# ON THE CUSPIDALIZATION PROBLEM FOR HYPERBOLIC CURVES OVER FINITE FIELDS

YASUHIRO WAKABAYASHI

Abstract. In this paper, we study some group-theoretic constructions associated to arithmetic fundamental groups of hyperbolic curves over finite fields. One of the main results of this paper asserts that any Frobenius-preserving isomorphism between the geometrically pro-$l$ fundamental groups of hyperbolic curves with one given point removed induces an isomorphism between the geometrically pro-$l$ fundamental groups of the hyperbolic curves obtained by removing other points. Finally, we apply this result to obtain results concerning certain cuspidalization problems for fundamental groups of (not necessarily proper) hyperbolic curves over finite fields.

## Contents



## Introduction

In the present paper, we consider the following problem:

**Problem.**
*Suppose that we are given a hyperbolic curve over a finite field in which $l$ is invertible. Then, given the geometrically pro-$l$ fundamental group of the curve obtained by removing a* **specific** *point from this hyperbolic curve, is it possible to reconstruct the geometrically pro-$l$ fundamental groups of the curves obtained by removing other points which* **vary** *"continuously" in a suitable sense?*


We shall formulate the above problem mathematically.
Let $l$ be a prime number, $X$ a hyperbolic curve over a finite field $K$ in which $l$ is invertible. For $n$ a positive integer, we denote by $X_n$ the $n$-th configuration space associated to $X$ (hence, $X_1 = X$), and write $\Pi_{X_n}$ for the geometrically pro-$l$ fundamental group of $X$. Here, the fiber of $X_2 \to X$ over a $K$-rational point $x \in X$ may be naturally identified with $X \backslash \{x\}$, so we may regard $X_2 \to X$ as a *continuous family of cuspidalizations of* $X$. Therefore, the above problem can be formulated as follows (where $Y$ denotes a hyperbolic curve over a finite field $L$ in which $l$ is also invertible, and we use similar notations for $Y$ to the notations used for $X$):

**Theorem A.**

*Let*

$$\alpha : \Pi_{X\backslash\{x\}} \xrightarrow{\sim} \Pi_{Y\backslash\{y\}}$$

*be a Frobenius-preserving isomorphism [cf. Definition 3.5] which maps the decomposition group $D_x$ of $x$ (well-defined up to $\Pi_{X\backslash\{x\}}$-conjugacy) onto the decomposition group $D_y$ of $y$ (well-defined up to $\Pi_{Y\backslash\{y\}}$-conjugacy). Here, we shall denote by $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$ (resp., $\overline{D}_x$, $\overline{D}_y$) the isomorphism (resp., the decomposition group of $x$ in $\Pi_X$, the decomposition group of $y$ in $\Pi_Y$) obtained by passing to the quotients $\Pi_{X\backslash\{x\}} \twoheadrightarrow \Pi_X$, $\Pi_{Y\backslash\{y\}} \twoheadrightarrow \Pi_Y$.*

*Then there exists a unique isomorphism*

$$\alpha_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$$

*which is compatible with the natural switching automorphisms up to an inner automorphism and fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y \end{array}$$

*that induces $\alpha$ by restricting $\alpha_2$ to the inverse images (via the vertical arrows) of $\overline{D}_x$ and $\overline{D}_y$.*

*In particular, if $x'$ (resp., $y'$) is a $K$-rational point of $X$ (resp., an $L$-rational point of $Y$), and we assume that the decomposition groups of $x'$, $y'$ correspond* *via $\alpha$, then we have an isomorphism*

$$\alpha' : \Pi_{X\setminus\{x'\}} \xrightarrow{\sim} \Pi_{Y\setminus\{y'\}}$$

*such that $\alpha$ and $\alpha'$ induce the same isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$.*

In Section 1, we recall the notion of the (log) configuration space associated to a hyperbolic curve and review group-theoretic properties of the various fundamental groups associated to such spaces. In particular, the splitting determined by the Frobenius action on the pro-$l$ étale fundamental group $\Delta_{X_n}$ of $X_n \otimes_K \overline{K}$ gives rise to an explicit description of the graded Lie algebra obtained by considering the weight filtration on $\Delta_{X_n}$ (cf. Definition 1.6). This explicit description will play an essential role in the proof of Theorem A.

In Section 2, we discuss a certain *specific choice* (among composites with inner automorphisms) of the morphism between geometrically pro-$l$ fundamental groups obtained by switching the two ordered marked points parametrized by the second configuration space. This choice will play a key role in the proof of Theorem A.

