Indeed, this follows from the fact that $(1 + \overline{\epsilon} \cdot \mathfrak{f})^p = 1$ is algebraizable.)

Remark 4.7. In light of the classical algebraization theory of formal schemes (for example, the theory considered in [4], §5), one might expect that *data of a finite nature on a compact object should be algebraizable.* However, as Remark 4.6 shows, this is not the case in the algebraization theory of log schemes. (Note that Kummerness of a morphism of log schemes is of a finite nature.)

By applying Theorem 4.5, we obtain the following corollary. Note that the corollary generalizes [22], Théorème 2.2, (a). (In [22], Théorème 2.2, (a), the underlying scheme of the base log scheme is assumed to be the spectrum of a complete discrete valuation ring.)

Corollary 4.8. *Let $S^{\log}$ be an fs log scheme whose underlying scheme S is the spectrum of a complete local ring whose maximal ideal (respectively, residue field) we denote by $\mathfrak{m}$ (respectively, k), $X^{\log}$ a log regular fs log scheme, and $X^{\log} \to S^{\log}$ a proper morphism. Then the strict closed immersion $X_0^{\log} \stackrel{\mathrm{def}}{=} X^{\log} \times_{S^{\log}} s^{\log} \to X^{\log}$ induces a natural equivalence of the category of ket coverings over $X^{\log}$ and the category of ket coverings over $X_0^{\log}$, where $s^{\log}$ is the log scheme obtained by equipping* $\mathrm{Spec}\, k$ *with the log structure induced by the log structure of $S^{\log}$ via the closed immersion $s \to S$ induced by the natural projection $A \to A/\mathfrak{m} \simeq k$. In particular, if $X^{\log}$ is connected, then $X_0^{\log}$ is also connected, and $\pi_1(X_0^{\log}) \xrightarrow{\sim} \pi_1(X^{\log})$.*

Proof. We may assume that $X^{\log}$ is connected. First, we prove that the functor is fully faithful. Let $Y^{\log} \to X^{\log}$ is a connected ket covering. Then if we denote by $Y \to S' \to S$ the Stein factorization of the underlying morphism of the composite $Y^{\log} \to X^{\log} \to S^{\log}$, then the connectedness of Y and the surjectivity of $Y \to S'$ implies that S' is connected. Since S is the spectrum of the complete ring and $S' \to S$ is finite, it thus follows that $Y \times_S \mathrm{Spec}\, k$, hence also, $Y^{\log} \times_{S^{\log}} s^{\log}$ is connected (note that $s^{\log} \to S^{\log}$ is strict). Therefore, by the general theory of Galois categories, the functor in question is fully faithful.

Next, we prove that the functor is essentially surjective. Let $Y_0^{\log} \to X_0^{\log}$ be a connected ket covering. Then it follows from [23], Théorème 0.1 that there exists a unique connected ket covering $Y_n^{\log} \to X_n^{\log} \stackrel{\mathrm{def}}{=} X^{\log} \times_{S^{\log}} S_n^{\log}$ such that $Y_n^{\log} \times_{S_n^{\log}} s^{\log} \simeq Y_0^{\log}$, where $S_n^{\log}$ is the log scheme obtained by equipping $\mathrm{Spec}\,(A/\mathfrak{m}^{n+1})$ with the log structure induced by the log structure of $S^{\log}$ via the closed immersion induced by the natural projection $A \to A/\mathfrak{m}^{n+1}$. Now we denote by $\mathfrak{Y}^{\log}$ the noetherian log formal scheme obtained by the system $\{Y_n^{\log}\}_n$. Note that by considering the characteristic $\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}_{\mathfrak{Y}}^*$ of $\mathfrak{Y}^{\log}$, one may conclude that the log structure of $\mathfrak{Y}^{\log}$ is fs; and that by the construction of $\mathfrak{Y}^{\log}$, the fiber product $\mathfrak{Y}^{\log} \times_{S^{\log}} S_n^{\log}$ is naturally isomorphic to $Y_n^{\log}$.

We denote by $\mathfrak{X}^{\log}$ the log completion of $X^{\log}$ along X_0. Now It follows from the properness of $X \to S$ and the fact that A is complete that X is excellent. Now since $Y_0^{\log} \to X_0^{\log}$ is Kummer, $\mathfrak{Y}^{\log} \to \mathfrak{X}^{\log}$ is also Kummer; moreover, since $Y_n \to X_n$ is finite, $\mathfrak{Y} \to \mathfrak{X}$ is also finite. Next, to see that $\mathfrak{Y}$ is reduced, by taking a geometric point $\overline{y} \to \mathfrak{Y}$ of $\mathfrak{Y}$, and replacing X by $\operatorname{Spec} \mathcal{O}_{X,\overline{x}}$ (where $\overline{x} \to X$ is the geometric point obtained by the composite $\overline{y} \to \mathfrak{Y} \to \mathfrak{X} \to X$), we may assume that X is the spectrum of a strictly henselian local ring. (Note that the finiteness of $\mathfrak{Y} \to \mathfrak{X}$ implies that there exists a strictly henselian local ring R_Y that is finite over $\mathcal{O}_{X,\overline{x}}$ such that $\mathfrak{Y} = \operatorname{Spf} \hat{R}_Y$, where $\hat{R}_Y$ is the completion of R_Y with respect to $\mathfrak{m}R_Y$.) Then it follows from the fact that $Y_0^{\log} \to X_0^{\log}$ is a ket covering and Proposition A.4 that there exists a diagram

$$\begin{array}{ccc} P_X & \longrightarrow & \mathcal{O}_{X,\overline{x}}/\mathfrak{m}\mathcal{O}_{X,\overline{x}} \\ \downarrow & & \downarrow \\ P_Y & \longrightarrow & R_Y/\mathfrak{m}R_Y \end{array},$$

where $P_X = (\mathcal{M}_{X_0}/\mathcal{O}^*_{X_0})_{\overline{x}}$, $P_Y = (\mathcal{M}_{Y_0}/\mathcal{O}^*_{Y_0})_{\overline{y}}$ and the horizontal arrows are clean charts such that the natural morphism $(\mathcal{O}_{X,\overline{x}}/\mathfrak{m}\mathcal{O}_{X,\overline{x}}) \otimes_{\mathbb{Z}[P_X]} \mathbb{Z}[P_Y] \to R_Y/\mathfrak{m}R_Y$ is an isomorphism. It follows from the fact that these clean charts lift to clean charts of X_n and Y_n that this isomorphism lifts to an isomorphism $\hat{\mathcal{O}}_{X,\overline{x}} \otimes_{\mathbb{Z}[P_X]} \mathbb{Z}[P_Y] \xrightarrow{\sim} \hat{R}_Y$ (where $\hat{\mathcal{O}}_{X,\overline{x}}$ is the completion of $\mathcal{O}_{X,\overline{x}}$ with respect to the ideal $\mathfrak{m}\mathcal{O}_{X,\overline{x}}$). Thus, by [11], Theorem 4.1, 8.2 and the log regularity of $X^{\log}$, we obtain that $\hat{R}_Y$ is normal, hence reduced.

Thus, by Theorem 4.5, there exists a unique finite Kummer fs log scheme $Y^{\log}$ over $X^{\log}$ whose log completion of along $Y \times_S s$ is naturally isomorphic to $\mathfrak{Y}^{\log}$. Moreover, it follows from the fact that $\hat{\mathcal{O}}_{X,\overline{x}} \otimes_{\mathbb{Z}[P_X]} \mathbb{Z}[P_Y] \xrightarrow{\sim} \hat{R}_Y$ (in the preceding paragraph) is an isomorphism that $Y^{\log} \to X^{\log}$ is a ket covering. □

5 Morphisms of type $\mathbb{N}^{\oplus n}$

In this section, we define the notion of a morphism of type $\mathbb{N}^{\oplus n}$ and consider fundamental properties of such a morphism.

Definition 5.1. Let $X^{\log}$ and $Y^{\log}$ be fs log schemes, $f^{\log} : Y^{\log} \to X^{\log}$ a morphism of log schemes.

(i) Let n be a natural number. We shall refer to $f^{\log} : Y^{\log} \to X^{\log}$ as a *morphism of type* $\mathbb{N}^{\oplus n}$ if

- the underlying morphism $f : Y \to X$ of schemes is an isomorphism;
- for any geometric point $\overline{x} \to X$ of X and any clean chart (an étale neighborhood $U \to X$ of $\overline{x} \to X$, $\alpha : P \to \mathcal{O}_U$) of $X^{\log}$ at $\overline{x} \to X$, there exists an étale morphism $V \to U$ and a clean chart $(V \to U \to X \overset{f^{-1}}{\simeq} Y, Q \to \mathcal{O}_V)$ of $Y^{\log}$ at the geometric point $\overline{x} \to X \overset{f^{-1}}{\simeq} Y$ (i.e., $V \to U \to X \overset{f^{-1}}{\simeq} Y$ is an étale neighborhood of the geometric point $\overline{x} \to X \overset{f^{-1}}{\simeq} Y$) such that there exists an isomorphism $\iota : Q \xrightarrow{\sim} P \oplus \mathbb{N}^{\oplus n}$, and the morphism $Q \to \mathcal{O}_V$ is given by

$$\begin{array}{ccccc} Q & \overset{\iota}{\underset{\sim}{\longrightarrow}} & P \oplus \mathbb{N}^{\oplus n} & \longrightarrow & \mathcal{O}_V \\ & & (p, m_1, \cdots, m_n) & \mapsto & \alpha(p)\,|_V \cdot 0^{m_1+\cdots+m_n} \,, \end{array}$$

and $f^{\log}$ is determined by the morphism of monoids:

$$\begin{array}{ccc} P & \longrightarrow Q \overset{\iota}{\underset{\sim}{\longrightarrow}} & P \oplus \mathbb{N}^{\oplus n} \\ p & \mapsto & (p, 0, \cdots, 0)\,. \end{array}$$

(ii) We shall refer to $f^{\log} : Y^{\log} \to X^{\log}$ as a morphism of type $\mathbb{N}^{\oplus *}$ if

- the underlying morphism $f : Y \to X$ is an isomorphism;
- for any geometric point $\overline{x} \to X$ of X, there exists an étale neighborhood $U \to X$ of $\overline{x} \to X$ such that the base-change $Y^{\log} \times_{X^{\log}} U^{\log} \to U^{\log}$ is morphism of type $\mathbb{N}^{\oplus n}$ for some natural number n. Here, $U^{\log}$ is the log scheme obtaind by equipping U with the log structure induced by the log structure of $X^{\log}$.

Remark 5.2. A typical example of a morphism of type $\mathbb{N}$ is as follows: Let X be a regular scheme, $D \subseteq X$ a prime divisor of X such that the closed immersion $D \hookrightarrow X$ is regular immersion of codimension 1. We denote by $X^{\log}$ the log scheme obtained by equipping X with the log structure associated to the divisor D, and by $D^{\log}$ the log scheme obtained by equipping D with the log structure induced by the log structure of $X^{\log}$ via $D \hookrightarrow X$. Then the morphism $D^{\log} \to D$ induced by the natural inclusion $\mathcal{O}_D^* \hookrightarrow \mathcal{M}_D$ is of type $\mathbb{N}$.

Remark 5.3. In this section, we often use the notation $\underline{X}^{\log} \to X^{\log}$ to denote a morphism of type $\mathbb{N}^*$. Moreover, we often identify the underlying scheme of $\underline{X}^{\log}$ with X via the underlying morphism of schemes of the morphism of type $\mathbb{N}^*$.

Remark 5.4. In the notation of Definition 5.1, there exists a splitting $Q \xrightarrow{\sim} P \oplus (Q/P)$; moreover, it is *canonical*. In fact, by the definition of a morphism of type $\mathbb{N}^{\oplus n}$, the quotient Q/P of Q by P is isomorphic to $\mathbb{N}^{\oplus n}$ (non-canonically). We denote by e_i the element of Q/P that corresponds to $(0, \cdots, \overset{i\text{-th}}{1}, \cdots, 0)$ under the isomorphism $Q/P \simeq \mathbb{N}^{\oplus n}$. Then, by the existence of the (non-canonical) isomorphism $Q \xrightarrow{\sim} P \oplus \mathbb{N}^{\oplus n}$, there exists a unique element $\tilde{e}_i$ of Q such that;

- $\tilde{e}_i$ modulo P is e_i,
- $\tilde{e}_i$ is irreducible element of P (Definition A.3).

Thus, the section

$$\begin{array}{ccc} Q/P & \longrightarrow & Q \\ e_i & \mapsto & \tilde{e}_i \end{array}$$

of the natural projection $Q \to Q/P$ induces a canonical splitting $Q \simeq P \oplus (Q/P)$. Moreover, the image of $\tilde{e}_i$ via the morphism which appears in the chart $Q \to \mathcal{O}_V$ is 0.

Lemma 5.5. *A morphism of type $\mathbb{N}^{\oplus n}$ is stable under base-change in the category of fs log schemes.*

Proof. Let $X^{\log}$ be a fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of type $\mathbb{N}^{\oplus n}$, and $Y^{\log} \to X^{\log}$ a morphism of fs log schemes. Let

$$Y \xleftarrow{\text{étale}} V \longrightarrow U \xrightarrow{\text{étale}} X$$

$$\begin{array}{ccc} Q & \longleftarrow & P \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \mathcal{O}_V & \longleftarrow & \mathcal{O}_U \end{array}$$

be an fs chart of $Y^{\log} \to X^{\log}$. Then the underlying scheme of the fiber product of $\underline{X}^{\log}$ and $Y^{\log}$ over $X^{\log}$ in the category of *arbitrary* log schemes is Y, and this fiber product has a chart

$$\begin{array}{ccc} Q \oplus \mathbb{N}^{\oplus n} & \longrightarrow & \mathcal{O}_V \\ (p, m_1, \cdots, m_n) & \mapsto & \alpha(p) \cdot 0^{m_1 + \cdots + m_n} . \end{array}$$

Now $Q \oplus \mathbb{N}^{\oplus n}$ is an fs monoid. Thus, this fiber product is also the fiber product in the category of the fs log schemes. Moreover, it follows immediately from the definition of a morphism of type $\mathbb{N}^{\oplus n}$ that the projection $\underline{X}^{\log} \times_{X^{\log}} Y^{\log} \to Y^{\log}$ is type $\mathbb{N}^{\oplus n}$. $\square$

Definition 5.6. Let X be a scheme, and $\mathcal{M}_1 \to \mathcal{O}_X$ and $\mathcal{M}_2 \to \mathcal{O}_X$ fs log structures on X. Let $X_1^{\log}$ (respectively, $X_2^{\log}$) be the log scheme obtained by equipping X with the log structure $\mathcal{M}_1 \to \mathcal{O}_X$ (respectively, $\mathcal{M}_2 \to \mathcal{O}_X$). Then the natural morphism $X_1^{\log} \times_X X_2^{\log} \to X$ induces an isomorphism between the underlying scheme of $X_1^{\log} \times_X X_2^{\log}$ and X. We shall denote by $\mathcal{M}_1 + \mathcal{M}_2 \to \mathcal{O}_X$ the log structure of $X_1^{\log} \times_X X_2^{\log}$ on X.

Remark 5.7.

(i) In the notation of Definition 5.6, for any geometric point $\overline{x} \to X$; there exist an étale neighborhood $U \to X$ of $\overline{x} \to X$, fs monoids P_1 and P_2, and morphisms of monoids $\alpha_1 : P_1 \to \mathcal{O}_U$ and $\alpha_2 : P_2 \to \mathcal{O}_U$ such that $\alpha_1 : P_1 \to \mathcal{O}_U$ (respectively, $\alpha_2 : P_2 \to \mathcal{O}_U$) is an fs chart of $\mathcal{M}_1$ (respectively, $\mathcal{M}_2$) at $\overline{x} \to X$. Then there exists an fs chart of the log structure $\mathcal{M}_1 + \mathcal{M}_2 \to \mathcal{O}_X$ at $\overline{x} \to X$ that is of the form

$$\begin{array}{ccc} P_1 \oplus P_2 & \longrightarrow & \mathcal{O}_U \\ (p_1, p_2) & \mapsto & \alpha_1(p_1) \cdot \alpha_2(p_2) . \end{array}$$

In particular, $(\mathcal{M}_1 + \mathcal{M}_2)/\mathcal{O}_X^* \simeq (\mathcal{M}_1/\mathcal{O}_X^*) \oplus (\mathcal{M}_2/\mathcal{O}_X^*)$.

(ii) In the notation of Definition 5.6, for any a morphism of scheme $f : Y \to X$, $f^*(\mathcal{M}_1 + \mathcal{M}_2) = f^*(\mathcal{M}_1) + f^*(\mathcal{M}_2)$ (where f^* denotes the pull-back of log structures, not of sheaves).

(iii) Let X be a regular scheme, and $D = \Sigma_{i=1}^n D_i \subseteq X$ a divisor with normal crossings. If we denote by $\mathcal{M}(D)$ (respectively, $\mathcal{M}(D_i)$) the log structure of X defined by the divisor with normal crossings D (respectively, D_i), then $\mathcal{M}(D) = \Sigma_{i=1}^n \mathcal{M}(D_i)$.

(iv) Clearly, $(\mathcal{M}_1 + \mathcal{M}_2) + \mathcal{M}_3 = \mathcal{M}_1 + (\mathcal{M}_2 + \mathcal{M}_3)$.

Remark 5.8. Let $X^{\log}$ be an fs log scheme, and $f^{\log} : \underline{X}^{\log} \to X^{\log}$ be a morphism of type $\mathbb{N}^{\oplus n}$. Then we have the following diagram:

$$\begin{array}{ccccc} \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{M}_X/\mathcal{O}_X^* \\ \| & & \downarrow & & \downarrow \\ \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_{\underline{X}} & \longrightarrow & \mathcal{M}_{\underline{X}}/\mathcal{O}_X^* \\ & & \downarrow & & \downarrow \\ & & \mathcal{M}_{\underline{X}}/\mathcal{M}_X & \xrightarrow{\sim} & \mathcal{C}_{f^{\log}} , \end{array}$$

where $\mathcal{C}_{f^{\log}}$ is the quotient of $\mathcal{M}_{\underline{X}}/\mathcal{O}_X^*$ by the subsheaf $\mathcal{M}_X/\mathcal{O}_X^*$. Then, by the definition of a morphism of type $\mathbb{N}^{\oplus n}$, $\mathcal{C}_{f^{\log}}$ is locally constant, and the stalk at any geometric point of X is non-canonically isomorphic to $\mathbb{N}^{\oplus n}$. (Indeed, this follows from the existence of the chart in Definition 5.1.) Moreover, by Remark 5.4, the sheaf $\mathcal{M}_{\underline{X}}/\mathcal{O}_X^*$ admits a canonical splitting $(\mathcal{M}_X/\mathcal{O}_X^*) \oplus \mathcal{C}_{f^{\log}}$.

Now the group $\mathrm{Aut}\,(\mathbb{N}^{\oplus n})$ is isomorphic to the symmetric group on n letters, hence, in particular, is finite. (Indeed, this follows from the fact that any automorphism of $\mathbb{N}^{\oplus n}$ preserves the irreducible elements of $\mathbb{N}^{\oplus n}$, together with the fact that the irreducible elements of $\mathbb{N}^{\oplus n}$ are the e_i's (where $e_i = (0, \cdots, 0, \overset{i\text{–th}}{1}, 0, \cdots, 0))$. More generally, by Proposition A.2, if P is a clean monoid, then $\mathrm{Aut}\,(P)$ is a finite group.) Since $\mathcal{C}_{f^{\log}}$ is locally constant, and the stalk at any geometric point of X is isomorphic to $\mathbb{N}^{\oplus n}$, it thus follows that there exists a finite étale covering $X' \to X$ such that the pull-back of $\mathcal{C}_{f^{\log}}$ to X' is constant. (Indeed, this follows from the fact that since the sheaf of sets of isomorphisms between $\mathcal{C}_{f^{\log}}$ and $\mathbb{N}_X^{\oplus n}$ on the étale site of X is locally constant, and has finite stalks, there exists a finite étale covering $X' \to X$ such that the restriction of the sheaf to X' is constant.) Moreover, since $\mathrm{Aut}\,(\mathbb{N})$ is trivial, if $n = 1$, then $\mathcal{C}_{f^{\log}}$ is always constant.

On the other hand, in the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, equal] & \mathcal{M}_X^{\mathrm{gp}} \arrow[r] \arrow[d] & \mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & \mathcal{M}_{\underline{X}}^{\mathrm{gp}} \arrow[r] \arrow[d] & \mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{O}_X^* \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{M}_X^{\mathrm{gp}} \arrow[r, "\sim"] \arrow[d] & \mathcal{C}_{\underline{X}}^{\mathrm{gp}} \arrow[d] & \\
 & & 0 & 0\,, &
\end{tikzcd}$$

all vertical and horizontal sequences are exact. Now the sheaf $\mathcal{C}_{\underline{X}}^{\mathrm{gp}}$ is locally constant, and the stalk at any geometric point is non-canonically isomorphic to $\mathbb{Z}_X^{\oplus n}$. By Remark 5.4, the sheaf $\mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{O}_X^*$ admits a canonical splitting $(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*) \oplus \mathcal{C}_{\underline{X}}^{\mathrm{gp}}$.

Lemma 5.9. *Let $X^{\log}$ be an fs log scheme, and $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of type $\mathbb{N}^{\oplus n}$. Then there exists a unique morphism $\underline{\underline{X}}^{\log} \to X$ of type*

$\mathbb{N}^{\oplus n}$ *and a unique morphism* $\underline{X}^{\log} \to \underline{\underline{X}}^{\log}$ *such that the resulting morphism* $\underline{X}^{\log} \to \underline{\underline{X}}^{\log} \times_X X^{\log}$ *is an isomorphism, i.e.,* $\mathcal{M}_{\underline{X}} = \mathcal{M}_X + \mathcal{M}_{\underline{\underline{X}}}$.

Proof. By Remark 5.8, we have a canonical section $\mathcal{C}_{f^{\log}} \to \mathcal{M}_{\underline{X}}/\mathcal{O}_X^*$. We define the sheaf $\mathcal{M}_{\underline{\underline{X}}}$ of monoids by the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{M}_{\underline{\underline{X}}} & \longrightarrow & \mathcal{C}_{f^{\log}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}_{\underline{X}} & \longrightarrow & \mathcal{M}_{\underline{X}}/\mathcal{O}_X^*. \end{array}$$

Then since the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_{\underline{X}}$ factors through $\mathcal{M}_{\underline{\underline{X}}}$, the composite $\mathcal{M}_{\underline{\underline{X}}} \to \mathcal{M}_{\underline{X}} \to \mathcal{O}_X$ (where the second morphism $\mathcal{M}_{\underline{X}} \to \mathcal{O}_X$ is the log structure of $\underline{X}^{\log}$) is a log structure on X; moreover, the injection $\mathcal{M}_{\underline{\underline{X}}} \to \mathcal{M}_{\underline{X}}$ induces the morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log}$ (where $\underline{\underline{X}}^{\log}$ is the log scheme obtained by equipping X with the log structure $\mathcal{M}_{\underline{\underline{X}}} \to \mathcal{O}_X$). On the other hand, it follows from the fact that the stalk of $\mathcal{C}_{f^{\log}}$ at any geometric point of X is isomorphic to $\mathbb{N}^{\oplus n}$, together with the fact that the image of $\tilde{e}_i$ via the morphism $Q \to \mathcal{O}_V$ is 0 in the notation of Remark 5.4 that the morphism $\underline{\underline{X}}^{\log} \to X$ induced by the natural inculusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_{\underline{\underline{X}}}$ is of type $\mathbb{N}^{\oplus n}$. Now, by construction and the the fact that $f^{\log}$ is of type $\mathbb{N}^{\oplus n}$, the resulting morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log} \times_X X^{\log}$ is an isomorphism. □

Definition 5.10. Let $X^{\log}$ be a locally noetherian connected fs log scheme.

(i) Let $f^{\log} : \underline{X}^{\log} \to X^{\log}$ be a morphism of type $\mathbb{N}^{\oplus n}$. Then we shall refer to $f^{\log}$ as a *morphism of constant type* $\mathbb{N}^{\oplus n}$ if $\mathcal{C}_{f^{\log}}$ is constant. Let $f^{\log}$ be a morphism of constant type $\mathbb{N}^{\oplus n}$. Then we shall refer to an isomorphism $\tau : \mathbb{N}_X^{\oplus n} \xrightarrow{\sim} \mathcal{C}_{f^{\log}}$ as a *trivialization* of $f^{\log}$. Note that, by the portion of Remark 5.8 concerning the case "$n = 1$", any morphism of type $\mathbb{N}$ is of constant type $\mathbb{N}$; moreover, such a morphism has a canonical trivialization.

(ii) For pairs $(f_i^{\log}, \tau_i)$ $(i = 1, 2)$, where $f_i^{\log} : X_i^{\log} \to X^{\log}$ is a morphism of constant type $\mathbb{N}^{\oplus n}$ and τ_i is a trivialization of $f_i^{\log}$, we shall say that $(f_1^{\log}, \tau_1)$ is equivalent to $(f_2^{\log}, \tau_2)$ if there exists an isomorphism of fs log schemes $g^{\log} : X_1^{\log} \to X_2^{\log}$ over $X^{\log}$ such that the trivialization of $f_1^{\log}$ induced by the isomorphism $g^* : \mathcal{M}_{X_2} \xrightarrow{\sim} \mathcal{M}_{X_1}$ and τ_2 coincides with τ_1.

(iii) We shall denote by $\mathbb{M}_{X^{\log}}$ the set of pairs $(f^{\log}, \tau)$, where $f^{\log}$ is a morphism of constant type $\mathbb{N}^{\oplus n}$ and τ is a trivialization of $f^{\log}$ modulo the equivalence defined in (ii).

(iv) We shall denote by ι the morphism $\mathbb{M}_{X^{\log}} \to \mathrm{Pic}(X)^{\oplus n}$ defined as follows:

Let $(f^{\log} : \underline{X}^{\log} \to X^{\log}, \tau)$ be an element of $\mathbb{M}_{X^{\log}}$. Then the middle horizontal sequence in the second diagram in Remark 5.8 determines a connecting morphism

$$\mathrm{H}^0_{\text{ét}}(X, \mathcal{M}^{\mathrm{gp}}_{\underline{X}}/\mathcal{O}^*_X) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathcal{O}^*_X)\,.$$

Now since one has natural isomorphisms $\mathcal{M}^{\mathrm{gp}}_{\underline{X}}/\mathcal{O}^*_X \simeq (\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X) \oplus \mathcal{C}^{\mathrm{gp}}_{f^{\log}}$ and $\mathrm{H}^1_{\text{ét}}(X, \mathcal{O}^*_X) \simeq \mathrm{Pic}(X)$, we obtain a morphism

$$\mathrm{H}^0_{\text{ét}}(X, \mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X) \oplus \mathrm{H}^0_{\text{ét}}(X, \mathcal{C}^{\mathrm{gp}}_{f^{\log}}) \longrightarrow \mathrm{Pic}(X)\,.$$

For the element $e_i = (0, \cdots, \overset{i-\text{th}}{1}, \cdots, 0)$ of $\mathrm{H}^0_{\text{ét}}(\mathbb{Z}^{\oplus n}_X) = \mathbb{Z}^{\oplus n}$, let us denote by $\mathcal{L}_i$ the image of e_i via the composite

$$\mathrm{H}^0_{\text{ét}}(X, \mathbb{Z}^{\oplus n}_X) \overset{\text{via}\,\tau^{\mathrm{gp}}}{\xrightarrow{\sim}} \mathrm{H}^0_{\text{ét}}(X, \mathcal{C}^{\mathrm{gp}}_{f^{\log}}) \longrightarrow \mathrm{H}^0_{\text{ét}}(X, \mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X) \oplus \mathrm{H}^0_{\text{ét}}(X, \mathcal{C}^{\mathrm{gp}}_{f^{\log}}) \longrightarrow \mathrm{Pic}(X)\,,$$

where the second arrow is $x \mapsto (0, x)$, and the third arrow is as above. Then we shall write $\iota(f^{\log}, \tau) = (\mathcal{L}_1, \cdots, \mathcal{L}_n)$.

(v) We shall denote by κ the morphism $\mathrm{Pic}(X^{\log})^{\oplus n} \to \mathbb{M}_{X^{\log}}$ defined as follows:

Let $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$ be an element of $\mathrm{Pic}(X^{\log})^{\oplus n}$. We denote by V_i the geometric line bundle defined by the invertible sheaf $\mathcal{L}_i$ (i.e., the spectrum of the symmetric algebra of $\mathcal{L}_i^{\otimes(-1)}$ over X), by $p_i : V_i \to X$ the natural morphism, by $s_i : X \to V_i$ the 0-section of p_i, by $p : V \overset{\text{def}}{=} V_1 \times_X \cdots \times_X V_n \to X$ the natural morphism and by $s : X \to V$ the section $s_1 \times_X \cdots \times_X s_n$ of p. Let $V^{\log}$ be the log scheme obtained by equipping V with the log structure $\mathcal{M}_V = p^*\mathcal{M}_X + \mathcal{M}(D_1) + \cdots \mathcal{M}(D_n)$, where D_i is the divisor on V defined by the following cartesian diagram

$$\begin{array}{ccc} D_i & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \mathrm{pr}_i \\ X & \xrightarrow{s_i} & V_i\,, \end{array}$$

and $\mathcal{M}(D_i)$ is a log structure defined by the divisor D_i. (See Remark 5.11 below.) Then we obtain a natural morphism of log schemes $p^{\log} : V^{\log} \to X^{\log}$ whose underlying morphism of schemes is p. If we denote by $\underline{X}^{\log}$ the log scheme obtained by equipping X with the log structure $s^*\mathcal{M}_V$, then it is immediate that the composite $\underline{X}^{\log} \stackrel{s^{\log}}{\to} V^{\log} \stackrel{p^{\log}}{\to} X^{\log}$ is of type $\mathbb{N}^{\oplus n}$. On the other hand, since

$$\mathcal{M}_{\underline{X}} = s^*(p^*\mathcal{M}_X + \mathcal{M}(D_1) + \cdots \mathcal{M}(D_n)) = \mathcal{M}_X + s^*\mathcal{M}(D_1) + \cdots + s^*\mathcal{M}(D_n)\,,$$

it follows that

$$\mathcal{C}_{f^{\log}} \simeq (s^*\mathcal{M}(D_1)/\mathcal{O}_X^*) \oplus \cdots \oplus (s^*\mathcal{M}(D_n)/\mathcal{O}_X^*)$$

(cf. Remark 5.7, (i)). Now, by the portion of Remark 5.8 concerning the case "$n = 1$", $s^*\mathcal{M}(D_i)/\mathcal{O}_X^*$ is constant, i.e., there exists a canonical isomorphism $\tau_i : \mathbb{N}_X \stackrel{\sim}{\to} s^*\mathcal{M}(D_i)/\mathcal{O}_X^*$. Thus, $\mathcal{C}_{f^{\log}}$ is constant. Let us define a trivialization τ of $p^{\log} \circ s^{\log}$ by

$$\begin{array}{ccc} \mathbb{N}_X^{\oplus n} & \stackrel{\tau}{\longrightarrow} & (s^*\mathcal{M}(D_1)/\mathcal{O}_X^*) \oplus \cdots \oplus (s^*\mathcal{M}(D_n)/\mathcal{O}_X^*) \\ (m_1, \cdots, m_n) & \mapsto & (\tau_1(m_1), \cdots, \tau_n(m_n))\,. \end{array}$$

Then we shall write $\kappa(\mathcal{L}_1, \cdots, \mathcal{L}_n) = (p^{\log} \circ s^{\log}, \tau)$.

Remark 5.11. For a positive Cartier divisor D on a locally noetherian scheme X, we denote by $\mathcal{M}(D)$ the log structure on X that is defined as follows:

Let us denote by $\mathcal{G}_D \in \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m)$ the $\mathbb{G}_m$-torsor sheaf on (the étale site of) X that is determined by D, and by $\mathcal{G}_D^i \in \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m)$ the $\mathbb{G}_m$-torsor sheaf on X that is obtained by applying a "change of structure of group" to $\mathcal{G}_D$ via the morphism

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ f & \mapsto & f^i. \end{array}$$

Write $\mathcal{M}(D)' = \sqcup_{i \in \mathbb{N}} \mathcal{G}_D^i$. Then the natural morphisms $\mathcal{G}_D^i \times \mathcal{G}_D^j \to \mathcal{G}_D^{i+j}$ determine a natural structure of sheaf of monoids on $\mathcal{M}(D)'$. Moreover, the composite $\mathcal{G}_D \hookrightarrow \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ (the first inclusion arises from the fact that the invertible sheaf determined by the $\mathbb{G}_m$-torsor sheaf $\mathcal{G}_D$ is naturally isomorphic to $\mathcal{O}_X(-D)$) induces a homomorphism $\mathcal{M}(D)' \to \mathcal{O}_X$ of sheaves of monoids. Then we define the log structure $\mathcal{M}(D)$ as the log structure associated to the above pre-log structure $\mathcal{M}(D)' \to \mathcal{O}_X$.

Note that, if X is regular, and D is a divisor with normal crossings, then this log structure $\mathcal{M}(D)$ coincides with the log structure defined in [10], 1.5.1.

Remark 5.12. Let $X^{\log}$ be a locally noetherian connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbb{N}^{\oplus n}$, and $\tau : \mathbb{N}_X^{\oplus n} \stackrel{\sim}{\to} \mathcal{C}_{f^{\log}}$

a trivialization. We write $\iota(f^{\log},\tau)=(\mathcal{L}_1,\cdots,\mathcal{L}_n)$. If we denote by $\mathcal{G}_i$ the subsheaf of $\mathcal{M}_{\underline{X}}$ defined by the following cartesian diagram

$$\begin{array}{ccc}
\mathcal{G}_i & \longrightarrow & 0\oplus\{e_{i,X}\} \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{M}_{\underline{X}} & \longrightarrow & (\mathcal{M}_{\underline{X}}/\mathcal{O}_X^* \simeq)\,(\mathcal{M}_X/\mathcal{O}_X^*)\oplus\mathcal{C}_{f^{\log}}
\end{array}$$

(where $\{e_{i,X}\}$ is the subsheaf of $\mathbb{N}_X^{\oplus n}$ whose sections correspond to $e_i=(0,\cdots,\overset{i-\mathrm{th}}{1},\cdots,0)\in\mathbb{N}^{\oplus n}$), then $\mathcal{G}_i$ is a $\mathbb{G}_m$-torsor sheaf on X. Moreover, it is a tautology that the invertible sheaf determined by the $\mathbb{G}_m$-torsor sheaf $\mathcal{G}_i$ is naturally isomorphic to $\mathcal{L}_i$.

Lemma 5.13. *Let $X^{\log}$ be a locally noetherian connected fs log scheme, $f^{\log}:\underline{X}^{\log}\to X^{\log}$ a morphism of constant type $\mathbb{N}^{\oplus n}$, and τ a trivialization of $f^{\log}$. Then there exist morphisms $f_i^{\log}:\underline{X}_i^{\log}\to X$ of type $\mathbb{N}$, whose canonical trivialization (see Definition 5.10, (i)) we denote by τ_i, such that the following hold:*

(*i*) $\mathcal{M}_{\underline{X}}=\mathcal{M}_X+\Sigma_{i=1}^n\mathcal{M}_{\underline{X}_i}$.

(*ii*) *The composite*

$$\mathcal{C}_{f^{\log}}\xrightarrow[\sim]{\text{via }(i)}\mathcal{C}_{f_1^{\log}}\oplus\cdots\oplus\mathcal{C}_{f_n^{\log}}\xrightarrow[\sim]{\tau_1\oplus\cdots\oplus\tau_n}\mathbb{N}^{\oplus n}$$

coincides with τ.

(*iii*) $\iota(f^{\log},\tau)=(\iota(f_1^{\log},\tau_1),\cdots,\iota(f_n^{\log},\tau_n))$.

Proof. Let us denote by $\mathcal{M}_i$ the subsheaf of $\mathcal{M}_{\underline{X}}$ defined by the following cartesian diagram (cf. the cartesian diagram of the proof of Lemma 5.9)

$$\begin{array}{ccc}
\mathcal{M}_i & \longrightarrow & 0\oplus\mathbb{N}_X \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{M}_{\underline{X}} & \longrightarrow & (\mathcal{M}_{\underline{X}}/\mathcal{O}_X^*\xrightarrow{\sim}(\mathcal{M}_X/\mathcal{O}_X^*)\oplus\mathcal{C}_{f^{\log}}\xrightarrow[\sim]{\mathrm{id}\oplus\tau})(\mathcal{M}_X/\mathcal{O}_X^*)\oplus\mathbb{N}_X^{\oplus n}\,,
\end{array}$$

where the right-hand vertical arrow is

$$\begin{array}{ccc}
0\oplus\mathbb{N}_X & \longrightarrow & (\mathcal{M}_X/\mathcal{O}_X^*)\oplus\mathbb{N}_X^{\oplus n} \\
(0,n_X) & \mapsto & (0,n\cdot e_{i,X})\,.
\end{array}$$

Then the composite $\mathcal{M}_i\to\mathcal{M}_{\underline{X}}\to\mathcal{O}_X$ is a log structure. Moreover, if we denote by $\underline{X}_i^{\log}$ the log scheme obtained by equipping X with the log structure $\mathcal{M}_i \to \mathcal{O}_X$ and by $f_i^{\log} : \underline{X}_i^{\log} \to X$ the morphism determined by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_i$, then the $f_i^{\log}$ satisfies conditions (i), (ii) and (iii) in the statement of Lemma 5.13. $\square$

Theorem 5.14. *ι is a bijection. The inverse of ι is κ.*

Proof. By Lemmas 5.5 and 5.9, the morphism $\mathbb{M}_X \to \mathbb{M}_{X^{\log}}$ induced by the morphism $X^{\log} \to X$ (determined by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$) is a bijection. Therefore, we may assume that the log structure of $X^{\log}$ is trivial. Moreover, by Lemma 5.13, we may assume $n = 1$.

First, we prove that $\kappa \circ \iota$ is the identity morphism. Let $f^{\log} : \underline{X}^{\log} \to X$ be a morphism of type $\mathbb{N}$. If we denote by $\mathcal{G}$ the $\mathbb{G}_m$-torsor sheaf defined in Remark 5.12, then it is a tautology that the restriction to X of the $\mathbb{G}_m$-torsor sheaf on V that corresponds to the invertible sheaf $\mathcal{O}_V(-X)$ (where we regard X as a Cartier divisor on V via the 0-section $X \to V$) is naturally isomorphic to $\mathcal{G}$. Therefore, the pull-back to X of the log structure on V associated to the divisor X (see Remark 5.11) is naturally isomorphic to $M_{\underline{X}}$.

Next, we prove that $\iota \circ \kappa$ is the identity morphism. Let $\mathcal{L}$ be an invertible sheaf on X. If we denote by $\mathcal{G}$ the $\mathbb{G}_m$-torsor sheaf that corresponds to $\mathcal{L}$, then it is a tautology that the restriction to X of the $\mathbb{G}_m$-torsor sheaf that corresponds to the invertible sheaf $\mathcal{O}_V(-X)$ (where we regard X as a Cartier divisor on V via the 0-section $X \to V$) is naturally isomorphic to $\mathcal{G}$. Therefore, the line bundle that corresponds to the $\mathbb{G}_m$-torsor sheaf obtained by the log structure on V associated to the divisor X (see Remark 5.12) is naturally isomorphic to $\mathcal{L}$. $\square$

Remark 5.15. In the notation of Remark 5.2, the invertible sheaf on D which corresponds to the morphism $D^{\log} \to D$ of type $\mathbb{N}$ is the normal sheaf $\mathcal{N}_{D/X}$ of D in X by the definition of ι.

Definition 5.16. Let $X^{\log}$ be a locally noetherian connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbb{N}^{\oplus n}$, $\tau : \mathbb{N}_X^{\oplus n} \xrightarrow{\sim} \mathcal{C}_{f^{\log}}$ a trivialization of $f^{\log}$, and $\iota(f^{\log}, \tau) = (\mathcal{L}_1, \cdots \mathcal{L}_n)$. We shall denote by $\pi_i : P_i \to X$ the $\mathbb{P}^1$-bundle associated to the locally free sheaf $\mathcal{L}_i^{\otimes(-1)} \oplus \mathcal{O}_X$ (see Remark 5.17), by $s_i^0 : X \to P_i$ (respectively, $s_i^\infty : X \to P_i$) is the section of π_i induced by the projection $\mathcal{L}_i^{\otimes(-1)} \oplus \mathcal{O}_X \to \mathcal{O}_X$ (respectively, $\mathcal{L}_i^{\otimes(-1)} \oplus \mathcal{O}_X \to \mathcal{L}_i^{\otimes(-1)}$), by $\pi : P \stackrel{\text{def}}{=} P_1 \times_X \cdots \times_X P_n \to X$ the natural morphism and by $s^0 : X \to P$ the section $s_i^0 \times_X \cdots \times_X s_i^0$ of π. We shall denote by $P^{\log}$ the log scheme obtained by equipping P with the log structure $\mathcal{M}_P = \pi^*\mathcal{M}_X + \mathcal{M}(D_1^0) + \cdots + \mathcal{M}(D_n^0) + \mathcal{M}(D_1^\infty) + \cdots + \mathcal{M}(D_n^\infty)$, where D_i^0 (respectively, D_i^∞) is the divisor on P defined by the following cartesian diagram

$$\begin{tikzcd} D_i^0 \arrow[r] \arrow[d] & P \arrow[d] \\ X \arrow[r, "s_i^0"] & P_i \end{tikzcd} \qquad \left(\text{respectively, } \begin{tikzcd} D_i^\infty \arrow[r] \arrow[d] & P \arrow[d] \\ X \arrow[r, "s_i^\infty"] & P_i \end{tikzcd}\right),$$

and $\mathcal{M}(D_i^0)$ (respectively, $\mathcal{M}(D_i^\infty)$) is the log structure defined by the divisor D_i^0 (respectively, D_i^∞). Then we obtain a natural morphism of log schemes $\pi^{\log} : P^{\log} \to X^{\log}$ whose underlying morphism of schemes is π; by Theorem 5.14, the log scheme obtained by equipping X with the log structure $(s^0)^*\mathcal{M}_P$ is isomorphic to $\underline{X}^{\log}$, and the composite $\underline{X}^{\log} \xrightarrow{(s^0)^{\log}} P^{\log} \xrightarrow{\pi^{\log}} X^{\log}$ is $f^{\log}$. We shall refer to $\pi^{\log} : P^{\log} \to X^{\log}$ as the *log $\mathbb{G}_m^{\times n}$-torsor associated to* $(f^{\log}, \tau)$ or, alternatively, *to* $(\mathcal{L}_1, \cdots \mathcal{L}_n)$. Note that $\pi^{\log}$ is projective and log smooth.

Remark 5.17. Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme X, $V \to X$ the geometric vector bundle associated to $\mathcal{E}$, and $P \to X$ (respectively, $P' \to X$) the $\mathbb{P}^n$-bundle (respectively, the $\mathbb{P}^{n-1}$-bundle) associated to the locally free sheaf $\mathcal{E}^\vee \oplus \mathcal{O}_X$ (respectively, $\mathcal{E}^\vee$) (where $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$), and $P' \hookrightarrow P$ the closed immersion over X determined by the projection $\mathcal{E}^\vee \oplus \mathcal{O}_X \to \mathcal{E}^\vee$. Then V is naturally isomorphic to the complement of P' in P.

Indeed, it follows immediately from construction that $P \setminus P' \to X$ is a vector bundle of rank n over X. Moreover, for an open subscheme $U \hookrightarrow X$ of X, a section of $(P \setminus P') \mid_U \to U$ corresponds to the isomorphic class of the following data:

- An invertible sheaf $\mathcal{L}$ on U.
- A surjection $\pi : \mathcal{E}^\vee \mid_U \oplus \mathcal{O}_U \to \mathcal{L}$ such that the composite $\mathcal{O}_U \hookrightarrow \mathcal{E}^\vee \mid_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathcal{L}$ does not vanish on U (we denote by $s \in \Gamma(U, \mathcal{L})$ the section of $\mathcal{L}$ determined by the above composite $\mathcal{O}_U \hookrightarrow \mathcal{E}^\vee \mid_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathcal{L}$).

It is immediate that then $\mathcal{O}_U \xrightarrow{s} \mathcal{L}$ is an isomorphism, and if we denote by $\phi_U(s)$ the section of $\Gamma(U, \mathcal{E} \mid_U)$ determined by the composite $\mathcal{E}^\vee \mid_U \hookrightarrow \mathcal{E}^\vee \mid_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathcal{L} \xrightarrow{s^{-1}} \mathcal{O}_U$ for the above data, then the assignment $(\mathcal{L}, \pi : \mathcal{E}^\vee \mid_U \oplus \mathcal{O}_U \to \mathcal{L}) \mapsto \phi_U(s)$ determines a bijection between the set of sections of $(P \setminus P') \mid_U \to U$ and $\Gamma(U, \mathcal{E} \mid_U)$; therefore, $P \setminus P' \to X$ is isomorphic to $V \to X$. Moreover, by the above correspondence ϕ_X between the set of sections of $P \setminus P' \to X$ and $\Gamma(X, \mathcal{E} \mid_X)$, the 0-section $X \to V$ of $V \to X$ corresponds to the pair $(\mathcal{O}_X, \mathcal{E}^\vee \oplus \mathcal{O}_X \xrightarrow{\mathrm{pr}_2} \mathcal{O}_X)$.

The main result of this section is the following theorem.

Theorem 5.18. *Let $X^{\log}$ be a locally noetherian connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbb{N}^{\oplus n}$, $\tau : \mathbb{N}_X^{\oplus n} \xrightarrow{\sim} \mathcal{C}_{f^{\log}}$ a trivialization of $f^{\log}$, and $\pi^{\log} : P^{\log} \to X^{\log}$ the log $\mathbb{G}_m^{\times n}$-torsor associated to $(f^{\log}, \tau)$. Then $(s^0)^{\log} : \underline{X}^{\log} \to P^{\log}$ induces a natural equivalence between the Galois category of ket coverings of $P^{\log}$ and the Galois category of ket coverings of $\underline{X}^{\log}$, i.e., $\pi_1((s^0)^{\log})$ is an isomorphism.*

Proof. (Step 1) If X is the spectrum of a field k, and the log structure of X is trivial, then $\pi_1((s^0)^{\log})$ is an isomorphism.

By base-changing, we may assume that k is separably closed. Moreover, by Proposition 3.4, we may assume $n = 1$. Then it follows from Lemma 5.19, (ii) below that $\pi_1((s^0)^{\log})$ is an isomorphism.

(Step 2) If X is the spectrum of a separably closed field k, then $\pi_1((s^0)^{\log})$ is surjective. (We denote by $\alpha : M \to k$ a clean chart of $X^{\log}$.)

We write $R = k[[M]]$, and $S = \operatorname{Spec} R$. Let $S^{\log}$ be the log scheme obtained by equipping S with the log structure associated to the chart given by the natural morphism $M \to R$. Then, by [11], Theorem 3.1, $S^{\log}$ is log regular. Write $(\underline{S}^{\log} \to S^{\log}, \tau_S) \overset{\text{def}}{=} \kappa(\mathcal{O}_S, \cdots, \mathcal{O}_S)$, and denote by $P_S^{\log} \to S^{\log}$ the log $\mathbb{G}_m^{\times n}$-torsor associated to $(\mathcal{O}_S, \cdots, \mathcal{O}_S)$, and by $(s^0)_S^{\log}$ the closed immersion $\underline{S}^{\log} \to P_S^{\log}$. Then we obtain the following cartesian diagram:

$$\begin{array}{ccc} \underline{X}^{\log} & \xrightarrow{(s^0)^{\log}} & P^{\log} \\ \downarrow & & \downarrow \\ \underline{S}^{\log} & \xrightarrow{(s^0)_S^{\log}} & P_S^{\log}. \end{array}$$

We denote by K the field of fractions of R, and by $\operatorname{Spec} K \to S^{\log}$ the strict morphism whose underlying morphism corresponds to the natural inclusion $R \hookrightarrow K$. Then we obtain the following diagram:

$$\begin{array}{ccc} \underline{X}^{\log} & \xrightarrow{(s^0)^{\log}} & P^{\log} \\ \downarrow & & \downarrow \\ \underline{S}^{\log} & \xrightarrow{(s^0)_S^{\log}} & P_S^{\log} \\ \uparrow & & \uparrow \\ \underline{S}^{\log} \times_{S^{\log}} \operatorname{Spec} K \overset{\text{def}}{=} \underline{(\operatorname{Spec} K)}^{\log} & \xrightarrow{(s^0)_K^{\log}} & P_K^{\log} \overset{\text{def}}{=} P_S^{\log} \times_{S^{\log}} \operatorname{Spec} K \end{array}$$

(where the two squares are cartesian).

Now, in the above diagram, the following hold:

(i) $\pi_1(\underline{(\operatorname{Spec} K)}^{\log}) \to \pi_1(P_K^{\log})$ is an isomorphism. (This follows from Step 1.)

(ii) $\pi_1(P_K^{\log}) \to \pi_1(P_S^{\log})$ is surjective. (This follows from the fact that if we denote by η_{P_S} the generic point of P_S [note that since $S^{\log}$ is log regular, P_S is also log regular], then $\pi_1(\eta_{P_S}) \to \pi_1(P_S^{\log})$ is surjective, together with the fact that $\eta_{P_S} \to P_S^{\log}$ factors through $P_K^{\log}$.)

(iii) $\pi_1(\underline{S}^{\log}) \to \pi_1(P_S^{\log})$ is surjective. (This follows from (i) and (ii).)

(iv) $\pi_1(\underline{X}^{\log}) \to \pi_1(\underline{S}^{\log})$ is an isomorphism. (This follows from Proposition A.8.)

(v) $\pi_1(P^{\log}) \to \pi_1(P_S^{\log})$ is an isomorphism. (This follows from Corollary 4.8.)

Therefore, by (iii), (iv) and (v), $\pi_1((s^0)^{\log})$ is surjective.

(Step 3) If X is the spectrum of a strictly henselian local ring A whose residue field is k, then $\pi_1((s^0)^{\log})$ is an isomorphism. (We denote by $(\operatorname{Spec} k =)\overline{x} \to X$ the closed point of X, and by $\alpha : M \to A$ a clean chart of $X^{\log}$.)

First, we prove that $\pi_1((s^0)^{\log})$ is surjective. Let $Q^{\log} \to P^{\log}$ be a connected ket covering of $P^{\log}$. If we denote by $Q \to X' \to X$ the Stein factorization of the composite $Q \to P \to X$, then since Q is connected, and $Q \to X'$ is surjective, we obtain that X' is connected. Now since X is the spectrum of a strictly henselian local ring, and X' is finite over X, $X' \times_X \overline{x}$, hence also $Q \times_X \overline{x}$ is connected. Thus, by base-changing by $\overline{x}^{\log} \to X^{\log}$, we may assume that X is the spectrum of a separably closed field k. Then the surjectivity in question follows from Step 2.

Next, we prove that $\pi_1((s^0)^{\log})$ is injective. Let $Y^{\log} \to \underline{X}^{\log}$ be a connected ket covering. Then, by Proposition A.4, $Y^{\log}$ is of the form $\operatorname{Spec}(A \otimes_{\mathbb{Z}[M \oplus \mathbb{N}^{\oplus n}]} \mathbb{Z}[N])$ for some fs monoid N and some Kummer morphism $M \oplus \mathbb{N}^{\oplus n} \to N$. If we denote by $W^{\log}$ the log scheme obtained by equipping $\operatorname{Spec}(A[t_1, \cdots, t_n] \otimes_{\mathbb{Z}[M \oplus \mathbb{N}^{\oplus n}]} \mathbb{Z}[N])$ (where the morphism $M \oplus \mathbb{N}^{\oplus n} \to A[t_1, \cdots, t_n]$ is given by

$$\begin{array}{ccc} M \oplus \mathbb{N}^{\oplus n} & \longrightarrow & A[t_1, \cdots, t_n] \\ (p, m_1, \cdots, m_n) & \mapsto & \alpha(p) \cdot t_1^{m_1} \cdots t_n^{m_n} \;) \end{array}$$

with the log structure induced by the chart given by the natural morphism $N \to A[t_1, \cdots, t_n] \otimes_{\mathbb{Z}[M \oplus \mathbb{N}^{\oplus n}]} \mathbb{Z}[N]$, then the natural morphism

$$W^{\log} = (\operatorname{Spec}(A[t_1, \cdots, t_n] \otimes_{\mathbb{Z}[M \oplus \mathbb{N}^{\oplus n}]} \mathbb{Z}[N]))^{\log} \to (\operatorname{Spec} A[t_1, \cdots, t_n])^{\log} = V^{\log} (\subseteq P^{\log})$$

(where the equality $\operatorname{Spec} A[t_1, \cdots, t_n] = V$ is obtained by regarding t_i as the "coordinate" of V determined by $\mathcal{L}_i[\simeq \mathcal{O}_X]$) is a connected ket covering, and $W^{\log} \times_{V^{\log}} \underline{X}^{\log}$ is $Y^{\log}$. Thus, the ket covering $Y^{\log}$ over $\underline{X}^{\log}$ extends to a ket covering $W^{\log}$ over $V^{\log}$. Therefore, we obtain the following diagram:

$$\begin{array}{ccccc} Y^{\log} & \longrightarrow & W^{\log} & & \\ \big\downarrow & & \big\downarrow & & \\ \underline{X}^{\log} & \longrightarrow & V^{\log} & \longrightarrow & P^{\log}. \end{array}$$

Now, by the log purity theorem, the connected ket covering $W^{\log} \to V^{\log}$ extends to a connected ket covering of $P^{\log}$. Thus, the morphism $\pi_1(\underline{X}^{\log}) \to \pi_1(P^{\log})$ is an isomorphism.

(Step 4) In general, $\pi_1((s^0)^{\log})$ is an isomorphism.

We will show that the functor $\text{Két}(P^{\log}) \to \text{Két}(\underline{X}^{\log})$ induced by the morphism $(s^0)^{\log} : \underline{X}^{\log} \to P^{\log}$ is an equivalence. First, we prove that the functor is fully faithful. It is immediate that the functor is faithful (indeed, this follows from the existence of a log geometric point of $P^{\log}$ that factors through $\underline{X}^{\log}$ and the general theory of Galois categories). Thus, it is enough to show that the functor is full. Let $Q_1^{\log} \to P^{\log}$ and $Q_2^{\log} \to P^{\log}$ be ket coverings over $P^{\log}$, and $g^{\log} : Y_1 \stackrel{\text{def}}{=} Q_1^{\log} \times_{P^{\log}} \underline{X}^{\log} \to Y_2 \stackrel{\text{def}}{=} Q_2^{\log} \times_{P^{\log}} \underline{X}^{\log}$ a morphism in $\text{Két}(\underline{X}^{\log})$. Then, by Step 3, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g'^{\log} : Y_1'^{\log} \stackrel{\text{def}}{=} Y_1^{\log} \times_{X^{\log}} X'^{\log} \to Y_2'^{\log} \stackrel{\text{def}}{=} Y_2^{\log} \times_{X^{\log}} X'^{\log}$ over $\underline{X}'^{\log} \stackrel{\text{def}}{=} \underline{X}^{\log} \times_{X^{\log}} X'^{\log}$ obtained as the base-change of $g^{\log}$ by $X'^{\log} \to X^{\log}$ extends to a morphism $\tilde{g}'^{\log} : Q_1'^{\log} \stackrel{\text{def}}{=} Q_1^{\log} \times_{X^{\log}} X'^{\log} \to Q_2'^{\log} \stackrel{\text{def}}{=} Y_2^{\log} \times_{X^{\log}} X'^{\log}$ over $P'^{\log} \stackrel{\text{def}}{=} P^{\log} \times_{X^{\log}} X'^{\log}$. (Indeed, by Step 3, for any geometric point of X, there exists an etale neighborhood $U \to X$ of the geometric point such that if we denote by $U^{\log} \to X^{\log}$ the strict morphism whose underlying morphism of schemes is the morphism $U \to X$, then the base-change of $g^{\log}$ by $U^{\log} \to X^{\log}$ extends to a morphism $Q_1^{\log} \times_{X^{\log}} U^{\log} \to Q_2^{\log} \times_{X^{\log}} U^{\log}$. Thus, if we denote by $X'^{\log}$ the disjoint union of such a $U^{\log}$'s, then $X'^{\log} \to X^{\log}$ satisfies the above condition.) Let us denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1-st (respectively, 2-nd) projection $P'^{\log} \times_{P^{\log}} P'^{\log} \to P'^{\log}$. Now it follows immediately from the fact that the functor $\text{Két}(P'^{\log} \times_{P^{\log}} P'^{\log}) \to \text{Két}(\underline{X}'^{\log} \times_{\underline{X}^{\log}} \underline{X}'^{\log})$ induced by the morphism $\underline{X}'^{\log} \times_{\underline{X}^{\log}} \underline{X}'^{\log} \to P'^{\log} \times_{P^{\log}} P'^{\log}$ determined by $(s^0)^{\log}$

is faithful that the following diagram commutes:

$$\begin{array}{ccc} q_1^{\log *}Q_1'^{\log} & \xrightarrow{q_1^*\tilde{g}'^{\log}} & q_1^{\log *}Q_2'^{\log} \\ \downarrow & & \downarrow \\ q_2^{\log *}Q_1'^{\log} & \xrightarrow{q_2^*\tilde{g}'^{\log}} & q_2^{\log *}Q_2'^{\log}, \end{array}$$

where $q_i^{\log *}$ denotes the pull-back of each object over $P'^{\log}$ to an object over $P'^{\log} \times_{P^{\log}} P'^{\log}$ via $q_i^{\log}$, and the vertical arrows are the isomorphisms that arise from the fact that $Q_i'^{\log} \to P'^{\log}$ arises from $Q_i^{\log} \to P^{\log}$. Thus, by Lemma 5.20 below, $\tilde{g}'^{\log}$ extends to a morphism $\tilde{g}^{\log} : Q_1^{\log} \to Q_2^{\log}$. Since the base-change of $\tilde{g}^{\log}$ by $\underline{X}'^{\log} \to P^{\log}$ is naturally isomorphic to $g'^{\log}$, we obtain that $\tilde{g}^{\log}$ is an extension of $g^{\log}$.