Section 3 is devoted to proving Theorem A. Roughly speaking, starting from a given geometrically pro-$l$ fundamental group $\Pi_{X\setminus\{x\}}$, we reconstruct group-theoretically a suitable topological group, i.e., $\Pi_{X_2}^{\mathrm{Lie}}$ (cf. Definition 3.1), which contains the geometrically pro-$l$ fundamental group of the second configuration space, by using the explicit description of graded Lie algebra studied in Section 1. Next, we reconstruct the automorphism on $\Pi_{X_2}^{\mathrm{Lie}}$ induced by the specific choice of the switching morphism studied in Section 2. Finally, we verify that $\Pi_{X_2}$ can be generated, as a subgroup of $\Pi_{X_2}^{\mathrm{Lie}}$, by the given fundamental group $\Pi_{X\setminus\{x\}}$ and the image of this fundamental group via the specific choice of the switiching morphism studied in Section 2; this allows us to reconstruct $\Pi_{X_2}$ as a subgroup of $\Pi_{X_2}^{\mathrm{Lie}}$.

In Section 4, as an application of (a slightly generalized version of) Theorem A, we give a group-theoretic construction of the cuspidalization of an affine hyperbolic curve $X$ over a finite field at a point *"infinitesimally close"* to the cusp $x$. That is to say, we give a construction, starting from the geometrically pro-$l$ fundamental group $\Pi_X$ of $X$, of the geometrically pro-$l$ fundamental group $\Pi_{\overline{X}_x^{\log}}$ of the log scheme obtained by gluing $X$ to a tripod (i.e., the projective line minus three points) at a cusp $x$ of $X$:

**Theorem B.**

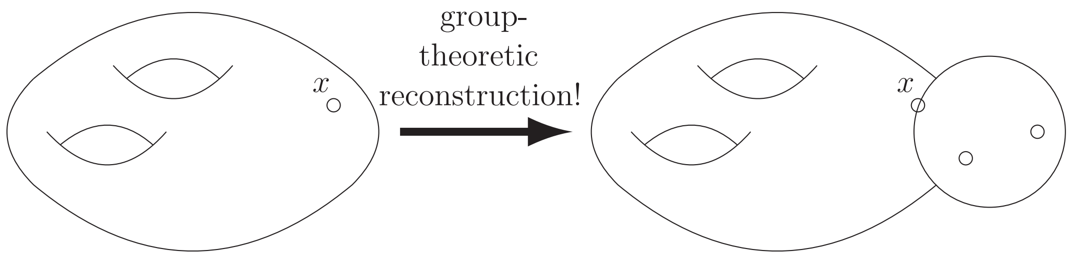
*Let $X$ (resp., $Y$) be an affine hyperbolic curve over a finite field $K$ (resp., $L$), $x$ a $K$-rational point of $\overline{X} \setminus X$ (resp., $y$ an $L$-rational point of $\overline{Y} \setminus Y$). Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism such that the decomposition groups of $x$ and $y$ (which are well-defined up to conjugacy) correspond via $\alpha$. Then there* *exists a unique isomorphism*

$$\tilde{\alpha} : \Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$$

*well-defined up to composition with an inner automorphism which maps the decomposition group (well-defined up to conjugacy) of $\tilde{x}$ in $\overline{X}_x^{\log}$ to that of $\tilde{y}$ in $\overline{Y}_y^{\log}$, and induces $\alpha$ by passing to the quotients $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}_y^{\log}} \twoheadrightarrow \Pi_Y$.*




Finally, we consider the cuspidalization problem for (geometrically pro-$l$) fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields (cf. Theorem 4.4):

**Theorem C.**
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let*

$$\alpha_1 : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists a unique isomorphism*

$$\alpha_n : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$$

*well-defined up to composition with an inner automorphism, which is compatible with the natural respective outer actions of the symmetric group on $n$ letters and makes the diagram*

$$\begin{array}{ccc} \Pi_{X_{n+1}} & \xrightarrow{\alpha_{n+1}} & \Pi_{Y_{n+1}} \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ \Pi_{X_n} & \xrightarrow{\alpha_n} & \Pi_{Y_n} \end{array}$$

*($i = 1, \cdots, n+1$) commute.*

This statement is already proved in [11] for the case where $n = 2$ and $X$ is proper, and in [4] for the case where $n \geq 3$ and $X$ is proper. On the other hand, by combining results obtained in this paper with the result obtained in [11], we obtain a shorter proof of the statement for $n \geq 3$ which includes, for the first time, the *affine case*.