Next, we prove that the functor is essentially surjective. Let $Y^{\log} \to \underline{X}^{\log}$ be a ket covering over $\underline{X}^{\log}$. Then, by Step 3, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the ket covering $Y'^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_{X^{\log}} X'^{\log} \to \underline{X}'^{\log} \stackrel{\mathrm{def}}{=} \underline{X}^{\log} \times_{X^{\log}} X'^{\log}$ extends to a ket covering $Q'^{\log} \to P'^{\log} \stackrel{\mathrm{def}}{=} P^{\log} \times_{X^{\log}} X'^{\log}$. Let us denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1-st (respectively, 2-nd) projection $P'^{\log} \times_{P^{\log}} P'^{\log} \to P'^{\log}$. Now, replacing the strict étale surjection $X'^{\log} \to X^{\log}$ by the composite $X''^{\log} \to X'^{\log} \to X^{\log}$, where $X''^{\log} \to X'^{\log}$ is a strict étale surjection, if necessary, we may assume that the isomorphism over $X'^{\log}$ that arises from the fact that $Y'^{\log} \to X'^{\log}$ arises from $Y^{\log} \to X^{\log}$ extends to an isomorphism $q_1^{\log *}Q'^{\log} \xrightarrow{\sim} q_2^{\log *}Q'^{\log}$, where $q_i^{\log *}$ denotes the pull-back of a ket covering over $P'^{\log}$ to an object over $P'^{\log} \times_{P^{\log}} P'^{\log}$ via $q_i^{\log}$. (It follows from Step 3 and a similar argument to the argument used in the proof that the functor in question is fully faithful [to show the existence of $X'^{\log} \to X^{\log}$] that such a strict étale surjection $X'^{\log} \to X^{\log}$ exists.) Moreover, since the functor $\mathrm{K\acute{e}t}(P'^{\log} \times_{P^{\log}} P'^{\log} \times_{P^{\log}} P'^{\log}) \to \mathrm{K\acute{e}t}(X'^{\log} \times_{X^{\log}} X'^{\log} \times_{X^{\log}} X'^{\log})$ induced by the morphism $X'^{\log} \times_{X^{\log}} X'^{\log} \times_{X^{\log}} X'^{\log} \to P'^{\log} \times_{P^{\log}} P'^{\log} \times_{P^{\log}} P'^{\log}$ determined by $(s^0)^{\log}$ is faithful, this isomorphism $q_1^{\log *}Q'^{\log} \xrightarrow{\sim} q_2^{\log *}Q'^{\log}$ satisfies the cocycle condition for being a descent datum. Thus, by Lemma 5.20 below, the ket covering $Q'^{\log} \to P'^{\log}$ extends to a ket covering $Q^{\log} \to P^{\log}$. Moreover, since $Q^{\log} \times_{P^{\log}} \underline{X}^{\log} \times_{\underline{X}^{\log}} \underline{X}'^{\log}$ equipped with descent data with respect to $X'^{\log} \to X^{\log}$ is naturally isomorphic to $Y'^{\log}$ equipped with descent data with respect to $X'^{\log} \to X^{\log}$, we obtain that $Q^{\log} \times_{P^{\log}} \underline{X}^{\log}$ is naturally ismorphic to $Y^{\log}$ over $X^{\log}$. □

Lemma 5.19. *Let k be a separably closed field whose (not necessarily positive) characteristic we denote by p, $(\mathbb{P}_k^1)^{\log}$ the log scheme obtaind by equipping the projective line $\mathbb{P}_k^1$ with the log structure associated to the divisor*

$\{0, \infty\} \subseteq \mathbb{P}^1_k$, $U \subseteq \mathbb{P}^1_k$ *the interior of* $(\mathbb{P}^1_k)^{\log}$ *(so* $U = \mathbb{G}_m$*), and* $(\operatorname{Spec} k)^{\log} \to (\mathbb{P}^1_k)^{\log}$ *the strict morphism for which the image of the underlying morphism of schemes is* $\{0\} \subseteq \mathbb{P}^1_k$*. Then the following hold:*

(*i*) *The morphism* $\pi_1(U) \to \pi_1((\mathbb{P}^1_k)^{\log})$ *is an isomorphism.*

(*ii*) *The morphism* $\pi_1((\operatorname{Spec} k)^{\log}) \to \pi_1((\mathbb{P}^1_k)^{\log})$ *is an isomorphism.*

Proof. First, we prove assertion (i). If we denote by η the geometric point of $\mathbb{P}^1_k$, then it follows from the fact that the natural morphism $\eta \to (\mathbb{P}^1_k)^{\log}$ induces a surjection $\pi_1(\eta) \to \pi_1((\mathbb{P}^1_k)^{\log})$, together with the fact that the natural morphism $\eta \to (\mathbb{P}^1_k)^{\log}$ factors through U that $\pi_1(U) \to \pi_1((\mathbb{P}^1_k)^{\log})$ is surjective. Moreover, since any connected finite étale covering over U is of the form

$$\begin{array}{ccccccc} U & = & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & = & U \\ & & f & \mapsto & f^n & & \end{array}$$

for some positive integer n that is prime to p, it is easily seen that any finite étale covering over U extends to a ket covering over $(\mathbb{P}^1_k)^{\log}$; thus, $\pi_1(U) \to \pi_1((\mathbb{P}^1_k)^{\log})$ is injective. Therefore, $\pi_1(U) \to \pi_1((\mathbb{P}^1_k)^{\log})$ is an isomoprhism.

Next, we prove assertion (ii). We denote by $(\mathbb{A}^1_k)^{\log} \to (\mathbb{P}^1_k)^{\log}$ the strict morphism whose underlying morphism of schemes is the natural open immersion $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ (where we regard $\mathbb{A}^1_k$ as $\mathbb{P}^1_k \setminus \{\infty\}$). By (i), the restriction to $(\mathbb{A}^1_k)^{\log}$ of any connected ket covering over $(\mathbb{P}^1_k)^{\log}$ is of the form $X^{\log} \xrightarrow{\sim} (\mathbb{A}^1_k)^{\log} \to (\mathbb{A}^1_k)^{\log}$, where $X^{\log}$ is the log scheme obtained by equipping $\mathbb{A}^1_k$ with the log structure associated to the divisor $\{0\} \subseteq \mathbb{A}^1_k$, and the underlying morphism of schemes of this ket covering $X^{\log} \to (\mathbb{A}^1_k)^{\log}$ is determined by the morphism

$$\begin{array}{ccc} k[t] & \longrightarrow & k[t] \\ t & \mapsto & t^n \end{array}$$

fot some positive integer n that is prime to p. It thus follows immediately from this fact and Proposition A.4 that $\pi_1((\operatorname{Spec} k)^{\log}) \to \pi_1((\mathbb{P}^1_k)^{\log})$ is an isomorphism. $\square$

Lemma 5.20. *Let* $X^{\log}$ *be a fs log scheme, and* $f^{\log} : Y^{\log} \to X^{\log}$ *a strict étale surjection. Then* $f^{\log}$ *induces a natural equivalence between the category of ket coverings of* $X^{\log}$ *and the category of ket coverings of* $Y^{\log}$ *equipped with descent data with respect to* $f^{\log}$*.*

Proof. This follows immediately from the fact that the property of being a ket covering is étale local, together with [23], Proposition 4.4. $\square$

The following corollary follows immediately from Theorem 3.3 and 5.18.

Corollary 5.21. *Let $X^{\log}$ be a log regular, connected, quasi-compact fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbb{N}^{\oplus n}$, $\tau : \mathbb{N}_X^{\oplus n} \xrightarrow{\sim} \mathcal{C}_{f^{\log}}$ a trivialization of $f^{\log}$, and $\pi^{\log} : P^{\log} \to X^{\log}$ the log $\mathbb{G}_m^{\times n}$-torsor associated to $(f^{\log}, \tau)$. Then for any strict geometric point $\overline{x}^{\log} \to X^{\log}$ of $X^{\log}$, the following sequence is exact:*

$$\varprojlim \pi_1(\underline{X}^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \xrightarrow{s} \pi_1(\underline{X}^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1\,.$$

Here the projective limit is over all reduced covering points $\overline{x}_\lambda^{\log} \to \overline{x}^{\log}$, and s is induced by the natural projections $\underline{X}^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log} \to \underline{X}^{\log}$. In, particular, by means of a natural isomorphism

$$\varprojlim \pi_1(\underline{X}^{\log} \times_{X^{\log}} \overline{x}_\lambda^{\log}) \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}$$

obtained in Remark 5.22 below, we obtain the following exact sequence:

$$\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n} \longrightarrow \pi_1(\underline{X}^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1\,,$$

where p is the characteristic of the residue field of the image of the underlying schemes of the strict geometric point, and $\hat{\mathbb{Z}}^{(p')}(1)$ is the pro-prime to p quotient of $\hat{\mathbb{Z}}(1)$.

Remark 5.22. Let k be a separably closed field whose (not necessarily positive) characteristic we denote by p, and $S^{\log}$ an fs log scheme whose underlying scheme S is the spectrum of k. Let $f^{\log} : \underline{S}^{\log} \to S^{\log}$ be a morphism of constant type $\mathbb{N}^{\oplus n}$, and τ a trivialization of $f^{\log}$.

Let $P \to k$, $Q \to k$ be respective clean charts of $S^{\log}$, $\underline{S}^{\log}$ given in Definition 5.1. Then, as is well-knouwn, the log fundamental group $\pi_1(S^{\log})$ (respectively, $\pi_1(\underline{S}^{\log})$) is naturally isomorphic to $\mathrm{Hom}(P^{\mathrm{gp}}, \hat{\mathbb{Z}}^{(p')}(1))$ (respectively, $\mathrm{Hom}(Q^{\mathrm{gp}}, \hat{\mathbb{Z}}^{(p')}(1))$), where $\hat{\mathbb{Z}}^{(p')}(1)$ is the pro-prime to p quotient of $\hat{\mathbb{Z}}(1)$ (cf. e.g., [8], Example 4.7). Moreover, the morphism $\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})$ induced by $f^{\log}$ is the morphism

$$\mathrm{Hom}(Q^{\mathrm{gp}}, \hat{\mathbb{Z}}^{(p')}(1)) \longrightarrow \mathrm{Hom}(P^{\mathrm{gp}}, \hat{\mathbb{Z}}^{(p')}(1))$$

induced by $P \to Q$ in Definition 5.1. In particular, the kernel of $\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})$ is naturally isomorphic to $\mathrm{Hom}(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, \hat{\mathbb{Z}}^{(p')}(1))$. Now the trivialization τ induces a natural isomorphism $\mathbb{Z}^{\oplus n} \xrightarrow{\sim} Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. Therefore, we obtain a *natural isomorphism*

$$(\varprojlim \pi_1(\underline{S}^{\log} \times_{S^{\log}} S_\lambda^{\log}) \xrightarrow{\sim})\mathrm{Ker}(\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})) \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}\,,$$

where the projective limit is over all reduced covering points $S_\lambda^{\log} \to S^{\log}$.

Proposition 5.23. *Let $X^{\log}$ be a log regular, connected, quasi-compact fs log scheme over a field k whose (not necessarily positive) characteristic we denote by p, $U_X \subseteq X$ the interior of $X^{\log}$, and $\mathcal{L}_1, \cdots, \mathcal{L}_n$ invertible sheaves on X. Let $\pi^{\log} : P^{\log} \to X^{\log}$ be the log $\mathbb{G}_m^{\times n}$-torsor associated to $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$. If the condition $(*)$ below is satisfied, then, in the following exact sequence obtained in Corollary 5.21*

$$(\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n} \simeq)\ \pi_1(P^{\log} \times_{X^{\log}} \overline{x}) \longrightarrow \pi_1(P^{\log}) \overset{\pi_1((s^0)^{\log})}{\longrightarrow} \pi_1(X^{\log}) \longrightarrow 1$$

(where $\overline{x} \to X$ is a geometric point of X), the first morphism is injective.

$(*)$ *For any integer i such that $1 \leq i \leq n$ and any positive integer N that is prime to p, there exists a covering $V \to U_X$ tamely ramified along $X \setminus U_X$ and an invertible sheaf $\mathcal{N}$ such that $\mathcal{N}^{\otimes N} \overset{\sim}{\to} \mathcal{L}_i\,|_V$.*

Proof. If we denote by $P_i^{\log} \to X^{\log}$ the log $\mathbb{G}_m$-torsor associated to $\mathcal{L}_i$ $(1 \leq i \leq n)$, then there exists a natural isomorphism $P^{\log} \overset{\sim}{\to} P_1^{\log} \times_{X^{\log}} \cdots \times_{X^{\log}} P_n^{\log}$ over $X^{\log}$. Thus, if the assertion in the case where $n = 1$ is verified, then the composite

$$\begin{array}{ccccc} \pi_1(P_i^{\log} \times_{X^{\log}} \overline{x}) & \longrightarrow & \prod_{k=1}^n \pi_1(P_k^{\log} \times_{X^{\log}} \overline{x}) & \overset{\prod_{k=1}^n \mathrm{pr}_k}{\overset{\sim}{\longleftarrow}} & \pi_1(P^{\log} \times_{X^{\log}} \overline{x}) \\ e & \mapsto & (0, \cdots, \overset{i\text{-th}}{e}, \cdots, 0) & & \\ \longrightarrow & \pi_1(P^{\log}) & \overset{\pi_1(\mathrm{pr}_j)}{\longrightarrow} & \pi_1(P_j^{\log}) & \end{array}$$

is injective (respectively, zero) if $i = j$ (if $i \neq j$). Therefore, to complete the proof of Proposition 5.23, we may assume that $n = 1$. Write $\mathcal{L} \overset{\text{def}}{=} \mathcal{L}_1$. Let N be a positive integer that is prime to p. Note that it is enough to show that the N-th (cyclic) ket covering over $P^{\log} \times_{X^{\log}} \overline{x}$ lifts to a ket covering $Q^{\log} \to P^{\log}$ over $P^{\log}$ to complete the proof of Proposition 5.23.

We denote by $Q_V^{\log} \to V$ the log $\mathbb{G}_m$-torsor associated to $\mathcal{N}$ (in the condition $(*)$), and by $Q_V \to P \times_X V$ the morphism determined by the following composite:

$$\begin{array}{ccccc} \mathcal{N} & \longrightarrow & \mathcal{N}^{\otimes N} & \overset{\sim}{\longrightarrow} & \mathcal{L}\,|_V \\ f & \mapsto & f^{\otimes N}. & & \end{array}$$

Then it follows from the definition of a log $\mathbb{G}_m$-torsor associated to an invertible sheaf that the morphism $Q_V \to P \times_X V$ extends to a morphism of log schemes $Q_V^{\log} \to P^{\log} \times_{X^{\log}} V$; thus, we obtain the following commutative diagram:

$$\begin{tikzcd}
Q_V^{\log} \times_{P^{\log}} U_P \arrow[rrr] \arrow[d] & & & U_P \arrow[d] & \\
Q_V^{\log} \arrow[rr] & & P^{\log} \times_{X^{\log}} V \arrow[r] \arrow[d] & P^{\log} \times_{X^{\log}} U_X \arrow[r] \arrow[d] & P^{\log} \arrow[d] \\
& & V \arrow[r] & U_X \arrow[r] & X^{\log},
\end{tikzcd}$$

where U_P is the interior of $P^{\log}$, and the three squares are cartesian. It follows immediately from the construction of $Q_V^{\log}$ that the log structure of $Q_V^{\log} \times_{P^{\log}} U_P$ is trivial, and that the top horizontal arrow $Q_V^{\log} \times_{P^{\log}} U_P = Q_V \times_P U_P \to U_P$ is finite étale.

Now I claim the normalization Q of U_P in $Q_V \times_P U_P$ is tamely ramified over P along $P \setminus U_P$. Indeed, this claim may be verified follows: Now every point a of $P \setminus U_P$ with $\dim \mathcal{O}_{P,a} = 1$ is either

(i) the generic point of a (reduced) divisor on P determined by s^0 or s^∞ (see Definition 5.16), or

(ii) the generic point of a (reduced) divisor on P which is the pull-back of a reduced divisor on X whose generic point x is a point of $X \setminus U_X$ with $\dim \mathcal{O}_{X,x} = 1$.

If a is the generic point of a (reduced) divisor on P determined by s^0 or s^∞ (i.e., of type (i)), then $\mathcal{O}_{P,a}$ is isomorphic to $K_X[t]_{(t)}$ (where K_X is the function field of X), and the base-change of $Q \to P$ by the natural morphism $(\mathrm{Spec}\, K_X[t]_{(t)} \xrightarrow{\sim})\mathrm{Spec}\, \mathcal{O}_{P,a} \to P$ is of the form $\mathrm{Spec}\, K_V[t^{1/N}]_{(t^{1/N})} \to \mathrm{Spec}\, K_X[t]_{(t)}$ (where K_V is the function field of V). Since K_V is a finite separable extension of K_X (by the tameness of $V \to U_X$), and N is prime to the characteristic of K_X (by the fact that N is prime to p whenever p is positive), we obtain that $Q \to P$ is tamely ramified at p. On the other hand, if a is the generic point of a (reduced) divisor on P which is the pull-back of a reduced divisor on X whose generic point x is a point of $X \setminus U_X$ with $\dim \mathcal{O}_{X,x} = 1$ (i.e., of type (ii)), then $\mathcal{O}_{P,a}$ is isomorphic to the localization $\mathcal{O}_{X,x}[t]_{(\pi)}$ of $\mathcal{O}_{X,x}[t]$ at the prime ideal (π) generated by a prime element π of the discrete valuation ring $\mathcal{O}_{X,x}$, and the base-change of $Q \to P$ by the natural morphism $(\mathrm{Spec}\,(\mathcal{O}_{X,x}[t]_{(\pi)}) \xrightarrow{\sim})\mathrm{Spec}\, \mathcal{O}_{P,a} \to P$ is of the form $\mathrm{Spec}\,(R[t^{1/N}]_{(\pi_R)}) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}[t]_{(\pi)})$ (where R is the normalization of $\mathcal{O}_{X,x}$ in K_V, and (π_R) is the prime ideal generated by a prime element π_R of the discrete valuation ring R). Since $\mathrm{Spec}\, R \to \mathrm{Spec}\, \mathcal{O}_{X,x}$ is tamely ramified (by the tameness of $V \to U_X$), and N is prime to the characteristic of K_X (by the fact that N is prime to p whenever p is positive), we obtain that $Q \to P$ is tamely ramified at a. This completes the proof that the normalization Q of U_P in $Q_V \times_P U_P$ is tamely ramified over P along $P \setminus U_P$.

Therefore, by the log purity theorem (cf. Remark 2.10), the covering extends to a ket covering $Q^{\log} \to P^{\log}$. Moreover, by the construction of the morphism $Q_V \to P \times_X V$, for any geometric point $\overline{x} \to X$ of X whose image lies on U_X, the restriction of the ket covering $Q^{\log} \times_{X^{\log}} \overline{x} \to P^{\log} \times_{X^{\log}} \overline{x}$ to any of the connected components of $Q^{\log} \times_{X^{\log}} \overline{x}$ is the N-th (cyclic) covering over $P^{\log} \times_{X^{\log}} \overline{x}$. $\square$

Definition 5.24. In the notation of Proposition 5.23, we shall refer to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}$

$$1 \longrightarrow \pi_1(P^{\log} \times_{X^{\log}} \overline{x}) \longrightarrow \pi_1(P^{\log}) \xrightarrow{\pi_1((s^0)^{\log})} \pi_1(X^{\log}) \longrightarrow 1$$

as *the extension of* $\pi_1(X^{\log})$ *by* $\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}$ *associated to* $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$. More generally, for a set of prime numbers Σ which does not contain p, we shall refer to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(\Sigma)}(1)^{\oplus n}$

$$1 \longrightarrow \pi_1(P^{\log} \times_{X^{\log}} \overline{x})/N \longrightarrow \pi_1(P^{\log})/N \xrightarrow{\text{via } \pi_1((s^0)^{\log})} \pi_1(X^{\log}) \longrightarrow 1$$

(where N is the kernel of the composite of the natural isomorphism $\pi_1(P^{\log} \times_{X^{\log}} \overline{x}) \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}$ and the surjection $\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n} \to \hat{\mathbb{Z}}^{(\Sigma)}(1)^{\oplus n}$ induced by the natural projection $\hat{\mathbb{Z}}^{(p')}(1) \to \hat{\mathbb{Z}}^{(\Sigma)}(1)$) naturally obtained from the extension of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(p')}(1)^{\oplus n}$ associated to $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$ as *the extension of* $\pi_1(X^{\log})$ *by* $\hat{\mathbb{Z}}^{(\Sigma)}(1)^{\oplus n}$ *associated to* $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$.

Remark 5.25. If we denote by $\mathcal{S}(\pi_1(U_X))$ (respectively, $U_{X\text{ét}}$) the classifying site of $\pi_1(U_X)$, (i.e., the site defined by considering the category of finite sets equipped with a continuous action of $\pi_1(U_X)$ [and coverings given by surjections of such sets]) (respectively, the étale site of U_X), then the natural morphism of sites

$$U_{X\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U_X))$$

induces a natural morphism

$$\mathrm{H}^n(\pi_1(U_X), \hat{\mathbb{Z}}^{(p')}(1)) \longrightarrow \mathrm{H}^n_{\text{ét}}(U_X, \hat{\mathbb{Z}}^{(p')}(1)).$$

If the morphism $\mathrm{H}^2(\pi_1(U_X), \hat{\mathbb{Z}}^{(p')}(1)) \to \mathrm{H}^2_{\text{ét}}(U_X, \hat{\mathbb{Z}}^{(p')}(1))$ is an *isomorphism*, then, by a similar argument to the argument used in the proof of [15], Lemma 4.3, any invertible sheaf on X satisfies the condition $(*)$ in Proposition 5.23. Moreover, if the morphism

$$\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbb{Z}}^{(p')}(1)) \longrightarrow \mathrm{H}^2(\pi_1(U_X), \hat{\mathbb{Z}}^{(p')}(1))$$

induced by the natural surjection $\pi_1(U_X) \to \pi_1(X^{\log})$ is an *isomorphism*, then, by a similar argument to the argument used in the proof of [15], Lemma 4.4, the extension of $\pi_1(X^{\log})$ associated to $\mathcal{L}$ is isomorphic to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(p')}(1)$ determined by the (étale-theoretic) first Chern class (see [15], Definition 4.1.) of the invertible sheaf $\mathcal{L}$ via the isomorphisms

$$\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbb{Z}}^{(p')}(1)) \xrightarrow{\sim} \mathrm{H}^2(\pi_1(U_X), \hat{\mathbb{Z}}^{(p')}(1)) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(U_X, \hat{\mathbb{Z}}^{(p')}(1))\,.$$

(Now, by means of the natural bijection in [19], Theorem 1.2.5, we identify the set of equivalence classes of extensions of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(p')}(1)$ with $\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbb{Z}}^{(p')}(1))$.) Moreover, then the extension of $\pi_1(X^{\log})$ associated to $(\mathcal{L}_1, \cdots, \mathcal{L}_n)$. is isomorphic to the fiber product of the extensions of $\pi_1(X^{\log})$ by $\hat{\mathbb{Z}}^{(p')}(1)$ determined by the (étale-theoretic) first Chern classes of the invertible sheaves $\mathcal{L}_i$ $(1 \leq i \leq n)$ over $\pi_1(X^{\log})$.

6 Log configuration schemes

In this section, we define the *log configuration scheme* of a curve over a field and consider the geometry of such log configuration schemes.

Throughout this section, we shall denote by X a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a field K whose (not necessarily positive) characteristic we denote by p, $\mathbb{P}^{\log}_K$ the log scheme obtained by equipping $\mathbb{P}^1_K$ with the log structure associated to the divisor $\{0, 1, \infty\} \subseteq \mathbb{P}^1_K$, and by $U_{\mathbb{P}}$ the interior of $\mathbb{P}^{\log}_K$.

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of r-pointed stable curves of genus g whose r sections are equipped with an ordering, and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing smooth curves ([12]). Then $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,r}$ ([12], Theorem 2.7). Let us write $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{g,0}$ and $\mathcal{M}_g = \mathcal{M}_{g,0}$. By considering the (1-)morphism $p^{\mathcal{M}}_{(r)r+1} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ obtained by forgetting the $(r+1)$-st section, we obtain a natural isomorphism of $\overline{\mathcal{M}}_{g,r+1}$ with the universal r-pointed stable curve over $\overline{\mathcal{M}}_{g,r}$ ([12], Corollary 2.6). Now we have a natural action of $\mathcal{S}_r$ (where $\mathcal{S}_r$ is the symmetric group on r letters) on $\overline{\mathcal{M}}_{g,r}$ which is given by permuting the sections. For $1 \leq i \leq r$, we shall denote by $p^{\mathcal{M}}_{(r)i} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ the (1-)morphism obtained by forgetting the i-th section.

Let us denote by $\overline{\mathcal{M}}^{\log}_{g,r}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$. Since the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ preserves the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$, the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ extends to an action on $\overline{\mathcal{M}}^{\log}_{g,r}$.

First, we define the *log configuration scheme* $X^{\log}_{(r)}$ as follows:

Definition 6.1. We define $X_{(r)}$ by the following (1-)commutative diagram

$$\begin{array}{ccc} X_{(r)} & \longrightarrow & \overline{\mathcal{M}}_{g,r} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{[X]} & \overline{\mathcal{M}}_{g}\,, \end{array}$$

where the bottom horizontal arrow $\operatorname{Spec} K \stackrel{[X]}{\to} \overline{\mathcal{M}}_g$ is the classifying (1-)morphism determined by the curve $X \to \operatorname{Spec} K$, the right-hand vertical arrow $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ the (1-)morphism obtained by forgetting the sections, and the (1-)commutative diagram is cartesian in the (2-)category of stacks. Since $\overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_g$ is representable, $X_{(r)}$ is a scheme. We shall denote by $X_{(r)}^{\log}$ the fs log scheme obtained by equipping $X_{(r)}$ with the log structure induced by the log structure of $\overline{\mathcal{M}}_{g,r}^{\log}$. We shall denote by $U_{X_{(r)}}$ the interior of $X_{(r)}^{\log}$, and by $D_{X_{(r)}}$ the complement of $U_{X_{(r)}}$ of $X_{(r)}$. Note that, by definition, the scheme $U_{X_{(r)}}$ is isomorphic to the usual r-th configuration space of X. For simplicity, we shall write $U_{(r)}$ (respectively, $D_{(r)}$) instead of $U_{X_{(r)}}$ (respectively, $D_{X_{(r)}}$) when there is no danger of confusion. By the definition of $X_{(r)}$ (respectively, $X_{(r)}^{\log}$), the action of $\mathcal{S}_r$ on $\overline{\mathcal{M}}_{g,r}$ (respectively, on $\overline{\mathcal{M}}_{g,r}^{\log}$) induces an action on $X_{(r)}$ (respectively, on $X_{(r)}^{\log}$).

As is well-known, the pull-back of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ via the (1-)morphism $p_{(r)r+1}^{\mathcal{M}} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ is a subdivisor of the divisor $\overline{\mathcal{M}}_{g,r+1} \setminus \mathcal{M}_{g,r+1}$ (cf. [12], the proof of Theorem 2.7). Thus, there exists a unique (1-)morphism $p_{(r)r+1}^{\mathcal{M}\log} : \overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ whose underlying morphism is the (1-)morphism $p_{(r)r+1}^{\mathcal{M}}$. Moreover, for an integer $1 \leq i \leq r$, since the composite of the automorphism of $\overline{\mathcal{M}}_{g,r}$ determined by the action of

$$(1, 2, \cdots, r) \mapsto (1, 2, \cdots, i-1, i+1, i+2, \cdots, r, i) \in \mathcal{S}_r$$

and $p_{(r)r+1}^{\mathcal{M}}$ coincides with the (1-)morphism $p_{(r)i}^{\mathcal{M}}$, the (1-)morphism $p_{(r)i}^{\mathcal{M}}$ also extends to a (1-)morphism $\overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$. We shall denote this (1-)morphism by $p_{(r)i}^{\mathcal{M}\log}$.

The (1-)morphism $p_{(r)i}^{\mathcal{M}} : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ (respectively, $p_{(r)i}^{\mathcal{M}\log} : \overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$) determines a morphism $X_{(r+1)} \to X_{(r)}$ (respectively, $X_{(r+1)}^{\log} \to X_{(r)}^{\log}$). We denote this morphism by $p_{X_{(r)}i}$ (respectively, $p_{X_{(r)}i}^{\log}$). Thus, we obtain the following (1-)cartesian diagrams:

$$\begin{array}{ccc} X_{(r+1)} & \xrightarrow{p_{X_{(r)}i}} & X_{(r)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,r+1} & \xrightarrow{p^{\mathcal{M}}_{(r)i}} & \overline{\mathcal{M}}_{g,r} \end{array} \qquad \begin{array}{ccc} X^{\log}_{(r+1)} & \xrightarrow{p^{\log}_{X_{(r)}i}} & X^{\log}_{(r)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^{\log}_{g,r+1} & \xrightarrow{p^{\mathcal{M}\log}_{(r)i}} & \overline{\mathcal{M}}^{\log}_{g,r} \, . \end{array}$$

Note that, by the definition of a stable curve, $p_{X_{(r)}i}$ is proper, flat, geometrically connected, and geometrically reduced. For simplicity, we shall write $p_{(r)i}$ (respectively, $p^{\log}_{(r)i}$) instead of $p_{X_{(r)}i}$ (respectively, $p^{\log}_{X_{(r)}i}$) when there is no danger of confusion.

Definition 6.2.

(i) Let $1 \leq i \leq r$ be integers. Then we shall denote by

$$\mathrm{pr}^{\log}_{X_{(r)}i} : X^{\log}_{(r)} \longrightarrow X$$

the composite

$$p^{\log}_{X_{(1)}2} \circ p^{\log}_{X_{(2)}2} \circ \cdots \circ p^{\log}_{X_{(r-i-1)}2} \circ p^{\log}_{X_{(r-i)}2} \circ p^{\log}_{X_{(r-i+1)}1} \circ \cdots \circ p^{\log}_{X_{(r-2)}1} \circ p^{\log}_{X_{(r-1)}1} \, ,$$

and by $\mathrm{pr}_{X_{(r)}i}$ the underlying morphism of schemes of $\mathrm{pr}^{\log}_{X_{(r)}i}$. For simplicity, we shall write $\mathrm{pr}^{\log}_{(r)i}$ (respectively, $\mathrm{pr}_{(r)i}$) instead of $\mathrm{pr}^{\log}_{X_{(r)}i}$ (respectively, $\mathrm{pr}_{X_{(r)}i}$) when there is no danger of confusion.

(ii) Let $1 \leq i \leq j \leq r$ be integers. Then we shall denote by

$$\mathrm{pr}^{\log}_{X_{(r)}i,j} : X^{\log}_{(r)} \longrightarrow X^{\log}_{(2)}$$

the composite

$$p^{\log}_{X_{(2)}3} \circ p^{\log}_{X_{(3)}3} \circ \cdots \circ p^{\log}_{X_{(r-j)}3} \circ p^{\log}_{X_{(r-j+1)}3} \circ p^{\log}_{X_{(r-j+2)}2} \circ \cdots$$

$$\cdots \circ p^{\log}_{X_{(r-i-1)}2} \circ p^{\log}_{X_{(r-i)}2} \circ p^{\log}_{X_{(r-i+1)}1} \circ \cdots \circ p^{\log}_{X_{(r-2)}1} \circ p^{\log}_{X_{(r-1)}1} \, ,$$

and by $\mathrm{pr}_{X_{(r)}i,j}$ the underlying morphism of schemes of $\mathrm{pr}^{\log}_{X_{(r)}i,j}$. For simplicity, we shall write $\mathrm{pr}^{\log}_{(r)i,j}$ (respectively, $\mathrm{pr}_{(r)i,j}$) instead of $\mathrm{pr}^{\log}_{X_{(r)}i,j}$ (respectively, $\mathrm{pr}_{X_{(r)}i,j}$) when there is no danger of confusion.

Next, let us consider the scheme-theoretic and log scheme-theoretic properties of $X^{\log}_{(r)}$ in more detail.

Lemma 6.3. *$X_{(r)}$ is connected.*

Proof. Since $X_{(0)} = \operatorname{Spec} K$ is connected, and the $p_{(r)i}$'s are proper and geometrically connected, it follows immediately that $X_{(r)}$ is connected. $\square$

Proposition 6.4. *The* (1-)*morphism $p^{\mathcal{M}\log}_{(r)r+1} : \overline{\mathcal{M}}^{\log}_{g,r+1} \to \overline{\mathcal{M}}^{\log}_{g,r}$ is log smooth.*

Proof. See [9], Section 4. $\square$

The following lemma follows immediately from Proposition 6.4.

Lemma 6.5. *$p^{\log}_{(r)i}$ is log smooth. In particular, since* $\operatorname{Spec} K$ (*equipped with the trivial log structure*) *is log regular, $X^{\log}_{(r)}$ is log regular.*

Proof. The assertion for $p^{\log}_{(r)r+1}$ follows immediately from Proposition 6.4. Since $p^{\log}_{(r)i}$ is a composite of an automorphism of $X^{\log}_{(r)}$ (obtained by permuting of the sections) and $p^{\log}_{(r)r+1}$, $p^{\log}_{(r)i}$ is also log smooth. $\square$

Remark 6.6. By Lemmas 6.3, 6.5 and Proposition A.10, $U_{(r)} \hookrightarrow X^{\log}_{(r)}$ induces a natural equivalence between the Galois category of ket coverings over $X^{\log}_{(r)}$ and the Galois category of coverings over $U_{(r)}$ tamely ramified along the divisor with normal crossings $D_{(r)} \subseteq X_{(r)}$. In particular, $\pi_1^{\text{tame}}(X_{(r)}, D_{(r)}) \simeq \pi_1(X^{\log}_{(r)})$. (Concerning $\pi_1^{\text{tame}}(X_{(r)}, D_{(r)})$, see [6], Corollary 2.4.4.)

Proposition 6.7. *Let $\overline{x}^{\log} \to X^{\log}_{(r)}$ be a strict geometric point. Then, for any integer $1 \leq i \leq r+1$, the following sequence is exact:*

$$\varprojlim \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda}) \xrightarrow{s} \pi_1(X^{\log}_{(r+1)}) \xrightarrow{\pi_1(p^{\log}_{(r)i})} \pi_1(X^{\log}_{(r)}) \longrightarrow 1\,.$$

Here, the projective limit is over all reduced covering points $\overline{x}^{\log}_{\lambda} \to \overline{x}^{\log}$, and s is induced by the natural morphism $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda} \to X^{\log}_{(r+1)}$.

Proof. This follows immediately from Lemma 6.3, 6.5 and Theorem 3.3. $\square$

Proposition 6.8. *Let $S^{\log}$ be a log regular fs log scheme, and $\overline{s} \to S$ a geometric point of S. If the characteristic $(\mathcal{M}_S/\mathcal{O}^*_S)_{\overline{s}}$ of $S^{\log}$ at $\overline{s} \to S$ is isomorphic to $\mathbb{N}^{\oplus n}$ for some n, then S is regular at the image of $\overline{s} \to S$, and the log structure of $S^{\log}$ is given by a divisor with normal crossings around the image of $\overline{s} \to S$.*

Proof. We take a clean chart $\alpha : \mathbb{N}^{\oplus n} \to \mathcal{O}_{S,\overline{s}}$ of $S^{\log}$ at $\overline{s} \to S$, and write $\alpha(e_i) = f_i \in \mathcal{O}_{S,\overline{s}}$ (where $e_i = (0, \cdots, 0, \overset{\text{i-th}}{1}, 0, \cdots, 0) \in \mathbb{N}^{\oplus n}$). Then, by the definition of log regularity, the following is satisfied:

(i) $\mathcal{O}_{S,\overline{s}}/(f_1, \ldots, f_n)$ is regular.

(ii) $(d \stackrel{\mathrm{def}}{=})\dim \mathcal{O}_{S,\overline{s}} = \dim (\mathcal{O}_{S,\overline{s}}/(f_1, \ldots, f_n)) + n$.

Thus, there exist elements $f_{n+1}, \ldots, f_d$ of $\mathcal{O}_{S,\overline{s}}$ such that $f_1, \ldots, f_d$ generate the maximal ideal of $\mathcal{O}_{S,\overline{s}}$. Therefore, $\mathcal{O}_{S,\overline{s}}$ is regular, and the log structure of $S^{\log}$ is given by the divisor with normal crossings defined by $f_1, \cdots, f_n$. $\square$

Lemma 6.9. *$X_{(r)}$ is regular, and the log structure of $X^{\log}$ is given by a divisor with normal crossings.*

Proof. Since the natural morphism $X_{(r)}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ is strict, for any geometric point $\overline{x} \to X_{(r)}$, the characteristic $(\mathcal{M}_{X_{(r)}}/\mathcal{O}^*_{X_{(r)}})_{\overline{x}}$ of $X_{(r)}^{\log}$ at $\overline{x} \to X_{(r)}$ is isomorphic to $\mathbb{N}^{\oplus n}$ for some n. Thus, the assertion follows immediately from Proposition 6.8. $\square$

Definition 6.10. Let $r \geq 2$ be a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^\# \geq 2$ equipped with an ordering. Then we shall denote by

$$(C_{(r)I} \longrightarrow X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1};\ s_1, \cdots, s_r : X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1} \longrightarrow C_{(r)I})$$

the r-pointed stable curve of genus g whose r sections are equipped with an ordering obtained by applying the clutching (1-)morphism ([12], Definition 3.8)

$$\beta_{0,g,I,\{1,2,\cdots,r\}\setminus I} : \overline{\mathcal{M}}_{0,I^\#+1} \times \overline{\mathcal{M}}_{g,r-I^\#+1} \to \overline{\mathcal{M}}_{g,r}$$

(where $\{1, 2, \cdots, r\} \setminus I$ is equipped with the natural ordering) to the $(I^\#+1)$-pointed stable curve of genus 0

$$X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+2} \longrightarrow X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,I^\#+2} \to \overline{\mathcal{M}}_{0,I^\#+1}$ over $\overline{\mathcal{M}}_{0,I^\#+1}$ and the $(r - I^\# + 1)$-pointed stable curve of genus g

$$X_{(r-I^\#+2)} \times_K \overline{\mathcal{M}}_{0,I^\#+1} \longrightarrow X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1}$$

obtained by base-changing $X_{(r-I^\#+2)} \stackrel{p_{X_{(r-I^\#+1)}}{}^{r-I^\#+2}}{\to} X_{(r-I^\#+1)}$. (Note that "the clutching locus" of

$$X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+2} \longrightarrow X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1}$$

$$[\text{respectively, } X_{(r-I^\#+2)} \times_K \overline{\mathcal{M}}_{0,I^\#+1} \longrightarrow X_{(r-I^\#+1)} \times_K \overline{\mathcal{M}}_{0,I^\#+1}]$$

is the $(I^{\#}+1)$-st [respectively, $(r-I^{\#}+1)$-st] section [cf. [12], Definition 3.8].)

Then it is immediate that the classifying (1-)morphism $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to \overline{\mathcal{M}}_{g,r}$ of this curve factors through $X_{(r)}$, and this morphism $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to X_{(r)}$ is a closed immersion (since it is a proper monomorphism). We shall denote by $\delta_{X_{(r)}I}$ this closed immersion, by $D_{X_{(r)}I}$ the scheme-theoretic image of $\delta_{X_{(r)}I}$, by $D^{\log}_{X_{(r)}I}$ the log scheme obtained by equipping $D_{X_{(r)}I}$ with the log structure induced by the log structure of $X^{\log}_{(r)}$ and by $\delta^{\log}_{X_{(r)}I} : D^{\log}_{X_{(r)}I} \to X^{\log}_{(r)}$ the strict closed immersion whose underlying morphism is $\delta_{X_{(r)}I}$. Note that by the construction of $D_{X_{(r)}I}$, the closed subscheme $D_{X_{(r)}I} \subseteq X_{(r)}$ does not depend on the imposed ordering of I. For simplicity, we shall write $D_{(r)I}$ (respectively, $D^{\log}_{(r)I}$; respectively, $\delta_{(r)I}$; respectively, $\delta^{\log}_{(r)I}$) instead of $D_{X_{(r)}I}$ (respectively, $D^{\log}_{X_{(r)}I}$; respectively, $\delta_{X_{(r)}I}$; respectively, $\delta^{\log}_{X_{(r)}I}$) when there is no danger of confusion.

Remark 6.11. Let $r \geq 2$ be a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2. By the definition of $D_{(r)I}$, $D_{(r)I}$ is irreducible. (Indeed, the log smoothness of $p^{\log}_{(s)s+1} : X^{\log}_{(s+1)} \to X^{\log}_{(s)}$ and the (1-)morphism $\overline{\mathcal{M}}^{\log}_{0,t+1} \to \overline{\mathcal{M}}^{\log}_{0,t}$ [obtained by forgetting the $(t+1)$-st section] [$s, t \in \mathbb{N}$] imply the log regularity [hence, in particular, the normality of the underlying scheme] of $X^{\log}_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\#}+1}$; moreover, by a similar argument to the argument used in the proof of Lemma 6.3, $D_{(r)I}$ is connected, hence, [in light of the normality just observed] irreducible.) Thus, $D_{(r)I}$ is an irreducible component of $D_{(r)}$. Moreover, $D_{(r)} = \bigcup_I D_{(r)I}$. (Indeed, if the image of a geometric point $\overline{x} \to X_{(r)}$ lies on $D_{(r)}$, then by considering the curve which corresponds to the composite $\overline{x} \to X_{(r)} \to \overline{\mathcal{M}}_{g,r}$, there exists a subset I of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2 such that the image of the geometric point $\overline{x} \to X_{(r)}$ lies on $D_{(r)I}$.) Therefore, the log structure of $X^{\log}_{(r)}$ is the log structure associated to the divisor with normal crossings $\bigcup_I D_{(r)I} \subseteq X_{(r)}$, i.e., if we denote by $\mathcal{M}(D_{(r)I})$ the log structure on $X_{(r)}$ associated to the divisor $D_{(r)I} \subseteq X_{(r)}$, then the log structure of $X^{\log}_{(r)}$ is $\Sigma_I \mathcal{M}(D_{(r)I})$ (see Definition 5.6).

Lemma 6.12. *Let $r \geq 2$ be a natural number, I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$ and $1 \leq i \leq r+1$ an integer.*

(*i*) *The closed subscheme of $X_{(r+1)}$ determined by the composite of the natural closed immersions* (*defined in Definition 6.10*)

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

is $D_{(r+1)I\cup\{r+1\}}$.

(*ii*) *The closed subscheme of* $X_{(r+1)}$ *determined by the composite of the natural closed immersions* (*defined in Definition 6.10*)

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

is $D_{(r+1)I}$.

(*iii*) *The inverse image of* $D_{(r)I} \subseteq X_{(r)}$ *via* $p_{(r)i}$ *is* $D_{(r+1)(I\cup\{r+1\})^{\sigma_i}} \cup D_{(r+1)I^{\sigma_i}}$, *where*

$$\sigma_i = ((1,2,\cdots,r+1) \mapsto (1,2,\cdots,i-1,i+1,i+2,\cdots,r+1,i)) \in \mathcal{S}_{r+1}\,,$$

and $I^{\sigma_i} = \{\sigma_i(k) \mid k \in I\}$.

(*iv*) *The closed subscheme* $D_{(r+1)\{i,j\}} \subseteq X_{(r+1)}$ $(j \neq i)$ *is the image of a section of* $p_{(r)i}$.

Proof. First, we prove assertion (i). By the definition of the r-pointed stable curve

$$(C_{(r)I} \longrightarrow D_{(r)I};\ s_1,\cdots,s_r : D_{(r)I} \longrightarrow C_{(r)I})\,,$$

the $(r+1)$-pointed stable curve determined by the closed immersion $C_{(r)I} \hookrightarrow X_{(r+1)}$ is obtained as the stabilization ([12], Definition 2.3) of the r-pointed stable curve of genus g

$$(C_{(r)I} \times_{D_{(r)I}} C_{(r)I} \xrightarrow{\mathrm{pr}_1} C_{(r)I};\ \tilde{s}_1,\cdots,\tilde{s}_r : C_{(r)I} \longrightarrow C_{(r)I} \times_{D_{(r)I}} C_{(r)I})\,,$$

(where $\tilde{s}_i$ is the section obtained by base-changing s_i) with the extra section obtained by the diagnal morphism $C_{(r)I} \to C_{(r)I} \times_{D_{(r)I}} C_{(r)I}$. Therefore, since the operation of stabilization commutes with base-change, the closed immersion in question

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \hookrightarrow C_{(r)I} \hookrightarrow X_{(r+1)}$$

determines the $(r+1)$-pointed stable curve obtained as the stabilization of the r-pointed stable curve of genus g

$$((X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} C_{(r)I} \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2};$$

$$s'_1,\cdots,s'_r : X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow (X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} C_{(r)I}) \quad (*_1)$$

(where s'_i is the section obtained by base-changing s_i) with the extra section induced by the diagonal morphism of $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$ over $D_{(r)I}$. On the other hand, since the operation of clutching commutes with the base-change, the r-pointed stable curve of genus g $(*_1)$ is obtained by applying the clutching (1-)morphism $\beta_{0,g,I,\{1,2,\cdots,r\}\setminus I}$ to the $(I^{\#}+1)$-pointed stable curve

$$(X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} (X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \quad (*_2)$$

obtained by base-changing the $(I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \to D_{(r)I}$ defined in Definition 6.10 and the $(r-I^{\#}+1)$-pointed stable curve

$$(X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}) \times_{D_{(r)I}} (X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1}) \xrightarrow{\mathrm{pr}_1} X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \quad (*_3)$$

obtained by base-changing the $(I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+1} \to D_{(r)I}$ defined in Definition 6.10. Note that then, by definition, the stable curve $(*_3)$ is isomorphic to the $(r-I^{\#}+1)$-pointed stable curve

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the $(r-I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+2)} \stackrel{p_{(r-I^{\#}+1)r-I^{\#}+2}}{\to} X_{(r-I^{\#}+1)}$. Moreover, since the image of the extra section of the r-pointed stable curve of genus g $(*_1)$ lies on the stable curve $(*_2)$, the $(r+1)$-pointed stable curve determined by the closed immersion in question is the $(r+1)$-pointed stable curve obtained by applying the clutching (1-)morphism $\beta_{0,g,I\cup\{r+1\},\{1,2,\cdots,r+1\}\setminus(I\cup\{r+1\})}$ to the $(I^{\#}+2)$-pointed stable curve

$$X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+3} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,I^{\#}+3} \to \overline{\mathcal{M}}_{0,I^{\#}+2}$ over $\overline{\mathcal{M}}_{0,I^{\#}+2}$ and the $(r-I^{\#}+1)$-pointed stable curve

$$X_{(r-I^{\#}+2)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2} \longrightarrow X_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}_{0,I^{\#}+2}$$

obtained by base-changing the $(r-I^{\#}+1)$-pointed stable curve $X_{(r-I^{\#}+2)} \stackrel{p_{(r-I^{\#}+1)r-I^{\#}+2}}{\to} X_{(r-I^{\#}+1)}$. This completes the proof of assertion (i).

Assertion (ii) follows from a similar argument to the argument used in the proof of assertion (i).

Assertion (iii) follows from assertion (i) and (ii), together with the fact that $p_{(r)i}$ coincides with the composite of the automorphism of $X_{(r+1)}$ determined by $\sigma_i \in \mathcal{S}_{r+1}$ and $p_{(r)r+1}$.

Finally, we prove assertion (iv). By the definition of $D_{(r+1)\{j,r+1\}}$, the composite

$$D_{(r+1)\{j,r+1\}} \overset{\delta_{(r+1)\{j,r+1\}}}{\longrightarrow} X_{(r+1)} \overset{p_{(r)r+1}}{\longrightarrow} X_{(r)}$$

is the classifying morphism of the r-pointed stable curve $X_{(r+1)} \overset{p_{(r)r+1}}{\to} X_{(r)}$. Thus, the composite $p_{(r)r+1} \circ \delta_{(r+1)\{j,r+1\}}$ is an isomorphism. This completes the proof of the assertion in the case where $i = r + 1$. In general, the assertion follows from the fact that $p_{(r)i}$ coincides with the composite of the automorphism of $X_{(r+1)}$ determined by $\sigma_i \in \mathcal{S}_{r+1}$ and $p_{(r)r+1}$. $\square$

Remark 6.13. Let $r \geq 2$ and $1 \leq i \leq r + 1$ be natural numbers, and σ_i the element of $\mathcal{S}_{r+1}$ defined in Lemma 6.12, (iii). Then one may verify easily that the image of the k-th section $(1 \leq k \leq r)$ of the r-pointed stable curve $p_{(r)r+1} : X_{(r+1)} \to X_{(r)}$ is $D_{(r+1)\{k,r+1\}}$ (see Lemma 6.12, (iv)). Therefore, by taking the composite of the sections of the r-pointed stable curve $p_{(r)r+1} : X_{(r+1)} \to X_{(r)}$ and the automorphism of $X_{(r+1)}$ determined by σ_i, we obtain a r-pointed stable curve $p_{(r)i} : X_{(r+1)} \to X_{(r)}$ such that the image of the k-th section $(1 \leq k \leq r)$ is

$$\begin{cases} D_{(r+1)\{k,i\}} & (\text{if } k \leq i-1) \\ D_{(r+1)\{i,k+1\}} & (\text{if } i \leq k)\,. \end{cases}$$

Thus, in particular, if $j \neq j'$ then $D_{(r+1)\{i,j\}} \cap D_{(r+1)\{i,j'\}}$ is empty. Moreover, we obtain $D_{(r+1)} = \bigcup_{j \neq i} D_{(r+1)\{i,j\}} \cup p^{-1}_{(r)i} D_{(r)}$. (See the proof of [12], Theorem 2.7. Note that the restriction of $S^{i,n+1}_{g,n+1}$ in the proof of [12], Theorem 2.7 to $X_{(n+1)}$ is $D_{(n+1)\{i,n+1\}}$.) On the other hand, the morphism $p^{\log}_{(r)i} : X^{\log}_{(r+1)} \to X^{\log}_{(r)}$ factors through the log scheme $(X_{(r+1)}, p^{-1}_{(r)i} D_{(r)})^{\log}$ obtained by equipping $X_{(r+1)}$ with the log structure associated to the divisor with normal crossings $p^{-1}_{(r)i} D_{(r)}$, the morphism $(X_{(r+1)}, p^{-1}_{(r)i} D_{(r)})^{\log} \to X^{\log}_{(r)}$ is log smooth, and the morphism $X^{\log}_{(r+1)} \to (X_{(r+1)}, p^{-1}_{(r)i} D_{(r)})^{\log}$ is obtained by "forgetting" the portion of the log structure of $X^{\log}_{(r+1)}$ defined by the divisors determined by the sections $D_{(r+1)\{i,j\}} \subseteq X_{(r+1)}$ $(j \neq i)$ (i.e., $\Sigma_{j \neq i} \mathcal{M}(D_{(r+1)\{i,j\}})$).

Lemma 6.14. *Let $r \geq 3$ be a natural number, and $i = 1$ or 2. Then the composite*

$$D^{\log}_{(r)\{i,i+1\}} \overset{\delta^{\log}_{(r)\{i,i+1\}}}{\longrightarrow} X^{\log}_{(r)} \overset{p^{\log}_{(r-1)i}}{\longrightarrow} X^{\log}_{(r-1)}$$

coincides with the composite

$$D^{\log}_{(r)\{i,i+1\}} \overset{\delta^{\log}_{(r)\{i,i+1\}}}{\longrightarrow} X^{\log}_{(r)} \overset{p^{\log}_{(r-1)i+1}}{\longrightarrow} X^{\log}_{(r-1)}\,.$$

Moreover, this is a morphism of type $\mathbb{N}$.

Proof. The assersion that $p^{\log}_{(r-1)i} \circ \delta^{\log}_{(r)\{i,i+1\}}$ coincides with $p^{\log}_{(r-1)i+1} \circ \delta^{\log}_{(r)\{i,i+1\}}$ follows from the fact that $p^{\log}_{(r-1)i+1}$ coincides with the composite of the automorphism of $X^{\log}_{(r)}$ determined by

$$\sigma = ((1,2,\cdots,r) \mapsto (1,2,\cdots,i-1,i+1,i,i+2,\cdots,r)) \in \mathcal{S}_r$$

and $p^{\log}_{(r-1)i}$, together with the fact that the restriction of the automorphism of $X^{\log}_{(r)}$ determined by σ to the closed subscheme $D^{\log}_{(r)\{i,i+1\}}$ is the identity morphism of $D^{\log}_{(r)\{i,i+1\}}$.

Now $p_{(r-1)i} \circ \delta_{(r)\{i,i+1\}}$ is an isomorphism by Lemma 6.12, (iv). Moreover, since $p^{\log}_{(r-1)i} \circ \delta^{\log}_{(r)\{i,i+1\}}$ is obtained by "forgetting" the portion of the log structure of $D^{\log}_{(r)\{i,i+1\}}$ that originates from $D_{(r)\{i,i+1\}} \subseteq X_{(r)}$ (i.e., $\mathcal{M}(D_{(r)\{i,i+1\}}) \mid_{D_{(r)\{i,i+1\}}}$) (see Remark 6.13), the composite $p^{\log}_{(r-1)i} \circ \delta^{\log}_{(r)\{i,i+1\}}$ is a morphism of type $\mathbb{N}$. $\square$

Definition 6.15. Let $r \geq 3$ be a natural number, and $i = 1$ or 2. Then we shall denote by $a^{\log}_{X_{(r)}\{i,i+1\}}$ the composite

$$D^{\log}_{X_{(r)}\{i,i+1\}} \xrightarrow{\delta^{\log}_{X_{(r)}\{i,i+1\}}} X^{\log}_{(r)} \xrightarrow{p^{\log}_{X_{(r-1)}i}} X^{\log}_{(r-1)},$$

and by $a_{X_{(r)}\{i,i+1\}}$ the underlying morphism of schemes of $a^{\log}_{X_{(r)}\{i,i+1\}}$. By Lemma 6.14, $a^{\log}_{X_{(r)}\{i,i+1\}}$ is a morphism of type $\mathbb{N}$.

We shall denote by $\mathcal{L}_{X_{(r)}\{i,i+1\}}$ the invertible sheaf on $D_{X_{(r)}\{i,i+1\}}$ which corresponds to $a^{\log}_{X_{(r)}\{i,i+1\}}$ under the bijection ι in Theorem 5.14. Note that, by the definition of ι and the proof of Lemma 6.14, $\mathcal{L}_{X_{(r)}\{i,i+1\}}$ is isomorphic to the normal sheaf of $D_{X_{(r)}\{i,i+1\}}$ in $X_{(r)}$ (cf. Remark 5.15).

We shall denote by $U_{X_{(r)}\{i,i+1\}}$ the open subscheme of $D_{X_{(r)}\{i,i+1\}}$ determined by the open immersion

$$U_{X_{(r-1)}} \hookrightarrow X_{(r-1)} \xrightarrow[\sim]{a^{-1}_{X_{(r)}\{i,i+1\}}} D_{X_{(r)}\{i,i+1\}}.$$

For simplicity, we shall write $a^{\log}_{(r)\{i,i+1\}}$ (respectively, $a_{(r)\{i,i+1\}}$; respectively, $\mathcal{L}_{(r)\{i,i+1\}}$; respectively, $U_{(r)\{i,i+1\}}$) instead of $a^{\log}_{X_{(r)}\{i,i+1\}}$ (respectively, $a_{X_{(r)}\{i,i+1\}}$; respectively, $\mathcal{L}_{X_{(r)}\{i,i+1\}}$; respectively, $U_{X_{(r)}\{i,i+1\}}$) when there is no danger of confusion.

Definition 6.16. Let $r \geq 3$ be a natural number, and $I = \{1,2\}$, $\{2,3\}$ or $\{1,3\}$. Then we shall denote by $D_{X_{(r)}I:\{1,2,3\}}$ the closed subscheme $D_{X_{(r)}I} \cap$

$D_{X_{(r)}\{1,2,3\}}$ of $D_{X_{(r)}I}$ and $D_{X_{(r)}\{1,2,3\}}$, and by $D^{\log}_{X_{(r)}I:\{1,2,3\}}$ the log scheme obtained by equipping $D_{X_{(r)}I:\{1,2,3\}}$ with the log structure induced by the log structure of $X^{\log}_{(r)}$. For simplicity, we shall write $D_{(r)I:\{1,2,3\}}$ (respectively, $D^{\log}_{(r)I:\{1,2,3\}}$) instead of $D_{X_{(r)}I:\{1,2,3\}}$ (respectively, $D^{\log}_{X_{(r)}I:\{1,2,3\}}$) when there is no danger of confusion.

Lemma 6.17. *Let $r \geq 3$ be a natural number. Then the composite*

$$D_{(r)\{1,2,3\}} \xrightarrow{\delta_{(r)\{1,2,3\}}} X_{(r)} \xrightarrow{p_{(r-1)1}} X_{(r-1)}$$

factors through $D_{(r-1)\{1,2\}}$. Moreover, this morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determines a trivial $\mathbb{P}^1$-bundle over $D_{(r-1)\{1,2\}}$, and $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$ determine sections of this $\mathbb{P}^1$-bundle.

Proof. The assertion that the composite $p_{(r-1)1} \circ \delta_{(r)\{1,2,3\}}$ factors through $D_{(r-1)\{1,2\}}$ follows from the fact that the inverse image of $D_{(r-1)\{1,2\}} \hookrightarrow X_{(r-1)}$ via $p_{(r-1)1}$ is $D_{(r)\{2,3\}} \cup D_{(r)\{1,2,3\}}$ (Lemma 6.12, (iii)). Moreover, by the proof of Lemma 6.12, (i), the natural morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determined by $p_{(r-1)1} \circ \delta_{(r)\{1,2,3\}}$ is isomorphic to the stable curve

$$X_{(r-2)} \times_K \overline{\mathcal{M}}_{0,4} \longrightarrow X_{(r-2)} \times_K \overline{\mathcal{M}}_{0,3}$$

obtained by base-changing the universal curve $\overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,3}$ over $\overline{\mathcal{M}}_{0,3}$, the natural morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determines a trivial $\mathbb{P}^1$-bundle. The assertion that $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$ determine sections of this $\mathbb{P}^1$-bundle follows from the fact that by the definition of the operation of clutching and Remark 6.13, the images of the 1-st and 2-nd sections of the natural morphism $D_{(r)\{1,2,3\}} \to D_{(r-1)\{1,2\}}$ determined by $p_{(r-1)1} \circ \delta_{(r)\{1,2,3\}}$ are $D_{(r)\{1,2\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$, respectively, together with the fact that by Lemma 6.12, (iii), the image of the 3-rd section (i.e., "the clutching locus" of the stable curve determined by the closed immersion $\delta_{(r-1)\{1,2\}}$) is $D_{(r)\{1,2,3\}} \cap D_{(r)\{2,3\}} = D_{(r)\{2,3\}:\{1,2,3\}}$. □

Definition 6.18. Let $r \geq 3$ be a natural number. Then we shall denote by $b_{X_{(r)}\{1,2,3\}}$ the isomorphism $D_{X_{(r)}\{1,2,3\}} \xrightarrow{\sim} X_{(r-2)} \times_K \mathbb{P}^1_K$ such that

- the composite

  $$D_{X_{(r)}\{1,2,3\}} \underset{\sim}{\xrightarrow{b_{X_{(r)}\{1,2,3\}}}} X_{(r-2)} \times_K \mathbb{P}^1_K \xrightarrow{\mathrm{pr}_1} X_{(r-2)}$$

  coincides with the composite

  $$D_{X_{(r)}\{1,2,3\}} \longrightarrow D_{X_{(r-1)}\{1,2\}} \underset{\sim}{\xrightarrow{a_{X_{(r-1)}\{1,2\}}}} X_{(r-2)}\,,$$

where the first morphism is the morphism determined by $p_{X_{(r-1)}1} \circ \delta_{X_{(r)}\{1,2,3\}}$; and

- the closed subscheme of $D_{X_{(r)}\{1,2,3\}}$ determined by the closed immersion

$$X_{(r-2)} \times_K \{0\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}}$$

$$\text{(respectively, } X_{(r-2)} \times_K \{1\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}}\,;$$

$$\text{respectively, } X_{(r-2)} \times_K \{\infty\} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}})$$

is $D_{X_{(r)}\{1,2\}:\{1,2,3\}}$ (respectively, $D_{X_{(r)}\{2,3\}:\{1,2,3\}}$; respectively, $D_{X_{(r)}\{1,3\}:\{1,2,3\}}$).

We shall denote by $U_{X_{(r)}\{1,2,3\}}$ the open subscheme of $D_{X_{(r)}\{1,2,3\}}$ determined by the open immersion

$$U_{(r-2)} \times_K U_{\mathbb{P}} \hookrightarrow X_{(r-2)} \times_K \mathbb{P}^1_K \overset{b^{-1}_{X_{(r)}\{1,2,3\}}}{\underset{\sim}{\longrightarrow}} D_{X_{(r)}\{1,2,3\}}\,.$$

For simplicity, we shall write $b_{(r)\{1,2,3\}}$ (respectively, $U_{(r)\{1,2,3\}}$) instead of $b_{X_{(r)}\{1,2,3\}}$ (respectively, $U_{X_{(r)}\{1,2,3\}}$) when there is no danger of confusion.