**Acknowledgement**

The author would like to express his sincere gratitude to Professors Shinichi Mochizuki and Yuichiro Hoshi for their warm encouragements, suggestions, and many helpful advices.

## Notations and Conventions

**Numbers:**

We shall denote by $\mathbb{Q}$ the field of *rational numbers*, by $\mathbb{Z}$ the ring of *rational integers*, and by $\mathbb{N} \subseteq \mathbb{Z}$ $\big($resp., $\mathbb{Z}_{\geq a} \subseteq \mathbb{Z}\big)$ the additive submonoid of integers $n \geq 0$ $\big($resp., the subset of integers $n \geq a$ for $a \in \mathbb{Z}\big)$. If $l$ is a prime number, then $\mathbb{Z}_l$ $\big($resp., $\mathbb{Q}_l\big)$ denotes the *l-adic completion* of $\mathbb{Z}$ $\big($resp., $\mathbb{Q}\big)$.

**Topological Groups:**

For an arbitrary Hausdorff topological group $G$, the notation

$$G^{\mathrm{ab}}$$

will be used to denote the *abelianization* of $G$, i.e., the quotient of $G$ by the closed subgroup of $G$ topologically generated by the commutators of $G$.

If $G$ is a center-free, then we have a natural exact sequence

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1$$

— where $\mathrm{Aut}(G)$ denotes the group of automorphisms of the topological group $G$; the injective (since $G$ is center-free) homomorphism $G \to \mathrm{Aut}(G)$ is obtained by letting $G$ act on $G$ by inner automorphisms; $\mathrm{Out}(G)$ is defined so as to render the sequence exact. If the profinite group $G$ is topologically finitely generated, then the groups $\mathrm{Aut}(G)$, $\mathrm{Out}(G)$ are naturally endowed with a profinite topology, and the above sequence may be regarded as an exact sequence of profinite groups.

If $J \to \mathrm{Out}(G)$ is a homomorphism of groups, then we shall write

$$G \overset{\mathrm{out}}{\rtimes} J := \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} J$$

for the "*outer semi-direct product of $J$ with $G$*". Thus, we have a natural exact sequence

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

It is verified (cf. [4], Lemma 4.10) that if an automorphism $\phi$ of $G \overset{\mathrm{out}}{\rtimes} J$ preserves the subgroup $G \subseteq G \overset{\mathrm{out}}{\rtimes} J$ and induces the identity morphism on $G$ and the quotient $J$, then $\phi$ is the identity morphism of $G \overset{\mathrm{out}}{\rtimes} J$.

**Log schemes:**

Basic references for the notion of *log scheme* are [7] and [6]. In this paper, log structures are always considered on the étale sites of schemes. For a log scheme $X^{\log}$, we shall denote by $X$ (resp., $\mathcal{M}_X$) the underlying scheme of $X^{\log}$ (resp., the sheaf of monoids defining the log structure of $X^{\log}$). Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ (where, by abuse of notation, we write $X$ for the log scheme obtained by equipping $X$ with the trivial log structure) induced by the natural inclusion $\mathcal{O}^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

We shall say that a log scheme $X^{\log}$ is *fs* if $\mathcal{M}_X$ is a sheaf of integral monoids, and locally for the étale topology, has a chart modeled on a finitely generated and saturated monoid. If $X^{\log}$ is *fs*, then, for $n$ a nonnegative integer, we shall refer to as the *n-interior* of $X^{\log}$ the open subset of $X$ on which the associated sheaf of groupifications of characteristic sheaf of $X^{\log}$ is of *rank* $\leq n$. Thus, the 0-interior of $X^{\log}$ is often referred to simply as the *interior* of $X^{\log}$.

**Curves:**

Let $f : X \to S$ be a morphism of schemes. Then we shall say that $f$ is a *family of curves of type (g,r)* if it factors $X \hookrightarrow \overline{X} \to S$ as the composite of an open immersion $X \hookrightarrow \overline{X}$ whose image is the complement $\overline{X} \setminus D$ of a relative divisor $D \subseteq \overline{X}$ which is finite étale over $S$ of relative degree $r$, and a morphism $\overline{X} \to S$ which is proper, smooth, and geometrically connected, and whose geometric fibers are one-dimensional of genus $g$. We shall refer to $\overline{X}$ as the *compactification* of $X$.

We shall say that $f$ is a *family of hyperbolic curves (resp., tripod)* if $f$ is a family of curves of type $(g,r)$ such that $(g,r)$ satisfies $2g-2+r > 0$ (resp., $(g,r) = (0,3)$ and the relative divisor $D$ is split over $S$).