Lemma 6.19. *Let $r \geq 3$ be a natural number. Then the isomorphism $b_{(r)\{1,2,3\}} : D_{(r)\{1,2,3\}} \xrightarrow{\sim} X_{(r-2)} \times_K \mathbb{P}^1_K$ extends to a unique morphism of log schemes $D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K$ of type $\mathbb{N}$.*

Proof. It is immediate that if $b_{(r)\{1,2,3\}}$ extends to such a morphism, then it is unique. Thus, it is enough to show that $b_{(r)\{1,2,3\}}$ extends to such a morphism.

By Remark 6.13, the morphism $D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-2)} \times_K \mathbb{P}^1_K$ determined by the composite

$$D^{\log}_{(r)\{1,2,3\}} \xrightarrow{\text{via } p^{\log}_{(r-1)1} \circ \delta^{\log}_{(r)\{1,2,3\}}} D^{\log}_{(r-1)\{1,2\}} \xrightarrow{a^{\log}_{(r-1)\{1,2\}}} X^{\log}_{(r-2)} \qquad (*)$$

and the composite

$$D^{\log}_{(r)\{1,2,3\}} \to D_{(r)\{1,2,3\}} \overset{b_{(r)\{1,2,3\}}}{\xrightarrow{\sim}} X_{(r-2)} \times_K \mathbb{P}^1_K \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1_K$$

is obtained by "forgetting" the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ defined by $D_{(r)\{1,2\}:\{1,2,3\}}$, $D_{(r)\{2,3\}:\{1,2,3\}}$ and $D_{(r)\{1,3\}:\{1,2,3\}}$ (i.e., $\mathcal{M}(D_{(r)\{1,2\}:\{1,2,3\}}+$

$D_{(r)\{2,3\}:\{1,2,3\}} + D_{(r)\{1,3\}:\{1,2,3\}}))$ and the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ that originates from $D_{(r)\{1,2,3\}} \subseteq X_{(r)}$ (i.e., $\mathcal{M}(D_{(r)\{1,2,3\}}) \mid_{D_{(r)\{1,2,3\}}}$). Therefore, the morphism $D^{\log}_{(r)\{1,2,3\}} \longrightarrow X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K$ determined by the above composite $(*)$ and the composite

$$D^{\log}_{(r)\{1,2,3\}} \longrightarrow D'^{\log}_{(r)\{1,2,3\}} \longrightarrow \mathbb{P}^{\log}_K$$

(where $D'^{\log}_{(r)\{1,2,3\}}$ is the log scheme obtained by equipping $D_{(r)\{1,2,3\}}$ with the log structure associated to the divisors

$$D_{(r)\{1,2\}:\{1,2,3\}},\ D_{(r)\{2,3\}:\{1,2,3\}} \ \text{and}\ D_{(r)\{1,3\}:\{1,2,3\}} \subseteq D_{(r)\{1,2,3\}}\,,$$

the first morphism is the natural morphism obtained by "forgetting" the portion of the log structure of $D^{\log}_{(r)\{1,2,3\}}$ that originates from the divisors other than

$$D_{(r)\{1,2\}:\{1,2,3\}},\ D_{(r)\{2,3\}:\{1,2,3\}} \ \text{and}\ D_{(r)\{1,3\}:\{1,2,3\}} \subseteq D_{(r)\{1,2,3\}}\,,$$

[among the divisors of the form $D_{(r)I} \mid_{D_{(r)\{1,2,3\}}}$ [where $I \subseteq \{1, 2, \cdots, r\}$ of cardinarity ≥ 2]] and the second morphism is the strict morphism induced by the natural morphism

$$D_{(r)\{1,2,3\}} \overset{b_{(r)\{1,2,3\}}}{\xrightarrow{\sim}} X_{(r-2)} \times_K \mathbb{P}^1_K \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1_K\ )$$

is an extension of $b_{(r)\{1,2,3\}}$ of the desired type. □

Definition 6.20. Let $r \geq 3$ be a natural number. Then we shall denote by $b^{\log}_{X_{(r)}\{1,2,3\}}$ the morphism

$$D^{\log}_{X_{(r)}\{1,2,3\}} \longrightarrow X^{\log}_{(r-2)} \times_K \mathbb{P}^{\log}_K\,,$$

obtained in Lemma 6.19. Note that this is a morphism of type $\mathbb{N}$ by Lemma 6.19.

We shall denote by $\mathcal{L}_{X_{(r)}\{1,2,3\}}$ the invertible sheaf on $D_{X_{(r)}\{1,2,3\}}$ which corresponds to the morphism $b^{\log}_{X_{(r)}\{1,2,3\}}$ under the bijection ι in Theorem 5.14. Note that, by the definition of ι and the proof of Lemma 6.19, $\mathcal{L}_{X_{(r)}\{1,2,3\}}$ is isomorphic to the normal sheaf of $D_{X_{(r)}\{1,2,3\}}$ in $X_{(r)}$ (cf. Remark 5.15). For simplicity, we shall write $b^{\log}_{(r)\{1,2,3\}}$ (respectively, $\mathcal{L}_{(r)\{1,2,3\}}$) instead of $b^{\log}_{X_{(r)}\{1,2,3\}}$ (respectively, $\mathcal{L}_{X_{(r)}\{1,2,3\}}$) when there is no danger of confusion.

Lemma 6.21. *Let $r \geq 2$ be a natural number.*

(*i*) $\mathcal{L}_{(r+1)\{1,2\}} \mid_{U_{(r+1)\{1,2\}}} \xrightarrow{\sim} (p^*_{(r)i}\mathcal{L}_{(r)\{1,2\}}) \mid_{U_{(r+1)\{1,2\}}}$ *for* $i \neq 1, 2$.

(*ii*) $\mathcal{L}_{(r+1)\{2,3\}} \mid_{U_{(r+1)\{2,3\}}} \xrightarrow{\sim} (p^*_{(r)1}\mathcal{L}_{(r)\{1,2\}}) \mid_{U_{(r+1)\{2,3\}}} \xrightarrow{\sim} (p^*_{(r)i}\mathcal{L}_{(r)\{2,3\}}) \mid_{U_{(r+1)\{2,3\}}}$ *for* $i \neq 1, 2, 3$.

(*iii*) $\mathcal{L}_{(r+1)\{1,2,3\}} \mid_{U_{(r+1)\{1,2,3\}}} \xrightarrow{\sim} (p^*_{(r)j}\mathcal{L}_{(r)\{1,2\}}) \mid_{U_{(r+1)\{1,2,3\}}} \xrightarrow{\sim} (p^*_{(r)i}\mathcal{L}_{(r)\{1,2,3\}}) \mid_{U_{(r+1)\{1,2,3\}}}$ *for* $j = 1, 2, 3$ *and* $i \neq 1, 2, 3$.

Proof. First, we prove assertion (i). It follows from the fact that $\mathcal{L}_{(r)\{1,2\}}$ is the normal sheaf of $D_{(r)\{1,2\}}$ in $X_{(r)}$, together with the flatness of $p_{(r)i}$ that $p^*_{(r)i}\mathcal{L}_{(r)\{1,2\}}$ is isomorphic to the normal sheaf of the closed subscheme of $X_{(r+1)}$ obtained as the fiber product of

$$\begin{tikzcd} & D_{(r)\{1,2\}} \arrow[d, "\delta_{(r)\{1,2\}}"] \\ X_{(r+1)} \arrow[r, "p_{(r)i}"] & X_{(r)} . \end{tikzcd}$$

Thus, by Lemma 6.12, (iii) and the fact that $\mathcal{L}_{(r+1)\{1,2\}}$ is the normal sheaf of $D_{(r+1)\{1,2\}}$ in $X_{(r+1)}$, together with the fact that the intersection of $D_{(r+1)\{1,2\}}$ and $D_{(r+1)\{1,2,i\}}$ is contained in $D_{(r+1)\{1,2\}} \setminus U_{(r+1)\{1,2\}}$, the restriction of $p^*_{(r)i}\mathcal{L}_{(r)\{1,2\}}$ to $U_{(r+1)\{1,2\}}$ is isomorphic to $\mathcal{L}_{(r+1)\{1,2\}} \mid_{U_{(r+1)\{1,2\}}}$. This completes the proof of (i).

Assertion (ii) and (iii) follow from a similar argument to the argument used in the proof of (i). □

7 Reconstruction of the fundamental groups of higher dimensional log configuration schemes

We continue with the notation of the preceding section. Let Σ be a (non-empty) set of prime numbers, and l a prime number that is invertible in K. (Thus, it makes sense to speak of Σ-integers.) Then we shall say that Σ is K-*innocuous* if

$$\Sigma = \begin{cases} \text{the set of all prime numbers or } \{l\} & \text{if } p = 0 \\ \{l\} & \text{if } p \geq 2 \, . \end{cases}$$

We shall fix a separable closure K^{sep} of K and denote by G_K the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K. Moreover, we shall denote by Λ the maximal pro-Σ quotient of $\hat{\mathbb{Z}}(1)$.

Definition 7.1.

(i) Let r be a positive natural number. We shall denote by $\Pi^{\log}_{X_{(r)}}$ the quotient of $\pi_1(X^{\log}_{(r)})$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(X^{\log}_{(r)} \times_K K^{\mathrm{sep}}) \to \pi_1(X^{\log}_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)}),$$

and write Π_X for $\Pi^{\log}_{X_{(1)}}$. For simplicity, we shall write $\Pi^{\log}_{(r)}$ instead of $\Pi^{\log}_{X_{(r)}}$ when there is no danger of confusion.

(ii) Let $r \geq 2$ a natural number, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality ≥ 2. We shall denote by $\Pi^{\log}_{X_{(r)}I}$ the quotient of $\pi_1(D^{\log}_{X_{(r)}I})$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(D^{\log}_{X_{(r)}I} \times_K K^{\mathrm{sep}}) \to \pi_1(D^{\log}_{X_{(r)}I} \times_K K^{\mathrm{sep}})^{(\Sigma)}).$$

For simplicity, we shall write $\Pi^{\log}_{(r)I}$ instead of $\Pi^{\log}_{X_{(r)}I}$ when there is no danger of confusion.

(iii) We shall denote by $\Pi^{\log}_{\mathbb{P}_K}$ the quotient of $\pi_1(\mathbb{P}^{\log}_K)$ by the closed normal subgroup

$$\mathrm{Ker}(\pi_1(\mathbb{P}^{\log}_K \times_K K^{\mathrm{sep}}) \to \pi_1(\mathbb{P}^{\log}_K \times_K K^{\mathrm{sep}})^{(\Sigma)}).$$

For simplicity, we shall write $\Pi^{\log}_{\mathbb{P}}$ instead of $\Pi^{\log}_{\mathbb{P}_K}$ when there is no danger of confusion.

Definition 7.2. Let $r \geq 3$ be a natural number. We shall denote by $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$ the graph of groups defined as follows:

$$\mathcal{G}^{\log}_{(r)}(\Sigma) = (\overset{\Pi^{\log}_{X_{(r)}\{1,2\}}}{\bullet} \underset{\{1\}}{-} \overset{\Pi^{\log}_{X_{(r)}\{1,2,3\}}}{\bullet} \underset{\{1\}}{-} \overset{\Pi^{\log}_{X_{(r)}\{2,3\}}}{\bullet}),$$

where $\{1\}$ is the trivial group; the symbols "$\bullet$" (respectively, "$-$") denote the vertices (respectively, the edges) of the underlying graphs; and the group that lies above a vertex (respectively, below an edge) denotes the group that corresponds to the vertex (respectively, edge). We shall denote by $\Pi^{\mathcal{G}}_{X_{(r)}}$ the profinite group

$$\varinjlim(\Pi^{\log}_{X_{(r)}\{1,2\}} \longleftarrow \{1\} \longrightarrow \Pi^{\log}_{X_{(r)}\{1,2,3\}} \longleftarrow \{1\} \longrightarrow \Pi^{\log}_{X_{(r)}\{2,3\}}),$$

where the inductive limit is taken in the category of profinite groups. For simplicity, we shall shall write $\mathcal{G}^{\log}_{(r)}(\Sigma)$ (respectively, $\Pi^{\mathcal{G}}_{(r)}$) instead of $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$ (respectively, $\Pi^{\mathcal{G}}_{X_{(r)}}$) when there is no danger of confusion.

Definition 7.3. Let G be a group. Then we shall denote by $G_\bullet$ the graph of groups whose underlying graph has one vertex that corresponds to G and no edges.

We shall denote by

$$f^{\log}_{X_{(r)}}(\Sigma) : \mathcal{G}^{\log}_{X_{(r)}}(\Sigma) \longrightarrow (\Pi^{\log}_{X_{(r)}})_\bullet$$

(cf. Definition 7.3) the morphism of graphs of groups determined by the morphisms $D^{\log}_{X_{(r)}I} \overset{\delta^{\log}_{X_{(r)}I}}{\to} X^{\log}_{(r)}$. For simplicity, we shall shall write $f^{\log}_{(r)}(\Sigma)$ instead of $f^{\log}_{X_{(r)}}(\Sigma)$ when there is no danger of confusion.

Let $I = \{1,2\}$, $\{2,3\}$ or $\{1,2,3\}$. Then, by the definition of $\mathcal{G}^{\log}_{X_{(r)}}(\Sigma)$, we have a natural morphism of graphs of groups

$$(\Pi^{\log}_{X_{(r)}I})_\bullet \longrightarrow \mathcal{G}^{\log}_{X_{(r)}}(\Sigma)\,.$$

We shall denote this morphism by $\delta^{\mathcal{G}\log}_{X_{(r)}I}$.

First, we will show the following theorem.

Theorem 7.4. *For a set of prime numbers Σ (which is not necessary K-innocuous), $f^{\log}_{(r)}(\Sigma)$ induces a surjection $\Pi^{\mathcal{G}}_{(r)} \to \Pi^{\log}_{(r)}$.*

Proof. First, we prove the assertion in the case where Σ is the set of all prime numbers. Since the morphism $p^{\log}_{(r)3}\,|_{D^{\log}_{(r+1)\{2,3\}}} = a^{\log}_{(r)\{2,3\}} : D^{\log}_{(r+1)\{2,3\}} \to X^{\log}_{(r-1)}$ is a morphism of type $\mathbb{N}$, the composite

$$\Pi^{\log}_{(r)\{2,3\}} \xrightarrow{\text{via } \delta^{\mathcal{G}\log}_{X_{(r)}\{2,3\}}} \Pi^{\mathcal{G}}_{(r)} \xrightarrow{\text{via } f^{\log}_{(r)}(\Sigma)} \Pi^{\log}_{(r)} \xrightarrow{\text{via } p^{\log}_{(r-1)3}} \Pi^{\log}_{(r-1)}$$

is surjective. Thus, the morphism

$$\Pi^{\mathcal{G}}_{(r)} \longrightarrow \Pi^{\log}_{(r-1)}$$

induced by the composite of $p^{\log}_{(r)3}$ and $f^{\log}_{(r)}(\Sigma)$ is surjective. In particular, it is enough to show that the image of the morphism $\Pi^{\mathcal{G}}_{(r)} \to \Pi^{\log}_{(r)}$ induced by $f^{\log}_{(r)}(\Sigma)$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$. Let $\overline{x}^{\log} \to X^{\log}_{(r-1)}$ be a strict geometric point of $X^{\log}_{(r-1)}$ such that the image of $\overline{x}^{\log} \to X^{\log}_{(r-1)}$ lies on $U_{(r-1)\{1,2\}}$. Then it follows from Proposition 6.7 that the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$ is generated by the image of the natural morphism $\pi_1(X^{\log}_{(r)\overline{x}^{\log}}) \to \Pi^{\log}_{(r)}$, where $X^{\log}_{(r)\overline{x}^{\log}}$ is the log scheme determined by the base-change of $p^{\log}_{(r)3} : X^{\log}_{(r)} \to X^{\log}_{(r-1)}$ via $\overline{x}^{\log} \to X^{\log}_{(r-1)}$. Let $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ (respectively, $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$) be the log scheme determined by the base-change of $p^{\log}_{(r)3} |_{D^{\log}_{(r)\{1,2\}}}: D^{\log}_{(r)\{1,2\}} \to X^{\log}_{(r-1)}$ (respectively, $p^{\log}_{(r)3} |_{D^{\log}_{(r)\{1,2,3\}}}: D^{\log}_{(r)\{1,2,3\}} \to X^{\log}_{(r-1)}$) via $\overline{x}^{\log} \to X^{\log}_{(r-1)}$, $D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}}$ the fiber product $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \times_{X^{\log}_{(r)}} D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}} = D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \times_{X^{\log}_{(r)\overline{x}^{\log}}} D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$; $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}$ the graph of groups defined by

$$\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} = (\overset{\pi_1(D^{\log}_{(r)\{1,2\}\overline{x}^{\log}})}{\bullet} \quad -_{\pi_1(D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}})} \quad \overset{\pi_1(D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}})}{\bullet});$$

and $\pi_1(\mathcal{G}^{\log}_{(r)\overline{x}^{\log}})$ the group definied by

$$\varinjlim(\pi_1(D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}) \longleftarrow \pi_1(D^{\log}_{(r)\{1,2\}:\{1,2,3\}\overline{x}^{\log}}) \longrightarrow \pi_1(D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}))$$

(where the inductive limit is taken in the category of profinite groups). Then the natural strict closed immersions $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}} \to X^{\log}_{(r)\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}} \to X^{\log}_{(r)\overline{x}^{\log}}$ (note that, by construction, the underlying schemes of $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$ are the irreducible components of the underlying scheme of $X^{\log}_{(r)\overline{x}^{\log}}$) induce a morphism of graphs of groups $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} & \longrightarrow & \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{G}^{\log}_{(r)}(\Sigma) & \xrightarrow{f^{\log}_{(r)}(\Sigma)} & \pi_1(X^{\log}_{(r)})_\bullet .
\end{array}$$

Now since the underlying schemes of $D^{\log}_{(r)\{1,2\}\overline{x}^{\log}}$ and $D^{\log}_{(r)\{1,2,3\}\overline{x}^{\log}}$ are the irreducible components of the underlying scheme of $X^{\log}_{(r)\overline{x}^{\log}}$, if we naturally regard $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}$ as a *graph of anabelioids* (cf. [16]), then the underlying graph of the graph of anabelioids determined as *the pull-back* of a ket covering $Y^{\log} \to X^{\log}_{(r)\overline{x}^{\log}}$ of $X^{\log}_{(r)\overline{x}^{\log}}$ via the morphism $\mathcal{G}^{\log}_{(r)\overline{x}^{\log}} \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})_\bullet$ coincides with the dual graph of the pointed stable curve Y_{red}. Thus, it is immediate that $\pi_1(\mathcal{G}^{\log}_{(r)\overline{x}^{\log}}) \to \pi_1(X^{\log}_{(r)\overline{x}^{\log}})$ is surjective. Therefore, since the image of $\pi_1(X^{\log}_{(r)\overline{x}^{\log}}) \to \Pi^{\log}_{(r)}$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$, the image of $\Pi^{\mathcal{G}}_{(r)}$ in $\Pi^{\log}_{(r)}$ via the morphism induced by

$f^{\log}_{(r)}(\Sigma)$ generates the kernel of the morphism $\Pi^{\log}_{(r)} \to \Pi^{\log}_{(r-1)}$ induced by $p^{\log}_{(r-1)3}$. This completes the proof of the desired surjectivity in the case where Σ is the set of all prime numbers.

In the general case, the assertion follows immediately from the assertion in the case where Σ is the set of all prime numbers. $\square$

Remark 7.5. Theorem 7.4 can be regarded as a logarithmic analogue of [13], Remark 1.2.

In the rest of this section, we assume that

Σ *is* K*-innocuous.*

Next, we prove fundamental facts concerning the fundamental groups of the log configuration schemes.

Lemma 7.6.

(*i*) *The natural morphism* $U_{(r)} \to X^{\log}_{(r)}$ *induces a natural isomorphism* $\pi_1(U_{(r)})^{(\Sigma)} \xrightarrow{\sim} \Pi^{\log}_{(r)}$, *where* $\pi_1(U_{(r)})^{(\Sigma)}$ *is the quotient of* $\pi_1(U_{(r)})$ *by the closed normal subgroup*

$$\mathrm{Ker}(\pi_1(U_{(r)} \times_K K^{\mathrm{sep}}) \to \pi_1(U_{(r)} \times_K K^{\mathrm{sep}})^{(\Sigma)})\,.$$

(*ii*) *The natural morphism* $U_{(r)\{1,2,3\}} \to X^{\log}_{(r)} \times_K \mathbb{P}^{\log}_K$ *induces a natural isomorphism* $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)} \xrightarrow{\sim} \Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathbb{P}}$, *where* $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)}$ *is the quotient of* $\pi_1(U_{(r)\{1,2,3\}})$ *by the closed normal subgroup*

$$\mathrm{Ker}(\pi_1(U_{(r)\{1,2,3\}} \times_K K^{\mathrm{sep}}) \to \pi_1(U_{(r)\{1,2,3\}} \times_K K^{\mathrm{sep}})^{(\Sigma)})\,.$$

(*iii*) *Let* $1 \le i \le r+1$ *be an integer, and* $\overline{x} \to X_{(r)}$ *a geometric point of* $X_{(r)}$ *whose image lies on* $U_{(r)}$. *Then the cartesian diagram*

$$\begin{tikzcd} X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x} \arrow[r] \arrow[d] & \overline{x} \arrow[d] \\ X^{\log}_{(r+1)} \arrow[r, "p^{\log}_{(r)i}"] & X^{\log}_{(r)} \end{tikzcd}$$

induces the following exact sequence:

$$1 \longrightarrow \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)} \longrightarrow \Pi^{\log}_{(r+1)} \xrightarrow{\text{via } p^{\log}_{(r)i}} \Pi^{\log}_{(r)} \longrightarrow 1\,.$$

(*iv*) *For a profinite group* Γ (*respectively, a scheme* S), *we shall denote by* $\mathcal{S}(\Gamma)$ (*respectively,* $S_{\text{ét}}$) *the classifying site of* Γ, (*i.e., the site defined by considering the category of finite sets equipped with a continuous action of* Γ [*and coverings given by surjections of such sets*]) (*respectively, the étale site of* S). *Then we have natural morphisms of sites*

$$U_{(r)\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U_{(r)}^{\log})^{(\Sigma)}) \longrightarrow \mathcal{S}(\Pi_{(r)}^{\log}) .$$

Let A *be a finite* $\Pi_{(r)}^{\log}$*-module whose order is a* Σ*-integer, and* n *an integer. Then the natural morphisms*

$$\mathrm{H}^n(\Pi_{(r)}^{\log}, A) \longrightarrow \mathrm{H}^n(\pi_1(U_{(r)}^{\log})^{(\Sigma)}, A) \longrightarrow \mathrm{H}^n_{\text{ét}}(U_{(r)}, \mathcal{F}_A)$$

induced by the above morphisms of sites are isomorphisms, where $\mathcal{F}_A$ *is the locally constant sheaf on* $U_{(r)}$ *determined by* A.

(*v*) *Let* A *be a finite* $\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log}$*-module whose order is a* Σ*-integer, and* n *an integer. Then the natural morphisms of sites*

$$U_{(r)\{1,2,3\}\text{ét}} \longrightarrow \mathcal{S}(\pi_1(U_{(r)\{1,2,3\}}^{\log})^{(\Sigma)}) \longrightarrow \mathcal{S}(\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log})$$

induce isomorphisms

$$\mathrm{H}^n(\Pi_{(r)}^{\log} \times_{G_K} \Pi_{\mathbb{P}}^{\log}, A) \xrightarrow{\sim} \mathrm{H}^n(\pi_1(U_{(r)\{1,2,3\}}^{\log})^{(\Sigma)}, A) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(U_{(r)\{1,2,3\}}, \mathcal{F}_A) ,$$

where $\mathcal{F}_A$ *is the locally constant sheaf determined by* A.

Proof. First, we prove (i). It is immediate that we may assume that K is separably closed. Let $V \to U_{(r)}$ be a Galois covering whose order is a Σ-integer (i.e., a Galois covering determined by an open normal subgroup of $\pi_1(U_{(r)}^{\log})^{(\Sigma)} = \pi_1(U_{(r)}^{\log})^{(\Sigma)})$, $Y \to X_{(r)}$ the normalization of $X_{(r)}$ in V, and $\overline{\eta} \to X_{(r)}$ a geometric point over the generic point of an irreducible component of $D_{(r)} = X_{(r)} \setminus U_{(r)} \subseteq X_{(r)}$. Then it follows from the Galoisness of $V \to U_{(r)}$ and the fact that the order of $V \to U_{(r)}$ is prime to p (whenever $p \geq 2$) that the base-change $Y \times_{X_{(r)}} \operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}} \to \operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}}$ is a tamely ramified covering (along the unique closed point of $\operatorname{Spec} \mathcal{O}_{X_{(r)},\overline{\eta}}$). Thus, by the log purity theorem, $Y \to X_{(r)}$ extends to a ket covering $Y^{\log} \to X_{(r)}^{\log}$. In particular, $\pi_1(U_{(r)}^{\log})^{(\Sigma)} \to \Pi_{(r)}^{\log}$ is injective, hence an isomorphism.

Next, we prove (ii). By Proposition 3.4, the natural morphism $\pi_1(X_{(r)}^{\log} \times_K \mathbb{P}_K^{\log}) \to \pi_1(X_{(r)}^{\log}) \times_{G_K} \pi_1(\mathbb{P}_K^{\log})$ is an isomorphism. Moreover, it is immediate that we may assume that K is separably closed. Therefore, by taking pro-Σ completions, $\pi_1(X_{(r)}^{\log} \times_K \mathbb{P}_K^{\log})^{(\Sigma)} \xrightarrow{\sim} (\pi_1(X_{(r)}^{\log}) \times \pi_1(\mathbb{P}_K^{\log}))^{(\Sigma)} \xrightarrow{\sim} \Pi_{(r)}^{\log} \times \Pi_{\mathbb{P}}^{\log}$. On the other hand, by a similar argument to the argument used in the proof of (i), we obtain an isomorphism $\pi_1(U_{(r)\{1,2,3\}})^{(\Sigma)} \xrightarrow{\sim} \pi_1(X^{\log}_{(r)} \times_K \mathbb{P}^{\log}_K)^{(\Sigma)}$. This completes the proof of (ii).

Next, we prove (iii). To prove (iii), we may assume that K is separably closed field. Moreover, if Σ is the set of all prime numbers, then this follows from [13], Lemma 2.4. Thus, we may assume that $\Sigma = \{l\}$ for a prime number l which is invertible in K. By [21], Proposition 2.7, we have an exact sequence

$$1 \longrightarrow \pi_1(U)^{(\Sigma)} \longrightarrow \pi_1(U_{(r+1)})^{(')} \overset{\text{via } p^{\log}_{(r)i}}{\longrightarrow} \pi_1(U_{(r)}) \longrightarrow 1\,,$$

where U is the interior of $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}$, and the profinite group $\pi_1(U_{(r+1)})^{(')}$ is the quotient of $\pi_1(U_{(r+1)})$ by the kernel of the natural surjection

$$\pi_1(U) \longrightarrow \pi_1(U)^{(\Sigma)}\,.$$

Now, by a similar argument to the argument used in the proof of (i), the group $\pi_1(U)^{(\Sigma)}$ is naturally isomorphic to $\pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})^{(\Sigma)}$. By the exactness of

$$1 \longrightarrow \pi_1(U)^{(\Sigma)} \longrightarrow \pi_1(U_{(r+1)})^{(')} \overset{\text{via } p^{\log}_{(r)i}}{\longrightarrow} \pi_1(U_{(r)}) \longrightarrow 1\,,$$

it is enough to show that the outer representation

$$\pi_1(U_{(r)}) \longrightarrow \mathrm{Out}(\pi_1(U)^{(\Sigma)})$$

induced by the above sequence factors through $\pi_1(U_{(r)})^{(\Sigma)}$ ([1], Proposition 3). On the other hand, by [13], Lemma 3.1, (i), the kernel of the natural morphism

$$\mathrm{Out}(\pi_1(U)^{(\Sigma)}) \longrightarrow \mathrm{Aut}((\pi_1(X \times_K K^{\mathrm{sep}})^{(\Sigma)})^{\mathrm{ab}})$$

is pro-Σ. Therefore, it is enough to show that the natural representation

$$\pi_1(U_{(r)}) \longrightarrow \mathrm{Aut}((\pi_1(X \times_K K^{\mathrm{sep}})^{(\Sigma)})^{\mathrm{ab}})$$

induced by the above outer representation factors through $\pi_1(U_{(r)})^{(\Sigma)}$. Now this is immediate. This completes the proof of assertion (iii).

Next, we prove (iv). The assertion that the first morphism is an isomorphism follows immediately from (i). Let $\overline{x} \to X_{(r)}$ be a geometric point of $X_{(r)}$ whose image lies on $U_{(r)}$. Then, by considering the Hochschild-Serre spectral sequence ([19], Theorem 2.1.5) associated to the exact sequence obtained in (iii)

$$1 \longrightarrow \pi \longrightarrow \Pi^{\log}_{(r+1)} \overset{\text{via } p^{\log}_{(r)r+1}}{\longrightarrow} \Pi^{\log}_{(r)} \longrightarrow 1$$

(where $\pi = \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x})$) and the Leray spectral sequence associated to the morphism $p_{(r)r+1} \mid_{U_{(r+1)}}$, we obtain the following morphism of spectral sequences:

$$\begin{array}{ccccc} E_2^{p,q} & = & \mathrm{H}^p(\Pi^{\log}_{(r)}, \mathrm{H}^q(\pi, A)) \Longrightarrow \mathrm{H}^{p+q}(\Pi^{\log}_{(r+1)}, A) & = & E^{p+q} \\ \downarrow & & & & \downarrow \\ {E'}_2^{p,q} & = & \mathrm{H}^p_{\text{ét}}(U_{(r)}, \mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(U_{(r+1)}, \mathcal{F}_A) & = & {E'}^{p+q} . \end{array}$$

Now, by considering the "compactification" of $p_{(r)r+1} \mid_{U_{(r+1)}}$

$$\begin{array}{ccc} U_{(r+1)} & \xrightarrow{p_{(r)r+1}|_{U_{(r+1)}} \times \mathrm{pr}_{(r+1)r+1}|_{U_{(r+1)}}} & U_{(r)} \times_K X \\ {\scriptstyle p_{(r)r+1}|_{U_{(r+1)}}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ U_{(r)} & = & U_{(r)} , \end{array}$$

it follows that the sheaf $\mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A$ is locally constant and constructible ([2], Corollary 10.3); moreover, the $\Pi_{(r+1)}$-module $(\mathbb{R}^q(p_{(r)r+1} \mid_{U_{(r)}})_* \mathcal{F}_A)_{\overline{x}}$ is naturally isomorphic to $\mathrm{H}^q(U, \mathcal{F}_A \mid_U)$ ([2], Theorem 7.3). Therefore, it is enough to show that the natural morphism

$$\mathrm{H}^n(\pi, A) \longrightarrow \mathrm{H}^n_{\text{ét}}(U, \mathcal{F}_A \mid_U)$$

is an isomorphism, where U is the interior of $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}$. Thus, one then verifies immediately that it is enough to verify that every étale cohomology class of U (with coefficients in $\mathcal{F}_A \mid_U$) vanishes upon pull-back to some (connected) finite étale Σ-covering $V \to U$. Moreover, by passing to an appropriate U, we may assume that $\mathcal{F}_A \mid_U$ is trivial. Then the vanishing assertion in question is immediate (respectively, a tautology) for $n = 0$ (respectively, $n = 1$). Moreover, the vanishing assertion in question is immediate for $n \geq 3$ by [2], Theorem 9.1. If U is *affine*, then since $\mathrm{H}^n_{\text{ét}}(U, \mathcal{F}_A \mid_U)$ vanishes for $n = 2$ ([2], Theorem 9.1), the assertion is immediate. If U is proper, then it is enough to take $V \to U$ so that the degree of $V \to U$ annihilates A (cf., e.g., the discussion at the bottom of [2], p. 136).

Finally, we prove (v). The assertion that the first morphism is an isomorphism follows from (i). Moreover, by a similar argument to the argument used in the proof (iv), the second morphism is also an isomorphism. □

Remark 7.7.

(i) By Lemma 7.6, (iv), (v), together with a similar argument to the argument used in [15], Lemma 4.3, any invertible sheaf on $X^{\log}_{(r)}$ or $X^{\log}_{(r)} \times_K \mathbb{P}^{\log}_K$ satisfies the condition $(*)$ in Proposition 5.23.

(ii) By (i) and Lemma 7.6, (iv), (v), the equivalence class of the extension of $\Pi^{\log}_{(r)}$ (respectively, $\Pi^{\log}_{(r)} \times_{G_K} \Pi^{\log}_{\mathbb{P}}$) associated to an invertible sheaf $\mathcal{L}$ on $X_{(r)}$ (respectively, $X_{(r)} \times_K \mathbb{P}^1_K$) (see Definition 5.24) depends only on the (étale-theoretic) first Chern class of $\mathcal{L} \mid_{U_{(r)}}$ (respectively, $\mathcal{L} \mid_{U_{(r)} \times_K U_{\mathbb{P}}}$). In particular, for example, the extension

$$1 \longrightarrow \Lambda \longrightarrow \Pi^{\log}_{(r+1)\{1,2\}} \xrightarrow{a^{\log}_{(r+1)\{1,2\}}} \Pi^{\log}_{(r)} \longrightarrow 1$$

of $\Pi^{\log}_{(r)}$ by Λ (i.e., the extension of $\Pi^{\log}_{(r)}$ associated to $(a^{-1}_{(r+1)\{1,2\}})^* \mathcal{L}_{(r+1)\{1,2\}}$) is isomorphic to the extension of $\Pi^{\log}_{(r)}$ by Λ associated to the invertible sheaf $(a^{-1}_{(r+1)\{1,2\}})^* (p_{(r)i} \mid_{D_{(r+1)\{1,2\}}})^* (\mathcal{L}_{(r)\{1,2\}})$ ($i \neq 1, 2$) (cf. Lemma 6.12, (iii)).

Lemma 7.8.

(*i*) *Let $r \geq 2$ be an integer and $2 \leq i \leq r$ an integer. Then the following diagram is cartesian:*

$$\begin{tikzcd}
\Pi^{\log}_{(r+1)\{1,2\}} \arrow[r, "\text{via } p^{\log}_{(r)i+1}"] \arrow[d, "\text{via } a^{\log}_{(r+1)\{1,2\}}"'] & \Pi^{\log}_{(r)\{1,2\}} \arrow[d, "\text{via } a^{\log}_{(r)\{1,2\}}"] \\
\Pi^{\log}_{(r)} \arrow[r, "\text{via } p^{\log}_{(r-1)i}"] & \Pi^{\log}_{(r-1)} .
\end{tikzcd}$$

(*ii*) *Let $r \geq 2$ be an integer. Then the following diagram is cartesian:*

$$\begin{tikzcd}
\Pi^{\log}_{(r+1)\{2,3\}} \arrow[r, "\text{via } p^{\log}_{(r)1}"] \arrow[d, "\text{via } a^{\log}_{(r+1)\{2,3\}}"'] & \Pi^{\log}_{(r)\{1,2\}} \arrow[d, "\text{via } a^{\log}_{(r)\{1,2\}}"] \\
\Pi^{\log}_{(r)} \arrow[r, "\text{via } p^{\log}_{(r-1)1}"] & \Pi^{\log}_{(r-1)} .
\end{tikzcd}$$

(*iii*) *Let $r \geq 3$ be an integer and $3 \leq i \leq r$ an integer. Then the following diagram is cartesian:*

$$\begin{tikzcd}
\Pi^{\log}_{(r+1)\{2,3\}} \arrow[r, "\text{via } p^{\log}_{(r)i+1}"] \arrow[d, "\text{via } a^{\log}_{(r+1)\{2,3\}}"'] & \Pi^{\log}_{(r)\{2,3\}} \arrow[d, "\text{via } a^{\log}_{(r)\{2,3\}}"] \\
\Pi^{\log}_{(r)} \arrow[r, "\text{via } p^{\log}_{(r-1)i}"] & \Pi^{\log}_{(r-1)} .
\end{tikzcd}$$

(*iv*) *Let $r \geq 2$ be an integer, and $j = 1$, 2 or 3. Then the following diagram is cartesian:*

$$\begin{tikzcd}
\Pi^{\log}_{(r+1)\{1,2,3\}} \arrow[r, "\text{via } p^{\log}_{(r)j}"] \arrow[d, "\text{via } b^{\log}_{(r+1)\{1,2,3\}}"'] & \Pi^{\log}_{(r)\{1,2\}} \arrow[d, "\text{via } a^{\log}_{(r)\{1,2\}}"] \\
\Pi^{\log}_{(r-1)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} \arrow[r, "\text{via } \mathrm{pr}_1"] & \Pi^{\log}_{(r-1)} .
\end{tikzcd}$$

(*v*) *Let $r \geq 3$ be an integer and $2 \leq i \leq r-1$ be an integer. Then the following diagram is cartesian:*

$$\begin{tikzcd}
\Pi^{\log}_{(r+1)\{1,2,3\}} \arrow[r, "\text{via } p^{\log}_{(r)i+2}"] \arrow[d, "\text{via } b^{\log}_{(r)\{1,2,3\}}"'] & \Pi^{\log}_{(r)\{1,2,3\}} \arrow[d, "\text{via } b^{\log}_{(r)\{1,2,3\}}"] \\
\Pi^{\log}_{(r-1)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} \arrow[r, "\text{via } p_{(r-2)i} \times \mathrm{id}_{\mathbb{P}^{\log}}"] & \Pi^{\log}_{(r-2)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} .
\end{tikzcd}$$

Proof. First, we prove assertion (i). By Remark 7.7, (ii), the extension

$$1 \longrightarrow \Lambda \longrightarrow \Pi^{\log}_{(r+1)\{1,2\}} \overset{\text{via } a^{\log}_{(r)\{1,2\}}}{\longrightarrow} \Pi^{\log}_{(r)} \longrightarrow 1$$

of $\Pi^{\log}_{(r)}$ by Λ is isomorphic to the extension of $\Pi^{\log}_{(r)}$ associated to $(p_{(r)j} \mid_{D_{(r+1)\{1,2\}}})^*\mathcal{L}_{(r)\{1,2\}}$ ($j \neq 1$, 2). On the other hand, by the commutativity of the diagram

$$\begin{tikzcd}
X_{(r)} \arrow[d, "p_{(r-1)i}"'] & D_{(r+1)\{1,2\}} \arrow[l, "a_{(r+1)\{1,2\}}"', "\sim"] \arrow[r, "\delta_{(r+1)\{1,2\}}"] \arrow[d] & X_{(r+1)} \arrow[d, "p_{(r)i+1}"] \\
X_{(r-1)} & D_{(r)\{1,2\}} \arrow[l, "a_{(r)\{1,2\}}"', "\sim"] \arrow[r, "\delta_{(r)\{1,2\}}"] & X_{(r)}
\end{tikzcd}$$

(cf. the definition of "$a_{(*)\{1,2\}}$" in Definition 6.15) implies that $(a^{-1}_{(r+1)\{1,2\}})^*(p_{(r)i+1} \mid_{D_{(r+1)\{1,2\}}})^*\mathcal{L}_{(r)\{1,2\}}$ is naturally isomorphic to $p^*_{(r-1)i}(a^{-1}_{(r)\{1,2\}})^*\mathcal{L}_{(r)\{1,2\}}$. Therefore, the fiber product of

$$\begin{tikzcd}
& \Pi^{\log}_{(r)\{1,2\}} \arrow[d, "\text{via } a^{\log}_{(r)\{1,2\}}"] \\
\Pi^{\log}_{(r)} \arrow[r, "\text{via } p^{\log}_{(r-1)i}"] & \Pi^{\log}_{(r-1)} ,
\end{tikzcd}$$

is isomorphic to the extension of $\Pi^{\log}_{(r)}$ associated to $(a^{-1}_{(r+1)\{1,2\}})^*(p_{(r)i+1} \mid_{D_{(r+1)\{1,2\}}})^*\mathcal{L}_{(r)\{1,2\}}$; thus, by Lemma 6.21, (i) (cf. also the argument in Remark 7.7, (ii)), this fiber product is isomorphic to $\Pi^{\log}_{(r+1)\{1,2\}}$.

Assertion (ii) follows from a similar argument to the argument used in the proof of assertion (i), Lemma 6.21, (ii) (cf. also the argument in Remark 7.7, (ii)), together with the commutativity of the following diagram:

$$\begin{tikzcd} X_{(r)} \arrow[d, "p_{(r-1)1}"'] & D_{(r+1)\{2,3\}} \arrow[l, "a_{(r+1)\{2,3\}}"', "\sim"] \arrow[r, "\delta_{(r+1)\{2,3\}}"] \arrow[d] & X_{(r+1)} \arrow[d, "p_{(r)1}"] \\ X_{(r-1)} & D_{(r)\{1,2\}} \arrow[l, "a_{(r)\{1,2\}}"', "\sim"] \arrow[r, "\delta_{(r)\{1,2\}}"] & X_{(r)} \end{tikzcd}$$

(cf. the definitions of "$a_{(*)\{1,2\}}$" and "$a_{(*)\{2,3\}}$" in Definition 6.15).

Assertion (iii) follows from a similar argument to the argument used in the proof of assertion (i), Lemma 6.21, (ii) (cf. also the argument in Remark 7.7, (ii)), together with the commutativity of the following diagram:

$$\begin{tikzcd} X_{(r)} \arrow[d, "p_{(r-1)i}"'] & D_{(r+1)\{2,3\}} \arrow[l, "a_{(r+1)\{2,3\}}"', "\sim"] \arrow[r, "\delta_{(r+1)\{2,3\}}"] \arrow[d] & X_{(r+1)} \arrow[d, "p_{(r)i+1}"] \\ X_{(r-1)} & D_{(r)\{2,3\}} \arrow[l, "a_{(r)\{2,3\}}"', "\sim"] \arrow[r, "\delta_{(r)\{2,3\}}"] & X_{(r)} \end{tikzcd}$$

(cf. the definition of "$a_{(*)\{2,3\}}$" in Definition 6.15).

Assertion (iv) follows from a similar argument to the argument used in the proof of assertion (i), Lemma 6.21, (iii) (cf. also the argument in Remark 7.7, (ii)), together with the commutativity of the following diagram:

$$\begin{tikzcd} X_{(r-1)} \times_K \mathbb{P}^1_K \arrow[d] & D_{(r+1)\{1,2,3\}} \arrow[l, "b_{(r+1)\{1,2,3\}}"', "\sim"] \arrow[r, "\delta_{(r+1)\{1,2,3\}}"] \arrow[d] & X_{(r+1)} \arrow[d, "p_{(r)j}"] \\ X_{(r-1)} & D_{(r)\{1,2\}} \arrow[l, "a_{(r)\{1,2\}}"', "\sim"] \arrow[r, "\delta_{(r)\{1,2\}}"] & X_{(r)}\ , \end{tikzcd}$$

where the left-hand vertical arrow is the first projection (cf. the definitions of "$a_{(*)\{1,2\}}$" and "$b_{(*)\{1,2,3\}}$" in Definition 6.15 and Definition 6.18).

Assertion (v) follows from a similar argument to the argument used in the proof of assertion (i), Lemma 6.21, (iii) (cf. also the argument in Remark 7.7, (ii)), together with the commutativity of the following diagram:

$$\begin{tikzcd} X_{(r-1)} \times_K \mathbb{P}^1_K \arrow[d, "p_{(r-2)i}\times \mathrm{id}_{\mathbb{P}^1_K}"'] & D_{(r+1)\{1,2,3\}} \arrow[l, "b_{(r+1)\{1,2,3\}}"', "\sim"] \arrow[r, "\delta_{(r+1)\{1,2,3\}}"] \arrow[d] & X_{(r+1)} \arrow[d, "p_{(r)i+2}"] \\ X_{(r-2)} \times_K \mathbb{P}^1_K & D_{(r)\{1,2,3\}} \arrow[l, "b_{(r)\{1,2,3\}}"', "\sim"] \arrow[r, "\delta_{(r)\{1,2,3\}}"] & X_{(r)} \end{tikzcd}$$

(cf. the definition of "$b_{(*)\{1,2,3\}}$" in Definition 6.18). $\square$

Lemma 7.9.

(*i*) *Let $r \geq 2$ be an integer, and $I = \{i, i+1\}$ $(i = 1,\ 2)$. Then the following diagram is cartesian:*

$$\begin{array}{ccc}
\Pi^{\log}_{(r)I} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)i,i+1}} & \Pi^{\log}_{(2)\{1,2\}} \\
\Big\downarrow{\scriptstyle \text{via } a^{\log}_{(r)I}} & & \Big\downarrow{\scriptstyle \text{via } a^{\log}_{(2)\{1,2\}}} \\
\Pi^{\log}_{(r-1)} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r-1)i}} & \Pi_X\ .
\end{array}$$

(*ii*) *Let $r \geq 3$ be an integer. Then the following diagram is cartesian:*

$$\begin{array}{ccccc}
\Pi^{\log}_{(r)\{1,2,3\}} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r)1,2}} & & & \Pi^{\log}_{(2)\{1,2\}} \\
\Big\downarrow{\scriptstyle \text{via } b^{\log}_{(r)\{1,2,3\}}} & & & & \Big\downarrow{\scriptstyle \text{via } a^{\log}_{(2)\{1,2\}}} \\
\Pi^{\log}_{(r-2)} \times_{G_K} \Pi^{\log}_{\mathbb{P}} & \xrightarrow{\mathrm{pr}_1} & \Pi^{\log}_{(r-2)} & \xrightarrow{\text{via } \mathrm{pr}^{\log}_{(r-1)1}} & \Pi_X\ .
\end{array}$$

Proof. Assertion (i) (respectively assertion (ii)) follows immediately from Lemma 7.8, (i), (ii) (respectively, (iv)), by induction on r. $\square$

Definition 7.10.

(i) Let $r \geq 2$ be an integer, and $I = \{i, i+1\}$ $(i = 1,\ 2)$. Then, by Lemma 7.9, (i), the morphism $\Pi^{\log}_{X_{(r)}I} \to \Pi^{\log}_{X_{(r-1)}}$ induced by $a^{\log}_{X_{(r)}I}$ and the morphism $\Pi^{\log}_{X_{(r)}I} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)}i,i+1}$ induces an isomorphism $\Pi^{\log}_{X_{(r)}I} \xrightarrow{\sim} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$. We shall denote this isomorphism by $\alpha^{\log}_{X_{(r)}I}$. For simplicity, we shall write $\alpha^{\log}_{(r)I}$ instead of $\alpha^{\log}_{X_{(r)}I}$ when there is no danger of confusion.

(ii) Let $r \geq 3$ be an integer. Then, by Lemma 7.9, (ii), the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(r-2)}} \times_{G_K} \Pi^{\log}_{\mathbb{P}_K}$ induced by $b^{\log}_{X_{(r)}\{1,2,3\}}$ and the morphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \to \Pi^{\log}_{X_{(2)}\{1,2\}}$ induced by $\mathrm{pr}^{\log}_{X_{(r)}1,2}$ induces an isomorphism $\Pi^{\log}_{X_{(r)}\{1,2,3\}} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-2)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$. We shall denote this isomorphism by $\beta^{\log}_{X_{(r)}\{1,2,3\}}$. For simplicity, we shall write $\beta^{\log}_{(r)\{1,2,3\}}$ instead of $\beta^{\log}_{X_{(r)}\{1,2,3\}}$ when there is no danger of confusion.

Definition 7.11. Let $* = 0$, 1 or ∞, and $D \subseteq \pi_1(\mathbb{P}^{\log}_K)$ the decompositon group at $* \in \mathbb{P}^1_K$ (well-defined up to conjugation by an element of $\pi_1(\mathbb{P}^{\log}_{K^{\mathrm{sep}}})$). Then we shall refer to the quotient of D by the kernel of the composite

$$D \hookrightarrow \pi_1(\mathbb{P}^{\log}_K) \longrightarrow \Pi^{\log}_{\mathbb{P}}$$

as *the pro-*($\underline{\Sigma}$) *decomposition group at* $* \in \mathbb{P}^1_K$.

Next, we will define the collection of data used in the *reconstruction of the fundamental groups of higher dimensional log configuration schemes* performed in Theorem 7.15 below.

Definition 7.12. Let $r \geq 2$ be an integer.

(i) We shall denote by $\mathcal{D}_X(\Sigma)$, or $\mathcal{D}_{X_{(1)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\log}_{X_{(2)}},\ \Pi_X,\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ G_K,\ \text{and } \Pi^{\log}_{\mathbb{P}_K}\ ;$$

- the morphisms

$$\Pi^{\log}_{X_{(2)}} \overset{\text{via } p^{\log}_{X_{(1)}i}}{\longrightarrow} \Pi_X\ (i = 1,\ 2),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(2)}}\ ,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ ,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\ ;\ \text{ and}$$

- the subgroups

$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$

determined by the pro-($\underline{\Sigma}$) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

(ii) We shall denote by $\mathcal{D}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\log}_{X_{(k)}}\ (2 \leq k \leq r+1),\ \ \Pi^{\log}_{X_{(2)}\{1,2\}},\ \Pi_X,\ G_K,\ \text{and } \Pi^{\log}_{\mathbb{P}_K}\ ;$$

- the morphisms

$$\Pi^{\log}_{X_{(k)}} \overset{\text{via } p^{\log}_{X_{(k-1)}i}}{\longrightarrow} \Pi^{\log}_{X_{(k-1)}} \ (2 \leq k \leq r+1,\ 1 \leq i \leq k),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \overset{\text{via } a^{\log}_{X_{(2)}\{1,2\}}}{\longrightarrow} \Pi_X\ ,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ ,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\ ;$$

- the composites

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}}{\overset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r)}1}}{\to} \Pi_X$),

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}}{\overset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{2,3\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r)}2}}{\to} \Pi_X$) and

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}}{\overset{\sim}{\longrightarrow}} \Pi^{\log}_{X_{(r+1)}\{1,2,3\}} \overset{\text{via } \delta^{\log}_{X_{(r+1)}\{1,2,3\}}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r-1)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r-1)}1}}{\to} \Pi_X$); and
- the subgroups

$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$

determined by the pro-($\underline{\Sigma}$) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

(iii) We shall denote by $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ the collection of data consisting of

- the profinite groups

$$\Pi^{\mathcal{G}}_{X_{(r+1)}},\ \Pi^{\log}_{X_{(k)}}\ (2 \leq k \leq r),\ \Pi^{\log}_{X_{(2)}\{1,2\}},\ \Pi_X,\ G_K,\ \text{and}\ \Pi^{\log}_{\mathbb{P}_K}\ ;$$

- the morphisms

$$\Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\text{via } p^{\log}_{X_{(r)}i} \circ f^{\log}_{X_{(r)}}(\Sigma)} \Pi^{\log}_{X_{(r)}}\ (1 \leq i \leq r+1),$$

$$\Pi^{\log}_{X_{(k)}} \xrightarrow{\text{via } p^{\log}_{X_{(k-1)}i}} \Pi^{\log}_{X_{(k-1)}}\ (2 \leq k \leq r,\ 1 \leq i \leq k),$$

$$\Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\text{via } a^{\log}_{X_{(2)}\{1,2\}}} \Pi_X\ ,$$

and the morphisms induced by the respective structure morphisms

$$\Pi_X \longrightarrow G_K\ ,$$

$$\Pi^{\log}_{\mathbb{P}_K} \longrightarrow G_K\ ;$$

- the composites

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{1,2\}})^{-1}}{\xrightarrow{\sim}} \Pi^{\log}_{X_{(r+1)}\{1,2\}} \xrightarrow{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{1,2\}}} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \xrightarrow{\text{via } \mathrm{pr}^{\log}_{X_{(r)}1}} \Pi_X$),

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}}{\xrightarrow{\sim}} \Pi^{\log}_{X_{(r+1)}\{2,3\}} \xrightarrow{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{2,3\}}} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r)}} \xrightarrow{\text{via } \mathrm{pr}^{\log}_{X_{(r)}2}} \Pi_X$) and

$$\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \overset{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}}{\xrightarrow{\sim}} \Pi^{\log}_{X_{(r+1)}\{1,2,3\}} \xrightarrow{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{1,2,3\}}} \Pi^{\mathcal{G}}_{X_{(r+1)}}$$

(where the morphism implicit in the fiber product $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ is $\Pi^{\log}_{X_{(r-1)}} \xrightarrow{\text{via } \mathrm{pr}^{\log}_{X_{(r-1)}1}} \Pi_X$); and

- the subgroups
$$\mathfrak{D}^{\log}_{K*} \subseteq \Pi^{\log}_{\mathbb{P}_K}$$
determined by the pro-(Σ) decomposition groups $\mathfrak{D}^{\log}_{K*}$ at $* \in \mathbb{P}^1_K$ ($* = 0$, 1 and ∞).

In the following, let Y be a smooth, proper, geometrically connected curve of genus $g_Y \geq 2$ over a field L whose (not necessarily positive) characteristic we denote by p_L, and $\mathbb{P}^{\log}_L$ the log scheme obtained by equipping $\mathbb{P}^1_L$ with the log structure associated to the divisor $\{0, 1, \infty\} \subseteq \mathbb{P}^1_L$. Moreover, we shall fix a separable closure L^{sep} of L and denote by G_L the absolute Galois group $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ of L.

Definition 7.13. Let $r \geq 2$ be an integer.

(i) We shall refer to isomorphisms
$$\phi^{\Pi^{\log}_{(2)}}_{(1)} : \Pi^{\log}_{X_{(2)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}} ;$$
$$\phi^{\Pi}_{(1)} : \Pi_X \xrightarrow{\sim} \Pi_Y ;$$
$$\phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(1)} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$
$$\phi^{G}_{(1)} : G_K \xrightarrow{\sim} G_L \text{ ; and}$$
$$\phi^{\Pi^{\log}_{\mathbb{P}}}_{(1)} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$
which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}_X(\Sigma)$ and $\mathcal{D}_Y(\Sigma_Y)$ as *an isomorphism of* $\mathcal{D}_X(\Sigma)$ *with* $\mathcal{D}_Y(\Sigma_Y)$.

(ii) We shall refer to isomorphisms
$$\phi^{\Pi^{\log}_{(k)}}_{(r)} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r+1);$$
$$\phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r)} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$
$$\phi^{G}_{(r)} : G_K \xrightarrow{\sim} G_L \text{ ; and}$$
$$\phi^{\Pi^{\log}_{\mathbb{P}}}_{(r)} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$
which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$ as *an isomorphism of* $\mathcal{D}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}_{Y_{(r)}}(\Sigma_Y)$.

(iii) We shall refer to isomorphisms

$$\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}} ;$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(k)}} : \Pi^{\log}_{(k)} \xrightarrow{\sim} \Pi^{\log}_{(k)} \ (1 \leq k \leq r);$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$

$$\phi_{(r)}^{\mathcal{G}\,G} : G_K \xrightarrow{\sim} G_L \,; \text{ and}$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

which are compatible with the morphisms and subgroups given in the definitions of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ and $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ as *an isomorphism of* $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma)$ *with* $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$.

Proposition 7.14. *Let $r \geq 2$ be an integer, and Σ_X (respectively, Σ_Y) a set of prime numbers that is innocuous in K (respectively, L). Let $\phi^{\mathcal{G}}_{(r)} : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ be an isomorphism. Then the following hold:*

(*i*) *There exists an isomorphism $F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)}) : \mathcal{D}_{X_{(r-1)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r-1)}}(\Sigma_Y)$. Moreover, the correspondence*

$$\phi^{\mathcal{G}}_{(r)} \mapsto F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})$$

is functorial.

(*ii*) *If $\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}}$ induces an isomorphism of the kernel of the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ with the kernel of the morphism $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$ induced by $f^{\log}_{Y_{(r+1)}}(\Sigma)$, then there exists an isomorphism $F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)}) : \mathcal{D}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r)}}(\Sigma_Y)$. Moreover, the correspondence*

$$\phi^{\mathcal{G}}_{(r)} \mapsto F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})$$

is functorial.

Proof. First, we prove assertion (i). If we write

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(k)}} \stackrel{\text{def}}{=} \phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(k)}} \ (1 \leq k \leq r) \,,$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(2)\{1,2\}}} \stackrel{\text{def}}{=} \phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}} ,$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{G} \overset{\text{def}}{=} \phi^{\mathcal{G}\,G}_{(r)}\,, \text{ and}$$

$$F^{\check{\mathcal{G}}}_{-1}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{\mathbb{P}}} \overset{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}}_{(r)}\,,$$

then we obtain an isomorphism $F^{\check{\mathcal{G}}}_{-1}(\phi_{(r)})$ of the desired type.

Next, we prove Assertion (ii). We denote by N_X (respectively, N_Y) the kernel of the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$) induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ (respectively, $f^{\log}_{Y_{(r+1)}}(\Sigma)$). Then, by the assumption, the isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$ induces an isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}|_{N_X}: N_X \xrightarrow{\sim} N_Y$. Therefore, the isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}$ induces an isomorphism $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\mathcal{G}}_{X_{(r+1)}}/N_X \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}/N_Y$. Since the morphism $\Pi^{\mathcal{G}}_{X_{(r+1)}} \to \Pi^{\log}_{X_{(r+1)}}$ (respectively, $\Pi^{\mathcal{G}}_{Y_{(r+1)}} \to \Pi^{\log}_{Y_{(r+1)}}$) induced by $f^{\log}_{X_{(r+1)}}(\Sigma)$ (respectively, $f^{\log}_{Y_{(r+1)}}(\Sigma)$) is surjective (Theorem 7.4), we obtain that $\phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r+1)}}$. Therefore, if we write

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(r+1)}} \overset{\text{def}}{=} \phi^{\mathcal{G}^{\log}_{(r+1)}}_{(r)}/N : \Pi^{\log}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(r+1)}}\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(k)}} \overset{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{(k)}}_{(r)} \ (1 \le k \le r)\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{(2)\{1,2\}}} \overset{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}}_{(r)}\,,$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{G} \overset{\text{def}}{=} \phi^{\mathcal{G}\,G}_{(r)}\,, \text{ and}$$

$$F^{\check{\mathcal{G}}}(\phi^{\mathcal{G}}_{(r)})^{\Pi^{\log}_{\mathbb{P}}} \overset{\text{def}}{=} \phi^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}}_{(r)}\,,$$

then we obtain an isomorphism $F^{\check{\mathcal{G}}}(\phi_{(r)})$ of the desired type. $\square$

Theorem 7.15. *Let $r \ge 3$ be an integer, and Σ_X (respectively, Σ_Y) a set of prime numbers that is K-innocuous (respectively, L-innocuous). Let $\phi_{(r-1)} : \mathcal{D}_{X_{(r-1)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}_{Y_{(r-1)}}(\Sigma_Y)$ be an isomorphism. Then there exists an isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)}) : \mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X) \xrightarrow{\sim} \mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ such that*

$$F^{\check{\mathcal{G}}}_{-1}(F^{\mathcal{G}}_{+1}(\phi_{(r-1)})) = \phi_{(r-1)}\,,$$

and that the isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}^{\log}_{(r+1)}}$ arises from an isomorphism of graphs of groups of $\mathcal{G}^{\log}_{X_{(r+1)}}(\Sigma)$ with $\mathcal{G}^{\log}_{Y_{(r+1)}}(\Sigma)$. Moreover, the correspondence

$$\phi_{(r-1)} \mapsto F^{\mathcal{G}}_{+1}(\phi_{(r-1)})$$

is functorial.

Proof. First, we define isomorphisms

$$\phi_{(r)}^{\mathcal{G}\,\Pi_{(r+1)\{1,2\}}^{\log}} : \Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \xrightarrow{\sim} \Pi_{Y_{(r)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{1,2\}}^{\log},$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi_{(r+1)\{2,3\}}^{\log}} : \Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{2,3\}}^{\log} \xrightarrow{\sim} \Pi_{Y_{(r)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{2,3\}}^{\log}, \text{ and}$$

$$\phi_{(r)}^{\mathcal{G}\,\Pi_{(r+1)\{1,2,3\}}^{\log}} : \Pi_{\mathbb{P}_K}^{\log} \times_{G_K} \Pi_{X_{(r-1)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \xrightarrow{\sim} \Pi_{\mathbb{P}_L}^{\log} \times_{G_L} \Pi_{Y_{(r-1)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{1,2\}}^{\log},$$

verify the desired compatibilities.

Now We show that if we denote by $\phi_{(r)}^{\mathcal{G}\,\Pi_{(r+1)\{1,2\}}^{\log}}$ the isomorphism

$$\phi_{(r-1)}^{\Pi_{(r)}^{\log}} \times_{\phi_{(r-1)}^{\Pi}} \phi_{(r-1)}^{\Pi_{(2)\{1,2\}}^{\log}} : \Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \xrightarrow{\sim} \Pi_{Y_{(r)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{1,2\}}^{\log}$$

(where $\Pi_{X_{(r)}}^{\log} \to \Pi_X$ [respectively, $\Pi_{Y_{(r)}}^{\log} \to \Pi_Y$] is $\Pi_{X_{(r)}}^{\log} \overset{\text{via } \mathrm{pr}_{X_{(r)}1}^{\log}}{\to} \Pi_X$ [respectively, $\Pi_{Y_{(r)}}^{\log} \overset{\text{via } \mathrm{pr}_{Y_{(r)}1}^{\log}}{\to} \Pi_Y$]), then, for any $1 \leq i \leq r+1$, the following diagram commutes:

$$\begin{array}{ccc}
\Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} & \xrightarrow[\sim]{\phi_{(r)}^{\mathcal{G}\,\Pi_{(r+1)\{1,2\}}^{\log}}} & \Pi_{Y_{(r)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{1,2\}}^{\log} \\
\downarrow & & \downarrow \\
\Pi_{X_{(r)}}^{\log} & \xrightarrow[\sim]{\phi_{(r-1)}^{\Pi_{(r)}^{\log}}} & \Pi_{Y_{(r)}}^{\log},
\end{array}$$

where the left-hand vertical arrow is

$$\Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \overset{(\alpha_{X_{(r)}\{1,2\}}^{\log})^{-1}}{\longrightarrow} \Pi_{X_{(r+1)}\{1,2\}}^{\log} \overset{\text{via } \delta_{X_{(r+1)}\{1,2\}}^{\mathcal{G}\log}}{\longrightarrow} \Pi_{X_{(r+1)}}^{\mathcal{G}} \overset{\text{via } p_{X_{(r)}i}^{\log} \circ f_{X_{(r+1)}}^{\log}(\Sigma)}{\longrightarrow} \Pi_{X_{(r)}}^{\log}$$

and the right-hand vertical arrow is

$$\Pi_{Y_{(r)}}^{\log} \times_{\Pi_Y} \Pi_{Y_{(2)}\{1,2\}}^{\log} \overset{(\alpha_{Y_{(r)}\{1,2\}}^{\log})^{-1}}{\longrightarrow} \Pi_{Y_{(r+1)}\{1,2\}}^{\log} \overset{\text{via } \delta_{Y_{(r+1)}\{1,2\}}^{\mathcal{G}\log}}{\longrightarrow} \Pi_{Y_{(r+1)}}^{\mathcal{G}} \overset{\text{via } p_{Y_{(r)}i}^{\log} \circ f_{Y_{(r+1)}}^{\log}(\Sigma)}{\longrightarrow} \Pi_{Y_{(r)}}^{\log}.$$

Indeed, if $i = 1$ or 2, then since the vertical arrows in the above diagram are the first projections (Lemma 7.9, (i)), the above diagram commutes. Moreover, if $i \geq 3$, then since the left-hand vertical arrow (respectively, the right-hand vertical arrow) in the above diagram is

$$\Pi_{X_{(r)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \overset{\text{via } p_{X_{(r-1)}i-1}^{\log} \times \mathrm{id}_{D_{X_{(2)}\{1,2\}}^{\log}}}{\longrightarrow} \Pi_{X_{(r-1)}}^{\log} \times_{\Pi_X} \Pi_{X_{(2)}\{1,2\}}^{\log} \overset{(\alpha_{X_{(r-1)}\{1,2\}}^{\log})^{-1}}{\longrightarrow}$$

$$\Pi^{\log}_{X_{(r)}\{1,2\}} \stackrel{\text{via } \delta^{\log}_{X_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{X_{(r)}}$$

(respectively,

$$\Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \stackrel{\text{via } p^{\log}_{Y_{(r-1)}^{i-1}} \times \mathrm{id}_{D^{\log}_{Y_{(2)}\{1,2\}}}}{\longrightarrow} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \stackrel{(\alpha^{\log}_{Y_{(r-1)}\{1,2\}})^{-1}}{\longrightarrow}$$

$$\Pi^{\log}_{Y_{(r)}\{1,2\}} \stackrel{\text{via } \delta^{\log}_{Y_{(r)}\{1,2\}}}{\longrightarrow} \Pi^{\log}_{Y_{(r)}} \;)$$

(Lemma 7.8, (i) and 7.9, (i)) the above diagram commutes. By a similar argument to the above argument, if we denote by $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{2,3\}}}$ the isomorphism

$$\phi_{(r-1)}^{\Pi^{\log}_{(r)}} \times_{\phi^{\Pi}_{(r-1)}} \phi_{(r-1)}^{\Pi^{\log}_{(2)\{1,2\}}} : \Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{\sim}{\longrightarrow} \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

(where $\Pi^{\log}_{X_{(r)}} \to \Pi_X$ [respectively, $\Pi^{\log}_{Y_{(r)}} \to \Pi_Y$] is $\Pi^{\log}_{X_{(r)}} \stackrel{\text{via } \mathrm{pr}^{\log}_{X_{(r)}^2}}{\to} \Pi_X$ [respectively, $\Pi^{\log}_{Y_{(r)}} \stackrel{\text{via } \mathrm{pr}^{\log}_{Y_{(r)}^2}}{\to} \Pi_Y$]), then, for any $1 \leq i \leq r+1$, the following diagram commutes:

$$\begin{array}{ccc}
\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} & \xrightarrow[\sim]{\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{2,3\}}}} & \Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \\
\downarrow & & \downarrow \\
\Pi^{\log}_{X_{(r)}} & \xrightarrow[\sim]{\phi_{(r-1)}^{\Pi^{\log}_{(r)}}} & \Pi^{\log}_{Y_{(r)}} ,
\end{array}$$

where the left-hand vertical arrow is

$$\Pi^{\log}_{X_{(r)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \stackrel{(\alpha^{\log}_{X_{(r)}\{2,3\}})^{-1}}{\longrightarrow} \Pi^{\log}_{X_{(r+1)}\{2,3\}} \stackrel{\text{via } \delta^{\mathcal{G}\log}_{X_{(r+1)}\{2,3\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{X_{(r+1)}} \stackrel{\text{via } p^{\log}_{X_{(r)}^{i}} \circ f^{\log}_{X_{(r+1)}}(\Sigma)}{\longrightarrow} \Pi^{\log}_{X_{(r)}}$$

and the right-hand vertical arrow is

$$\Pi^{\log}_{Y_{(r)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \stackrel{(\alpha^{\log}_{Y_{(r)}\{2,3\}})^{-1}}{\longrightarrow} \Pi^{\log}_{Y_{(r+1)}\{2,3\}} \stackrel{\text{via } \delta^{\mathcal{G}\log}_{Y_{(r+1)}\{2,3\}}}{\longrightarrow} \Pi^{\mathcal{G}}_{Y_{(r+1)}} \stackrel{\text{via } p^{\log}_{Y_{(r)}^{i}} \circ f^{\log}_{Y_{(r+1)}}(\Sigma)}{\longrightarrow} \Pi^{\log}_{Y_{(r)}}$$

by Lemma 7.8, (ii), (iii) and 7.9, (i). Moreover, by a similar argument to the above argument, if we denote by $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}$ the isomorphism

$$\phi_{(r-1)}^{\Pi^{\log}_{\mathbb{P}}} \times_{\phi^{G}_{(r-1)}} \phi_{(r-1)}^{\Pi^{\log}_{(r-1)}} \times_{\phi^{\Pi}_{(r-1)}} \phi_{(r-1)}^{\Pi^{\log}_{(2)\{1,2\}}} : \Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}}$$

$$\xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}}$$

(where $\Pi^{\log}_{X_{(r-1)}} \to \Pi_X$ [respectively, $\Pi^{\log}_{Y_{(r)}} \to \Pi_Y$] is $\Pi^{\log}_{X_{(r-1)}} \overset{\text{via } \mathrm{pr}^{\log}_{X_{(r-1)}1}}{\to} \Pi_X$ [respectively, $\Pi^{\log}_{Y_{(r-1)}} \overset{\text{via } \mathrm{pr}^{\log}_{Y_{(r-1)}1}}{\to} \Pi_Y$]), then, for any $1 \leq i \leq r+1$, the following diagram commutes:

$$\begin{array}{ccc}
\Pi^{\log}_{\mathbb{P}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} & \xrightarrow[\sim]{\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}} & \Pi^{\log}_{\mathbb{P}_L} \times_{G_L} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \\
\downarrow & & \downarrow \\
\Pi^{\log}_{X_{(r)}} & \xrightarrow[\sim]{\phi_{(r-1)}^{\Pi^{\log}_{(r)}}} & \Pi^{\log}_{Y_{(r)}} ,
\end{array}$$

where the left-hand vertical arrow is

$$\Pi_{\mathbb{P}^{\log}_K} \times_{G_K} \Pi^{\log}_{X_{(r-1)}} \times_{\Pi_X} \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{(\beta^{\log}_{X_{(r)}\{1,2,3\}})^{-1}} \Pi^{\log}_{X_{(r+1)}\{1,2,3\}}$$
$$\xrightarrow{\text{via } \delta^{\mathcal{G}\,\log}_{X_{(r+1)}\{1,2,3\}}} \Pi^{\mathcal{G}}_{(r+1)} \xrightarrow{\text{via } p^{\log}_{X_{(r)}i} \circ f^{\log}_{X_{(r+1)}}(\Sigma)} \Pi^{\log}_{X_{(r)}}$$

and the right-hand vertical arrow is

$$\Pi_{\mathbb{P}^{\log}_K} \times_{G_K} \Pi^{\log}_{Y_{(r-1)}} \times_{\Pi_Y} \Pi^{\log}_{Y_{(2)}\{1,2\}} \xrightarrow{(\beta^{\log}_{Y_{(r)}\{1,2,3\}})^{-1}} \Pi^{\log}_{Y_{(r+1)}\{1,2,3\}}$$
$$\xrightarrow{\text{via } \delta^{\mathcal{G}\,\log}_{Y_{(r+1)}\{1,2,3\}}} \Pi^{\mathcal{G}}_{(r+1)} \xrightarrow{\text{via } p^{\log}_{Y_{(r)}i} \circ f^{\log}_{Y_{(r+1)}}(\Sigma)} \Pi^{\log}_{Y_{(r)}}$$

from Lemma 7.8, (iv), (v) and 7.9, (ii).

By the above arguments, the isomorphisms $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2\}}}$, $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{2,3\}}}$, and $\phi_{(r)}^{\mathcal{G}\,\Pi^{\log}_{(r+1)\{1,2,3\}}}$ induce an isomorphism

$$\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}}$$

such that, for any $1 \leq i \leq r+1$, the following diagram commutes:

$$\begin{array}{ccc}
\Pi^{\mathcal{G}}_{X_{(r+1)}} & \xrightarrow[\sim]{\phi_{(r)}^{\mathcal{G}^{\log}_{(r+1)}}} & \Pi^{\mathcal{G}}_{Y_{(r+1)}} \\
\text{via } p^{\log}_{X_{(r)}i} \circ f^{\log}_{X_{(r+1)}}(\Sigma) \downarrow & & \downarrow \text{via } p^{\log}_{Y_{(r)}i} \circ f^{\log}_{Y_{(r+1)}}(\Sigma) \\
\Pi^{\log}_{X_{(r)}} & \xrightarrow[\sim]{\phi_{(r-1)}^{\Pi^{\log}_{(r)}}} & \Pi^{\log}_{Y_{(r)}} .
\end{array}$$

Therefore, the isomorphisms

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}^{\log}_{(r+1)}} : \Pi^{\mathcal{G}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathcal{G}}_{Y_{(r+1)}} ;$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{(k)}} \stackrel{\mathrm{def}}{=} \phi^{\Pi^{\log}_{(k)}}_{(r-1)} : \Pi^{\log}_{X_{(k)}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(k)}} \ (1 \leq k \leq r);$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{(2)\{1,2\}}} \stackrel{\mathrm{def}}{=} \phi^{\Pi^{\log}_{(2)\{1,2\}}}_{(r-1)} : \Pi^{\log}_{X_{(2)}\{1,2\}} \xrightarrow{\sim} \Pi^{\log}_{Y_{(2)}\{1,2\}} ;$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,G} \stackrel{\mathrm{def}}{=} \phi^{G}_{(r-1)} : G_K \xrightarrow{\sim} G_L \text{ ; and}$$

$$F^{\mathcal{G}}_{+1}(\phi_{(r-1)})^{\mathcal{G}\,\Pi^{\log}_{\mathbb{P}}} \stackrel{\mathrm{def}}{=} \phi^{\Pi^{\log}_{\mathbb{P}}}_{(r-1)} : \Pi^{\log}_{\mathbb{P}_K} \xrightarrow{\sim} \Pi^{\log}_{\mathbb{P}_L}$$

form an isomorphism $F^{\mathcal{G}}_{+1}(\phi_{(r-1)})$ of $\mathcal{D}^{\mathcal{G}}_{X_{(r)}}(\Sigma_X)$ with $\mathcal{D}^{\mathcal{G}}_{Y_{(r)}}(\Sigma_Y)$ of the desired type. $\square$

A Appendix

In this section, we prove the well-known fact that the category of ket coverings of a connected locally noetherian fs log scheme is *a Galois category*; this implies, in particular, the existence of log fundamental groups.

Definition A.1. Let P be a monoid. We shall say that P is *clean* if P is an fs monoid and $P^* = \{0\}$ (P^* is the set of invertible elements of P).

For example,

- $\mathbb{N}^{\oplus n}$
- the characteristic of an fs log scheme at any geometric point

are clean.

Definition A. 2. Let P be a torsion-free fs monoid. We shall denote by $(1/n)P$ the monoid $\{p \in P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \mid np \in \mathrm{Im}(P \hookrightarrow P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q})\}$. Then $(1/n)P$ is a torsion-free fs monoid. Moreover, if P is clean, then $(1/n)P$ is so.

Note that the natural inclusion $P \hookrightarrow P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ factors through $(1/n)P$. Thus, we always assume that $(1/n)P$ is a P-monoid via the natural inclusion $P \hookrightarrow (1/n)P$.

Note that the morphism

$$\begin{array}{ccc} (1/n)P & \longrightarrow & (1/n)P \\ p & \mapsto & np \end{array}$$

factors through P ($\subseteq (1/n)P$). Moreover, the resulting morphism $(1/n)P \to P$ is an isomorphism. We shall denote by $(1/n)_P$ the inverse isomorphism $P \to (1/n)P$.

Proposition A.1. *Let P be a torsion-free fs monoid, and Q a monoid. Then for any Kummer morphism $f : P \to Q$, there exists a positive number n such that the natural inclusion $P \hookrightarrow (1/n)P$ factors as a composite $P \xrightarrow{f} Q \xrightarrow{g} (1/n)P$. Moreover, then $n \cdot (1/n)P \subseteq \mathrm{Im} g$. If Q is torsion-free, then g is injective. In particular, g is Kummer.*

Proof. Since f is Kummer, there exists a positive natural number n such that $n \cdot Q \subseteq \mathrm{Im} f$. Thus, it follows from the injectivity of f that for any $q \in Q$, there exists a *unique* element $p_q \in P$ such that $nq = f(p_q)$. Now define $g : Q \to (1/n)P$ by $q \mapsto (1/n)_P(p_q)$. It is immediate that g is a homomorphism of monoids and $g \circ f(p) = p$ for any $p \in P$. Moreover, for any $(1/n)_P(p) \in (1/n)P$, $n((1/n)_P(p)) = p = g \circ f(p)$; hence $n((1/n)_P(p)) \in \mathrm{Im}\, g$.

It remains to show that if Q is torsion-free, then g is Kummer. If $g(q) = g(q')$, then $nq = nq'$. Since Q is torsion-free, $q = q'$; thus, g is injective. $\square$

Definition A.3. Let P be a monoid. We shall refer to an element $p \in P$ as *irreducible* if p satisfies the following:

If $p = p_1 + p_2$, then $p_1 = 0$ or $p_2 = 0$.

Proposition A.2. *Let P be a clean monoid.*

- (i) *The set of irreducible elements is the smallest set which generates P. In particular, the set is finite.*
- (ii) *The group of automorphisms of P is finite.*

Proof. First, we prove assertion (i). It follows immediately from the definition of irreducible elements that the set of irreducible elements is contained in any subset of P which generates P. Let $\{p_1, \cdots, p_r\} \subseteq P$ be a minimal set which generates P. Assume p_i is not irreducible. Then there exist natural numbers $n_i, \cdots, n_r$ such that $p_i = n_1p_1 + \cdots + n_rp_r$, and $2 \leq n_1 + \cdots + n_r$. If $n_i \neq 0$, then $n_1p_1 + \cdots + (n_i - 1)p_i + \cdots + n_rp_r = 0$. However, since $P^* = \{0\}$, we obtain a contradiction. Thus, $n_i = 0$. However, since we are operating under the assumption that $\{p_1, \cdots, p_r\} \subseteq P$ is a minimal set which generates P, we obtain a contradiction. Therefore, p_i is irreducible. This complete the proof of assertion (i).

Next, we prove assertion (ii). Since any automorphism of P preserves the irreducible elements of P, we obtain a natural homomorphism from the group of automorphisms of P to the group of permutations of the set of irreducible elements of P. Since the set of irreducible elements of P generates P by (i), this homomorphism is injective. On the other hand, since the set of irreducible elements of P is finite by (i), we conclude that the group of automorphism of P is also finite. $\square$

Proposition A.3.

(*i*) *Let L be a torsion-free finitely generated abelian group, and P a finitely generated submonoid of L. Then the submonoid $\tilde{P} = \{l \in L \mid nl \in P$ for some $n \in \mathbb{N}\}$ of L is finitely generated.*

(*ii*) *Let P be a torsion-free fs monoid, and Q a torsion-free saturated monoid. Let $f : P \to Q$ be a Kummer morphism. Then Q is finitely generated.*

Proof. First we prove assertion (i). Let us fix elements $p_1, \cdots, p_r \in P$ of P which generate P. We denote by C_P the cone in $L_{\mathbb{R}} \stackrel{\text{def}}{=} L \otimes_{\mathbb{Z}} \mathbb{R}$ generated by P (i.e., $C_P = \{c_1 p_1 + \cdots c_r p_r \in L_{\mathbb{R}} \mid c_i \in \mathbb{R}_{\geq 0}\}$). Then it is immediate that $\tilde{P} \subseteq C_P \cap L$ (in $L_{\mathbb{R}}$). Therefore, for any $l \in \tilde{P}$ there exist $n_i \in \mathbb{N}$ and $c_i \in [0, 1) \cap \mathbb{Q}$ such that

$$l = (n_1 + c_1) \cdot p_1 + \cdots (n_r + c_r) \cdot p_r \, .$$

Here, since the set $S = \{c_1 p_1 + \cdots c_r p_r \in \tilde{P} \mid c_i \in [0, 1)\}$ is contained in the intersection of L and a bounded subset of C_P, S is finite. Moreover, any element of $\tilde{P}$ is written by a sum of an element of P and an element of S; therefore, since $\tilde{P}$ is generated by $p_1, \cdots, p_r$ and this finite set S, $\tilde{P}$ is finitely generated.

Next, we prove assertion (ii). By Proposition A.1, the natural inclusion $P \to (1/n)P$ factors as a composite $P \xrightarrow{f} Q \xrightarrow{g} (1/n)P$ of f and a Kummer morphism g. By taking the groups associated to P, Q and $(1/n)P$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{f} & Q & \xrightarrow{g} & (1/n)P \\ \downarrow & & \downarrow & & \downarrow \\ P^{\mathrm{gp}} & \xrightarrow{f^{\mathrm{gp}}} & Q^{\mathrm{gp}} & \xrightarrow{g^{\mathrm{gp}}} & (1/n)P^{\mathrm{gp}} \, . \end{array}$$

Note that the all arrows in the above diagram are injective, and that Q^{gp} is a torsion-free finitely generated abelian group. Now we denote by $\tilde{P}$ the submonoid $\{q \in Q^{\mathrm{gp}} \mid nq \in P$ for some $n \in \mathbb{N}\}$ of Q^{gp}. I claim that $\tilde{P} = Q$. Indeed, if $\tilde{p} \in \tilde{P}$, then $\tilde{p} \in Q^{\mathrm{gp}}$ and $n\tilde{p} \in P \subseteq Q$. By the saturatedness of Q implies that $\tilde{p} \in Q$. If $q \in Q$, then by the Kummerness of f, $q \in \tilde{P}$; therefore $\tilde{P} = Q$. Thus, by (i), $\tilde{P} = Q$ is finitely generated. $\square$

Proposition A.4. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring A. Let us fix a global clean chart $P \to \mathcal{O}_X$ (see Definition 2.3). Then any connected ket covering of $X^{\log}$ is of the form $(X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])^{\log} \to X^{\log}$ where, $P \to Q$ is a Kummer morphism of fs monoids such that $nQ \subseteq \mathrm{Im}(P \to Q)$ for some integer n invertible on X, and the log structure of $(X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])^{\log}$ is induced by the natural morphism $Q \to \mathcal{O}_X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$. Conversely, if $Y^{\log} \to X^{\log}$ has this form, then it is a ket covering.*

Proof. The last assertion is immediate from the definition. Let $Y^{\log} \to X^{\log}$ be a connected ket covering. Since $Y \to X$ is finite, Y is affine. Let us write $Y = \mathrm{Spec}\, B$. Since $A \to B$ is finite and Y is connected, B is a strictly henselian local ring. By [10], Theorem 3.5, there exists an fs chart $Q \to B$ of $Y^{\log}$ and a chart

$$\begin{array}{ccc} \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, A \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, \mathbb{Z}[Q] & \longrightarrow & \mathrm{Spec}\, \mathbb{Z}[P] \end{array}$$

of $X^{\log} \to Y^{\log}$ such that the following conditions hold:

(i) $P \to Q$ is injective, and the cokernel of $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is finite and of order n invertible on A.

(ii) $\mathrm{Spec}\, B \to \mathrm{Spec}\, A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is étale.

(iii) $P \to Q/(Q \to B)^{-1}(B^*)$ is Kummer.

By conditions (i) and (iii), $P \to Q$ is Kummer, and satisfies $nQ \subseteq \mathrm{Im}(P \to Q)$.

Since $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ is finite, $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is a strictly henselian local ring. Thus, it follows from the fact that $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B$ is finite étale that $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is isomorphic to B. □

Proposition A.5. *Let X, Y and Z be locally noetherian fs log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ and $g^{\log} : Y^{\log} \to Z^{\log}$ morphisms, write $g^{\log} \circ f^{\log} = h^{\log}$. Then if $g^{\log}$ and $h^{\log}$ are ket coverings, then so is $f^{\log}$.*

Proof. The finiteness of f is clear. For the log étaleness of $f^{\log}$, we consider the following commutative diagram

$$\begin{tikzcd}
T'^{\log} \arrow[r, "s'^{\log}"] \arrow[d, "i^{\log}"'] & X^{\log} \arrow[d, "f^{\log}"] \\
T^{\log} \arrow[r, "s^{\log}"] & Y^{\log} \arrow[d, "g^{\log}"] \\
 & Z^{\log},
\end{tikzcd}$$

where $T'^{\log} \stackrel{i^{\log}}{\to} T^{\log}$ is an exact closed immersion defined by a quasi-coherent nilpotent $\mathcal{O}_T$-ideal. Since $h^{\log}$ is log étale, there exists $t^{\log} : T^{\log} \to X^{\log}$ such that $s'^{\log} = t^{\log} \circ i^{\log}$ and $g^{\log} \circ s^{\log} = h^{\log} \circ t^{\log} (= g^{\log} \circ f^{\log} \circ t^{\log})$. Now $g^{\log}$ is log étale; thus, $s^{\log} = f^{\log} \circ t^{\log}$. Therefore $f^{\log}$ is log étale.

For the Kummerness of $f^{\log}$, we take a geometric point $\overline{x} \to X$ of X. Let us write $P = (\mathcal{M}_X/\mathcal{O}_X^*)_{\overline{x}}$, $Q = (\mathcal{M}_Y/\mathcal{O}_Y^*)_{\overline{f(x)}}$ and $R = (\mathcal{M}_Z/\mathcal{O}_Z^*)_{\overline{h(x)}}$. Thus, we obtain the following diagram:

$$R \stackrel{(g^{\log})^*}{\longrightarrow} Q \stackrel{(f^{\log})^*}{\longrightarrow} P$$

Assume that $(f^{\log})^*(q) = (f^{\log})^*(q')$. Since it follows from the Kummerness of $(g^{\log})^*$ that there exsit a positive integer n and elements r, $r' \in R$ such that $(g^{\log})^*(r) = nq$ and $(g^{\log})^*(r') = nq'$, this implies that $(h^{\log})^*(r) = (h^{\log})^*(r')$. Thus, the injectivity of $(h^{\log})^*$ and the torsion-freeness of Q imply that $q = q'$. Hence $(f^{\log})^*$ is injective. Next, we take $p \in P$. Then it follows from the Kummerness of $h^{\log}$ that there exists an integer n such that $np \in \mathrm{Im}\,(h^{\log})^*$, hence $np \in \mathrm{Im}\,(f^{\log})^*$. Therefore, f^* is Kummer. $\square$

Proposition A.6. *A ket covering is an open and closed map. In particular, a connected ket covering over a connected fs log scheme is a surjection.*

Proof. This follows from Proposition A.4 and [8], Proposition 3.2. $\square$

Proposition A. 7. *Let $X^{\log}$ and $Y^{\log}$ be connected fs log schemes whose underlying schemes are the spectra of strictly henselian local rings, and $f^{\log}$: $X^{\log} \to Y^{\log}$ a ket covering. If the ket covering $f^{\log} : X^{\log} \to Y^{\log}$ has a section, then $f^{\log}$ is an isomorphism.*

Proof. This follows immediately from Proposition A.4. $\square$

Proposition A. 8. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring A whose residue field is*

k, and $(\operatorname{Spec} k)^{\log} = \overline{x}^{\log} \to X^{\log}$ *a strict geometric point over a geometric point of X for which the image of the underlying morphism of schemes is the closed point of X. Then $\overline{x}^{\log} \to X^{\log}$ induces an equivalence between the category of ket coverings of $X^{\log}$ and the category of ket coverings of $\overline{x}^{\log}$.*

Proof. It follows immediately from Proposition A.4 that the functor in question is essentially surjective, and full. Thus, we prove that the functor is faithful. Let $Y_1^{\log} \to X^{\log}$ and $Y_2^{\log} \to X^{\log}$ be ket coverings. Our claim is that the morphism

$$\phi : \operatorname{Hom}_{X^{\log}}(Y_1^{\log}, Y_2^{\log}) \longrightarrow \operatorname{Hom}_{\overline{x}^{\log}}(Y_1^{\log} \times_{X^{\log}} \overline{x}^{\log}, Y_2^{\log} \times_{X^{\log}} \overline{x}^{\log})$$

is injective. To show the injectivety of ϕ, we consider morphisms $f^{\log}$, $g^{\log} : Y_1^{\log} \to Y_2^{\log}$ over $X^{\log}$ which satisfy $f^{\log} \times_{X^{\log}} \overline{x}^{\log} = g^{\log} \times_{X^{\log}} \overline{x}^{\log} : Y_1^{\log} \times_{X^{\log}} \overline{x}^{\log} \to Y_2^{\log} \times_{X^{\log}} \overline{x}^{\log}$. Then, by Proposition A. 5, $f^{\log}$ and $g^{\log}$ are ket coverings. It is immediate that we may assume that Y_1 and Y_2 are connected. Now write

$$\Gamma_{f^{\log}} \stackrel{\mathrm{def}}{=} \mathrm{id}_{Y_1^{\log}} \times_{X^{\log}} f^{\log} : Y_1^{\log} \longrightarrow Y_1^{\log} \times_{X^{\log}} Y_2^{\log} ;$$

$$\Gamma_{g^{\log}} \stackrel{\mathrm{def}}{=} \mathrm{id}_{Y_1^{\log}} \times_{X^{\log}} g^{\log} : Y_1^{\log} \longrightarrow Y_1^{\log} \times_{X^{\log}} Y_2^{\log} .$$

Then since $\Gamma_{f^{\log}}$ and $\Gamma_{g^{\log}}$ are sections of the projection $Y_1^{\log} \times_{X^{\log}} Y_2^{\log} \to Y_1^{\log}$, and the projection is a ket covering (Proposition A. 5), $\Gamma_{f^{\log}}$ (respectively, $\Gamma_{g^{\log}}$) determines an isomorphism of $Y_1^{\log}$ with a connected component of $Y_1^{\log} \times_X Y_2^{\log}$ (Proposition A.6 and A.7). Thus, since $f^{\log} \times_{X^{\log}} \overline{x}^{\log} = g^{\log} \times_{X^{\log}} \overline{x}^{\log}$ we obtain $f^{\log} = g^{\log}$. $\square$

Proposition A.9. *Let $X^{\log}$ be an fs log scheme, $f^{\log} : Y^{\log} \to X^{\log}$ a ket covering, and $U_X \subseteq X$ (respectively, $U_Y \subseteq Y$) the interior of $X^{\log}$ (respectively, $Y^{\log}$). Then the projection $Y^{\log} \times_{X^{\log}} U_X \to Y^{\log}$ induces an isomorphism $Y^{\log} \times_{X^{\log}} U_X \simeq U_Y$.*

Proof. Since $U_X \to X^{\log}$ is a strict open immersion, $Y^{\log} \times_{X^{\log}} U_X \to Y^{\log}$ is an open immersion. Now since the log structure of U_X is trivial, the Kummerness of $Y^{\log} \times_{X^{\log}} U_X \to U_X$ implies that the log structure of $Y^{\log} \times_{X^{\log}} U_X$ is trivial. Thus, the open immersion $Y^{\log} \times_{X^{\log}} U_X \to Y^{\log}$ factors through U_Y. On the other hand, since the Kummerness of $f^{\log}$ implies that $f^{\log} |_{U_Y}$ factors through U_X, we obtaine that $Y^{\log} \times_{X^{\log}} U_X \simeq U_Y$. $\square$

Proposition A.10. *Let $X^{\log}$ be a log regular, quasi-compact fs log scheme and $U_X \subseteq X$ be the interior of $X^{\log}$. Then the morphism $U_X \to X^{\log}$ induces an equivalence of the category of ket coverings of $X^{\log}$ and the category of* *tamely ramified covering of U_X along $D_X = X \setminus U_X$. (We shall say that $V \to U_X$ is a tamely ramified covering along D_X, if $V \to U_X$ is finite étale, and at all points x of D_X with $\dim \mathcal{O}_{X,x} = 1$, the normalization of X in V is tamely ramified over x.)*

Proof. First, we prove that the morphism $U_X \hookrightarrow X^{\log}$ induces a functor from the category of ket coverings of $X^{\log}$ to the category of tamely ramified covering of U_X along D_X. Let $Y^{\log} \to X^{\log}$ be a connected ket covering and $\overline{x} \to X$ a geometric point of X. It follows from the Kummerness of $Y^{\log} \to X^{\log}$ that if the log structure of $X^{\log}$ at $\overline{x}$ is trivial, then the log structure of $Y^{\log}$ at any geometric points over $\overline{x}$ is trivial. Therefore, since a log étale morphism from a log scheme equipped with the trivial log structure to a log scheme equipped with the trivial log structure is étale, $Y^{\log} \times_{X^{\log}} U_X \to U_X$ is finite étale. Next, we will prove the tameness of $Y^{\log} \times_{X^{\log}} U_X \to U_X$. By base-changing, we may assume that X is the spectrum of a strictly henselian discrete valuation ring. Then it follows immediately from Proposition A.4 that $Y^{\log} \to X^{\log}$ is tamely ramified. This completes the proof of that the morphism $U_X \hookrightarrow X^{\log}$ induces a functor from the category of ket coverings of $X^{\log}$ to the category of tamely ramified covering of U_X along D_X.

Next, we show that this functor is fully faithful. Let $Y_1^{\log} \to X^{\log}$ and $Y_2^{\log} \to X^{\log}$ be ket coverings. Our claim is that the morphism

$$\phi : \mathrm{Hom}_{X^{\log}}(Y_1^{\log}, Y_2^{\log}) \longrightarrow \mathrm{Hom}_{U_X}(Y_1^{\log} \times_{X^{\log}} U_X, Y_2^{\log} \times_{X^{\log}} U_X) = \mathrm{Hom}_{U_X}(U_{Y_1}, U_{Y_2})$$

is an isomorphism, where U_{Y_1} (respectively, U_{Y_2}) is the interior of Y_1 (respectively, Y_2). The last equation follows from Proposition A.9. To show the injectivity of ϕ, let $f^{\log}$, $g^{\log} : Y_1^{\log} \to Y_2^{\log}$ be ket coverings over $X^{\log}$ such that $f^{\log} |_{U_{Y_1}} = g^{\log} |_{U_{Y_1}} : U_{Y_1} \to U_{Y_2}$. Now since $X^{\log}$ is log regular, and $Y_1^{\log} \to X^{\log}$ and $Y_2^{\log} \to X^{\log}$ are log étale, $Y_1^{\log}$ $Y_2^{\log}$ are log regular ([11], Theorem 8.2). Therefore, $U_{Y_1} \subseteq Y_1$ (respectively, $U_{Y_2} \subseteq Y_2$) is dense open subset of Y_1 (respectively, Y_2). Thus, $f^{\log} |_{U_{Y_1}} = g^{\log} |_{U_{Y_1}}$ implies $f = g$. Now since $Y_1^{\log}$ (respectively, $Y_2^{\log}$) is log regular, the log structure of $Y_1^{\log}$ (respectively, $Y_2^{\log}$) is $\mathcal{O}_{Y_1} \cap (U_{Y_1} \hookrightarrow Y_1)_* \mathcal{O}^*_{U_{Y_1}} \hookrightarrow \mathcal{O}_{Y_1}$ (respecrively, $\mathcal{O}_{Y_2} \cap (U_{Y_2} \hookrightarrow Y_2)_* \mathcal{O}^*_{U_{Y_2}} \hookrightarrow \mathcal{O}_{Y_2}$). Therefore, a morphism from Y_1 to Y_2 of log schemes determined by the underlying morphism of schemes. In other words, $f = g$ implies $f^{\log} = g^{\log}$; we thus conclude that ϕ is injective. Next, to show the surjectivity of ϕ, Let $f_U : U_{Y_1} \to U_{Y_2}$ be a morphism over U_X. Since the normalization of X in U_{Y_1} (respecrively, U_{Y_2}) is Y_1 (respecrively, Y_2), the morphism f_U extends to a morphism $f : Y_1 \to Y_2$. By an argument similar to the argument used to prove the injectivity of ϕ, a morphism from $Y_1^{\log}$ to $Y_2^{\log}$ of log schemes determined by the underlying morphism of schemes. Therefore $f : Y_1 \to Y_2$ extends to a morphism $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ of log schemes. We thus conclude that ϕ is surjective.

Finally, we show the essential surjectivity of this functor. Let $V \to U_X$ be a tamely ramified covering along D_X. Then, by the log purity theorem in [14], this covering extends to a ket covering over $X^{\log}$. (See Remark 2.10.) $\square$

Proposition A.11. *Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log}, g^{\log} : X^{\log} \to Y^{\log}$ morphisms of log schemes such that $f = g$. Let $\overline{x} \to X$ be a geometric point of X (we denote the image by $x \in X$). If there exist a log scheme ${X'}^{\log}$, a morphism $h^{\log} : {X'}^{\log} \to X^{\log}$ and a geometric point $\overline{x}' \to X'$ (we denote the image by $x' \in X'$) for which the image of the composite $\overline{x}' \to X' \xrightarrow{h} X$ is x such that the following conditions hold, then $f^{\log}$ coincides with $g^{\log}$ on an étale neighborhood of $\overline{x} \to X$:*

(*i*) *h is flat at $x' \in X'$.*

(*ii*) *The homomorphism $(\mathcal{M}_X/\mathcal{O}_X^*)_{\overline{x}} \to (\mathcal{M}_{X'}/\mathcal{O}_{X'}^*)_{\overline{x}'}$ induced by $h^{\log}$ is injective.*

(*iii*) *$f^{\log} \circ h^{\log}$ coincides with $g^{\log} \circ h^{\log}$ on an étale neighborhood of $\overline{x}' \to X'$.*

Proof. We denote by $\overline{y} \to Y$ the geometric point determined by the composite $\overline{x} \to X \xrightarrow{f=g} Y$. Then it is immediate that it is enough to show that the homomorphism $\mathcal{M}_{Y,\overline{y}} \to \mathcal{M}_{X,\overline{x}}$ induced by $f^{\log}$ coincides with the homomorphism $\mathcal{M}_{Y,\overline{y}} \to \mathcal{M}_{X,\overline{x}}$ induced by $g^{\log}$. Now, in the following diagram induced by $h^{\log}$

$$\begin{array}{ccccc}
\mathcal{O}^*_{X,\overline{x}} & \longrightarrow & \mathcal{M}_{X,\overline{x}} & \longrightarrow & (\mathcal{M}_X/\mathcal{O}^*_X)_{\overline{x}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}^*_{X',\overline{x}'} & \longrightarrow & \mathcal{M}_{X',\overline{x}'} & \longrightarrow & (\mathcal{M}_{X'}/\mathcal{O}^*_{X'})_{\overline{x}'}\,,
\end{array}$$

since the left-hand vertical arrow is injective (by assumption (i)), and the right-hand vertical arrow is injective (by assumption (ii)), we obtain that the homomorphism $\mathcal{M}_{X,\overline{x}} \to \mathcal{M}_{X',\overline{x}'}$ is injective. Therefore, by assumption (iii), the homomorphism $\mathcal{M}_{Y,\overline{y}} \to \mathcal{M}_{X,\overline{x}}$ induced by $f^{\log}$ coincides with the homomorphism $\mathcal{M}_{Y,\overline{y}} \to \mathcal{M}_{X,\overline{x}}$ induced by $g^{\log}$. $\square$

Proposition A.12. *A strict étale surjection is a strict epimorphism in the category of log schemes.*

Proof. Let $X^{\log}$, $Y^{\log}$ and $Z^{\log}$ be log schemes, $f^{\log} : Y^{\log} \to X^{\log}$ a strict, étale surjection, and $p_1^{\log}$ (respectively, $p_2^{\log}$) the 1-st (respectively, 2-nd) projection $Y^{\log} \times_{X^{\log}} Y^{\log} \to Y^{\log}$. Note that our claims are

(i) the morphism $\mathrm{Hom}(X^{\log}, Z^{\log}) \to \mathrm{Hom}(Y^{\log}, Z^{\log})$ induced by $f^{\log}$ is injective; and

(ii) if a morphism $g^{\log} : Y^{\log} \to Z^{\log}$ satisfies the equality $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, then $g^{\log}$ extends to a morphism $X^{\log} \to Z^{\log}$.

(i) follows immediately from Proposition A.11. (ii) may be verified as follows: Since $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, we obtain that $g \circ p_1 = g \circ p_2$. Since a strict étale morphism is a strict epimorphism in the category of schemes, it thus follows that there exists an extension $\tilde{g} : X \to Z$ of g (i.e., $g \circ f = \tilde{g}$). Moreover, since $\mathcal{M}_X$ is a sheaf on the étale site of X, and $Y^{\log} \to X^{\log}$ strict étale surjection, it thus follows from the fact that the morphism $(g \circ p_1)^{-1}\mathcal{M}_Z \to \mathcal{M}$ (where $\mathcal{M}$ is the sheaf of monoids which determines the log structure of $Y^{\log} \times_{X^{\log}} Y^{\log}$) coincides with the morphism $(g \circ p_2)^{-1}\mathcal{M}_Z \to \mathcal{M}$ that the morphism $g^{-1}\mathcal{M}_Z \to \mathcal{M}_Y$ extends to a morphism $\tilde{g}^{-1}\mathcal{M}_Z \to \mathcal{M}_X$. This completes the proof of (ii). □

Proposition A.13. *Let $X^{\log}$ be a locally noetherian fs log scheme. Then for a morphism $f^{\log}$ in the category of ket coverings of $X^{\log}$, $f^{\log}$ is a strict epimorphism in the category of ket coverings of $X^{\log}$ if and only if $f^{\log}$ is a surjection.*

Proof. It is immediate that if $f^{\log}$ is not surjective, then $f^{\log}$ is not a strict epimorphism in the category of ket coverings of $X^{\log}$. Thus, assume $f^{\log}$ is surjective.

(Step 1) The case where X is the spectrum of a strictly henselian ring.

Then, by Proposition A.8, by base-changing, we may assume that X is the spectrum of a separably closed field k. Let us fix a clean chart $P \to k$ of $X^{\log}$. Now we denote by $\hat{X}^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k[[P]]$ with the log structure defined by the natural morphism $P \to k[[P]]$. Then the following hold:

- $\hat{X}^{\log}$ is log regular ([11], Theorem 3.1)
- The natural surjection $k[[P]] \to k[[P]]/\mathfrak{m} \simeq k$ (where $\mathfrak{m} \subseteq k[[P]]$ is the maximal ideal of $k[[P]]$) induces the strict morphism $X^{\log} \to \hat{X}^{\log}$.
- The strict morphism $X^{\log} \to \hat{X}^{\log}$ induces a natural equivalence between $\mathrm{K\acute{e}t}(\hat{X}^{\log})$ and $\mathrm{K\acute{e}t}(X^{\log})$ (Proposition A.8).

Thus, by replacing $X^{\log}$ by $\hat{X}^{\log}$, we may assume that $X^{\log}$ is log regular. Moreover, if we denote by $U_X \subseteq X$ the interior of $X^{\log}$, then the strict morphism $U_X \to X^{\log}$ induces a natural equivalence between $\mathrm{Két}(X^{\log})$ and $\mathrm{Két}(U_X)$ (Proposition A.10). In $\mathrm{Két}(U_X)$, a surjection is faithfully flat, thus, strict eqimorphim (in the category of ket coverings of U_X).

(Step 2) The general case.

Let $Y_1^{\log} \to X^{\log}$, $Y_2^{\log} \to X^{\log}$ and $Z^{\log} \to X^{\log}$ ket coverings, $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ a surjection over $X^{\log}$, and $p_1^{\log}$ (respectively, $p_2^{\log}$) the 1-st (respectively, 2-nd) projection $Y_1^{\log} \times_{Y_2^{\log}} Y_1^{\log} \to Y_1^{\log}$. Note that our claims are

(i) the morphism $\mathrm{Hom}_{X^{\log}}(Y_2^{\log}, Z^{\log}) \to \mathrm{Hom}_{X^{\log}}(Y_1^{\log}, Z^{\log})$ induced by $f^{\log}$ is injective;

(ii) if a morphism $g^{\log} : Y_1^{\log} \to Z^{\log}$ satisfies the equality $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, then $g^{\log}$ extends to a morphism $Y_2^{\log} \to Z^{\log}$.

First, we prove assertion (i). Let $g_1^{\log}$ and $g_2^{\log} : Y_2^{\log} \to Z^{\log}$ be morphisms over $X^{\log}$ such that $g_1^{\log} \circ f^{\log} = g_2^{\log} \circ f^{\log}$. Then, by Step 1, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g_1'^{\log}$ obtained by base-changing of $g_1^{\log}$ by $X'^{\log} \to X^{\log}$ coincides with the morphism $g_2'^{\log}$ obtained by base-changing of $g_2^{\log}$ by $X'^{\log} \to X^{\log}$. On the other hand, since a strict étale surjection is a strict epimorphism (by Proposition A.12), we obtain that $g_1^{\log} = g_2^{\log}$. This completes the proof of assertion (i).

Next, we prove assertion (ii). By Step 1, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g'^{\log}$ obtained by base-changing of $g^{\log}$ by $X'^{\log} \to X^{\log}$ extends to a morphism $\tilde{g'}^{\log} : Y_2'^{\log} \overset{\mathrm{def}}{=} Y_2^{\log} \times_{X^{\log}} X'^{\log} \to Z'^{\log} \overset{\mathrm{def}}{=} Z^{\log} \times_{X^{\log}} X'^{\log}$. Now if we denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1-st (respectively, 2-nd) projection $Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \to Y_2'^{\log}$, then the composite

$$Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \xrightarrow{q_1^{\log}} Y_2'^{\log} \xrightarrow{\tilde{g'}^{\log}} Z'^{\log} \longrightarrow Z^{\log}$$

coincides with the composite

$$Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \xrightarrow{q_2^{\log}} Y_2'^{\log} \xrightarrow{\tilde{g'}^{\log}} Z'^{\log} \longrightarrow Z^{\log}.$$

Therefore, by Proposition A.12, the composite $Y_2'^{\log} \overset{\tilde{g'}^{\log}}{\to} Z'^{\log} \to Z^{\log}$ extends to a morphism $\tilde{g}^{\log} : Y_2^{\log} \to Z^{\log}$ (note that $Y_2'^{\log} \to Y_2^{\log}$ is a strict étale surjection). This complete the proof of assertion (ii). $\square$

Theorem A.1. *Let $X^{\log}$ be a connected locally noetherian fs log scheme and $\tilde{x}^{\log} \to X^{\log}$ a log geometric point of $X^{\log}$. For the category Két($X^{\log}$) of ket coverings of $X^{\log}$ and $X^{\log}$-morphisms, we denote by $F = F_{\tilde{x}^{\log}}$ the functor*

$$\begin{array}{ccc} \text{Két}(X^{\log}) & \longrightarrow & \text{(the category of finite sets)} \\ (Y^{\log} \to X^{\log}) & \mapsto & \{\text{log geometric points of } \mathrm{Y}^{\log} \text{ over } \tilde{x}^{\log} \to X^{\log}\}. \end{array}$$

Then (Két($X^{\log}$), F) is a Galois category.

Note that, by Proposition A.4, the set

$$\{\text{log geometric points of } \mathrm{Y}^{\log} \text{ over } \tilde{x}^{\log} \to X^{\log}\}$$

is finite. We must verify that (Két($X^{\log}$), F) satisfies the conditions $(\mathcal{G}_1), \ldots, (\mathcal{G}_5)$ and $(\mathcal{G}_6)$ in definition of Galois category in [7], Exposé V, 4.

$(\mathcal{G}_1)$ Két($X^{\log}$) has a final object and there exists a fiber product in Két($X^{\log}$).

Proof. It is immediate that $X^{\log}$ is a final object of Két($X^{\log}$). Next, we will prove the existence of a fiber product. Since any object $Y^{\log}$ of Két($X^{\log}$) is a fs log scheme, for the existence of a fiber product, it is enough to show that finiteness, log étaleness and Kummerness is stable under composition and base-change. The assertion for finiteness is classical, the assertion for log étaleness and Kummerness follows immediately from the definition. $\square$

$(\mathcal{G}_2)$ There exists a finite sum in Két($X^{\log}$). Moreover, if $f^{\log} : Y^{\log} \to X^{\log}$ is an object of Két($X^{\log}$) and G is a finite group of automorphisms of $Y^{\log}$ in Két($X^{\log}$), then there exists a quotient $Y^{\log}/G$ of $Y^{\log}$ by G in Két($X^{\log}$), and the natural morphism $Y^{\log} \to Y^{\log}/G$ is a strict epimorphism.

Proof. The existence of finite sums is immediate by the definition of a ket covering. In the following, we prove the existence of quotients. By Lemma 5.20, by base-changing, we may assume that the underlying scheme X of $X^{\log}$ is the spectrum of a strictly henselian local ring. Moreover, by a similar argument to the argument used in the proof of Proposition A.13, (Step 1), we may assume that there exists a separably closed field k and a clean monoid P such that the underlying scheme X of $X^{\log}$ is the spectrum of $k[[P]]$, and the log structure of $X^{\log}$ is the log structure induced by the natural morphism $P \to k[[P]]$. Moreover, by taking a connected component of Y and the stabilizer of the connected component with respect to the action of G on the set of connected components of Y, we may assume that Y is connected. Then, by Proposition A.4, there exists a clean monoid Q, and a Kummer morphism $u : P \to Q$ such that Y is isomorphic to $\operatorname{Spec}(k[[P]] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) \simeq \operatorname{Spec} k[[Q]]$ ($k[[P]] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \simeq k[[Q]]$ follows from the Kummerness of u), the log structure of $Y^{\log}$ is the log structure induced by the natural morphism $Q \to k[[Q]]$ and the morphism $Y^{\log} \to X^{\log}$ is determined by u. Now we have a following commutative diagram:

$$\begin{array}{ccccccc}
P & = & \mathcal{M}_X(X)/k[[P]]^* & \longleftarrow & \mathcal{M}_X(X) & \longrightarrow & k[[P]] \\
& & \downarrow & & \downarrow & & \downarrow \\
u\downarrow & & \tilde{Q} & \longleftarrow & \mathcal{M}_Y(Y)^G & \xrightarrow{\alpha^G} & k[[Q]]^G \\
& & \downarrow & & \downarrow & & \downarrow \\
Q & = & \mathcal{M}_Y(Y)/k[[Q]]^* & \longleftarrow & \mathcal{M}_Y(Y) & \longrightarrow & k[[Q]]
\end{array}$$

where $\tilde{Q} \stackrel{\text{def}}{=} \mathcal{M}_Y(Y)^G/(\alpha^G)^{-1}(k[[Q]]^G)^*$.

Let $Q \to \mathcal{M}_Y(Y)$ be a clean chart of $Y^{\log}$. Then the chart induces a (non-canonical) *splitting* $k[[Q]]^* \oplus Q \xrightarrow{\sim} \mathcal{M}_Y(Y)$. Since the action of G on $Y^{\log}$ is *over* $X^{\log}$, and $u : P \to Q$ is Kummer, for any $g \in G$, there exists $\sigma_g(q) \in k[[Q]]^*$ such that $(f, q)^g = (\sigma_g(q) \cdot f^g, q)$ ($(f, q) \in k[[Q]]^* \oplus Q \xrightarrow{\sim} \mathcal{M}_Y(Y)$); therefore, for $(f, q) \in \mathcal{M}_Y(Y)$, $(f, q) \in \mathcal{M}_Y(Y)^G$ if and only if $f = \sigma_g(q) \cdot f^g$ for any $g \in G$. Note that it is immediate that

$$\begin{array}{ccc}
Q & \longrightarrow & k[[Q]]^* \\
q & \mapsto & \sigma_g(q)
\end{array}$$

is a homomorphism; moreover, since $\sigma_g(p) = 1$ for any P, we obtain that $\sigma_g(q)$ is a root of $1 \in k[[Q]]$.

Now I claim that

$$\mathcal{M}_Y(Y)^G = \{(f, q) \mid f \in (k[[Q]]^*)^G,\ \sigma_g(q) = 1 \text{ for any } g \in G\},$$

i.e., if we denote by $Q^{[G]}$ the submonoid of Q of elements which satisfy $\sigma_g(q) = 1$ for any $g \in G$, then $\mathcal{M}_Y(Y)^G = (k[[Q]]^*)^G \oplus Q^{[G]}$, and the natural surjection $\mathcal{M}_Y(Y)^G \to \tilde{Q}$ induces an isomorphism $Q^{[G]} \xrightarrow{\sim} \tilde{Q}$. Indeed, since $k[[Q]]$ is a local k-algebra whose residue field is k, we have a *split* exact sequence:

$$0 \longrightarrow \mathfrak{m} \longrightarrow k[[Q]] \longrightarrow k \longrightarrow 0\,,$$

where $\mathfrak{m}$ is the maximal ideal of $k[[Q]]$; i.e., $\mathfrak{m} \oplus k \xrightarrow{\sim} k[[Q]]$. Thus, for $f \in k[[Q]]^*$, there exists $t \in \mathfrak{m}$ and $a \in k$ such that $f = t + a$. Let g be an element of G. Then since the action of G on $Y^{\log}$ is *over* $X^{\log}$, $f^g = t^g + a$

and $t^g \in \mathfrak{m}$. If $(f, q) \in \mathcal{M}_Y(Y)^G$, then $f^g = \sigma_g(q) \cdot f$. Thus, $t^g + a = \sigma_g(q) \cdot (a + t)$; therefore, $\sigma_g(q) = 1$ and $t^g = t$. (Here we use the fact that since $\sigma_g(q)$ is a root of $1 \in k[[Q]]$; in particular, $\sigma_g(q) \in k^*$.) This completes the proof of the above claim. In particular, $\mathcal{M}_Y(Y)^G \to k[[Q]]^G$ is a log structure on $\operatorname{Spec} k[[Q]]^G$ (i.e., $(\alpha^G)^{-1}(k[[Q]]^G)^* = (k[[Q]]^G)^*$), whose characteristic $\tilde{Q}$ $[= Q^{[G]}]$ is a submonoid of Q (thus, $\tilde{Q}$ is integral and torsion-free) and the log structure coincides with the log structure induced by the morphism $u^{[G]} : Q^{[G]} \hookrightarrow \mathcal{M}_Y(Y)^G \to k[[Q]]^G$. Now we shall denote by $Y'^{\log}$ the log scheme obtained by equipping $\operatorname{Spec} k[[Q]]^G$ with this log structure $\mathcal{M}_Y(Y)^G \to [[Q]]^G$. Note that it follows from the definition of $Q^{[G]}$ that $Q^{[G]}$ is saturated. Therefore, by Proposition A.3, (ii), $Q^{[G]}$ is fs; thus, $Y'^{\log}$ is an fs log scheme.

Next, I claim that the (clean) chart $u^{[G]} : Q^{[G]} \hookrightarrow \mathcal{M}_Y(Y)^G \to k[[Q]]^G$ obtained as above induces an isomorphism $v : k[[Q^{[G]}]] \xrightarrow{\sim} k[[Q]]^G$. Since $Q^{[G]}$ and Q are Kummer over P, to show this, it is enough to show that the natural morphism $v' : k[Q^{[G]}] \to k[Q]^G$, which satisfies $v' \otimes_{k[P]} k[[P]] = v$, is an isomorphism. Indeed, the claim may be verified as follows: As a k-module, $k[Q^{[G]}]$ (respectively, $k[Q]$) is freely generated by $q' \in Q^{[G]}$ (respectively, $q \in Q$). On the other hand, by the definition of σ_g, for $q \in k[Q]$, we obtain that $q^g = \sigma_g(q) \cdot q$. Then the above claim follows from this observation.

Therefore, we conclude that the fs log scheme $Y'^{\log}$ is the log scheme obtained by equipping $\operatorname{Spec} k[[Q^{[G]}]]$ with the log structure induced by the natural morphism $Q^{[G]} \to k[[Q^{[G]}]]$. In particular, by Proposition A. 4, $Y'^{\log} \to X^{\log}$ is a ket covering. Moreover, by the construction of $Y'^{\log}$, it is immediate that the ket covering $Y'^{\log} \to X^{\log}$ is a quotient of the action of G on the ket covering $Y^{\log} \to X^{\log}$ in $\text{Két}(X^{\log})$. Finally, by Proposition A.13, the natural morphism $Y^{\log} \to Y^{\log}/G$ is strict epimorphism. □

($\mathcal{G}_3$) Any morphism $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ in $\text{Két}(X^{\log})$ admits a factorization $Y_1^{\log} \xrightarrow{f'^{\log}} Y_2'^{\log} \xrightarrow{g^{\log}} Y_2^{\log}$, where $f'^{\log}$ is a strictly epimorphism and $g^{\log}$ is a monomorphism. Moreover, then $Y_2^{\log} = Y_2'^{\log} \sqcup Z^{\log}$ (disjoint union) for some object $Z^{\log}$ of $\text{Két}(X^{\log})$.

Proof. This follows immediately from Proposition A.6 and A.13. □

($\mathcal{G}_4$) F is left exact.

Proof. Let $Y^{\log}$ be an object of $\text{Két}(X^{\log})$ and $\overline{y} \to Y$ a geometric point of Y. Then any log geometric point $\tilde{y}^{\log}$ of $Y^{\log}$ over the geometric point $\overline{y} \to Y$ factors through a reduced covering point $\overline{y}_1^{\log} \to Y^{\log}$ over the geometric point $\overline{y} \to Y$. Thus, since a fiber product in $\text{Két}(X^{\log})$ is a fiber product in the category of fs log schemes and $F(Y^{\log})$ is finite, F commutes with the operation of taking fiber product. $\square$

($\mathcal{G}_5$) F commutes with the operation of taking a finite sum and the quotient by a action of a finite group (cf, ($\mathcal{G}_2$)). Moreover, if $f^{\log}$ is a strict epimorphism, then $F(f^{\log})$ is surjective.

Proof. The assertion for a finite sum is immediate. The assertion for quotient follows from a similar argument to the argument used in the proof of ($\mathcal{G}_4$). The assertion for a strict epimorphism follows from Proposition A.13 and the definition of a log geometric point. $\square$

($\mathcal{G}_6$) If $f^{\log}$ is a morphism in $\mathrm{K\acute{e}t}(X^{\log})$, then $f^{\log}$ is an isomorphism if and only if $F(f^{\log})$ is an isomorphism.

Proof. For this assertion, by base-changing, we may assume that X is the spectrum of a strictly henselian local ring, and the image of the underlying morphism of scheme of the log geometric point $\tilde{x}^{\log} \to X^{\log}$ is a closed point of X. Then the assertion follows immediately from Proposition A.4. $\square$

Theorem A. 2. *Let $X^{\log}$ and $Y^{\log}$ be connected locally noetherian fs log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then the functor*

$$\begin{array}{ccc} \mathrm{K\acute{e}t}(Y^{\log}) & \xrightarrow{(f^{\log})^*} & \mathrm{K\acute{e}t}(X^{\log}) \\ (Y'^{\log} \to Y^{\log}) & \mapsto & (Y'^{\log} \times_{Y^{\log}} X^{\log} \to X^{\log}) \end{array}$$

induced by $f^{\log}$ is exact. In particular, (by [7], Exposé V, Corollaire 6.2) for any log geometric point $\tilde{x}^{\log} \to X^{\log}$ of $X^{\log}$, the functor $(f^{\log})^*$ induces a morphism

$$\pi_1(f^{\log}) : \pi_1(X^{\log}, \tilde{x}^{\log}) \to \pi_1(Y^{\log}, f^{\log}(\tilde{x}^{\log}))\,,$$

where $f^{\log}(\tilde{x}^{\log}) \to Y^{\log}$ is the log geometric point obtained as the composite $\tilde{x}^{\log} \to X^{\log} \xrightarrow{f^{\log}} Y^{\log}$.

Proof. Let $\tilde{x}^{\log} \to X^{\log}$ be a log geometric point of $X^{\log}$. Then, by [7], Exposé V, Proposition 6.1, it is enough to show that the composite of functor

$$\mathrm{K\acute{e}t}(Y^{\log}) \xrightarrow{(f^{\log})^*} \mathrm{K\acute{e}t}(X^{\log}) \xrightarrow{F_{\tilde{x}^{\log}}} (\text{the category of finite sets})$$

is a fundamental functor over $\mathrm{K\acute{e}t}(Y^{\log})$. Now, by the definitions of $(f^{\log})^*$ and $F_{\tilde{x}^{\log}}$, for any ket covering $Y'^{\log} \to Y^{\log}$, $F_{\tilde{x}^{\log}} \circ (f^{\log})^*(Y'^{\log} \to Y^{\log}) = F_{f^{\log}(\tilde{x}^{\log})}(Y'^{\log} \to Y^{\log})$, i.e., $F_{\tilde{x}^{\log}} \circ (f^{\log})^* = F_{f^{\log}(\tilde{x}^{\log})}$. By Theorem A.1, the functor $F_{f^{\log}(\tilde{x}^{\log})}$ is a fundamental functor over $\mathrm{K\acute{e}t}(Y^{\log})$. This completes the proof of Thereom A.2. $\square$

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