We shall denote by

$$\overline{\mathcal{M}}_{g,[r]+s}$$

the moduli stack of $r+s$-pointed stable curves of genus $g$ for which $s$ sections are equipped with an ordering. This moduli stack may be obtained as the quotient of the moduli stack of ordered $(r+s)$-pointed stable curves of genus $g$ (cf. [8] for an exposition of the theory of such curves) by a suitable symmetric group action on $r$ letters. We shall denote by $\overline{\mathcal{M}}^{\log}_{g,[r]+s}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,[r]+s}$ with the log structure associated to the divisor with normal crossings which parametrizes singular curves.

**Fundamental Groups:**

A basic reference for the notion of *Kummer étale covering* is [6]. For a locally Noetherian, connected scheme $X$ (resp., a locally Noetherian, connected, fs log scheme $X^{\log}$) equipped with a geometric point $\overline{x} \to X$ (resp., log geometric point $\tilde{x}^{\log} \to X^{\log}$), we shall denote by $\pi_1(X, \overline{x})$ (resp., $\pi_1(X^{\log}, \tilde{x}^{\log})$) the étale fundamental group of $X$ (resp., logarithmic fundamental group of $X^{\log}$). Since one knows that the étale and logarithmic fundamental groups are determined up to inner automorphisms independently of the choice of basepoint, we shall omit the basepoint, and write $\pi_1(X)$ (resp., $\pi_1(X^{\log})$ ).

For a scheme $X$ (resp., fs log scheme $X^{\log}$) which is geometrically connected and of finite type over a field $K$ in which a prime number $l$ is invertible, we shall refer to the quotient $\Pi_X$ of $\pi_1(X)$ (resp., the quotient $\Pi_{X^{\log}}$ of $\pi_1(X^{\log})$) by the closed normal subgroup obtained as the kernel of the natural projection from $\pi_1(X \otimes_K \overline{K})$ (resp., $\pi_1(X^{\log} \otimes_K \overline{K})$) (where $\overline{K}$ is a separable closure of $K$) to its maximal pro-$l$ quotient $\Delta_X$ (resp., $\Delta_{X^{\log}}$) as the *geometrically pro-$l$ étale fundamental group* of $X$ (resp., *geometrically pro-$l$ logarithmic fundamental group* of $X^{\log}$). Thus, (if we write $G_K$ for the Galois group of a separable closure of $K$ over $K$, then) we have a natural exct sequence

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1$$

$$(\text{resp.}, 1 \longrightarrow \Delta_{X^{\log}} \longrightarrow \Pi_{X^{\log}} \longrightarrow G_K \longrightarrow 1).$$

Note that if the log structure of $X^{\log}$ is trivial, then we have natural isomorphisms $\Delta_X \cong \Delta_{X^{\log}}, \Pi_X \cong \Pi_{X^{\log}}$.

If $K$ is finite, then write $G_K^\dagger \subseteq G_K$ for the *maximal pro-$l$ subgroup of* $G_K$ (so $G_K^\dagger \cong \mathbb{Z}_l$). Also, we shall use the notation

$$\Pi^\dagger := \Pi \times_{G_K} G_K^\dagger \subseteq \Pi$$

— where $\Pi$ denotes either the geometrically pro-$l$ étale or logarithmic fundamental group of $X$ — as the *restricted pro-$l$ étale or logarithmic fundamental group* of $X$.

## 1. Fundamental groups of (log) configuration spaces

The purpose of this section is to recall the notion of the (log) configuration space associated to a curve and review group-theoretic properties of the various fundamental groups associated to such spaces.

Let $l$ be a prime number, $K$ a field in which $l$ is invertible, $\overline{K}$ a separable closure of $K$ — where we shall denote by $G_K$ the Galois group of $\overline{K}$ over $K$ — and $X$ a hyperbolic curve over $K$ of type $(g, r)$.

**Definition 1.1.**

(i) For $n \in \mathbb{Z}_{\geq 1}$, Write $X^{\times n}$ for the fiber product of $n$ copies of $X$ over $K$. We shall denote by

$$X_n (\subseteq X^{\times n})$$

the *n-th configuration space associated to* $X$, i.e., the scheme which represents the open subfunctor

$$S \mapsto \big\{(f_1, \cdots, f_n) \in X^{\times n}(S) \;\big|\; f_i \neq f_j \text{ if } i \neq j\big\}$$

of the functor represented by $X^{\times n}$.

(ii) Let us denote by $\overline{X}_n^{\log}$ the *n-th log configuration space associated to* $X$ (cf. [14]), i.e.,

$$\overline{X}_n^{\log} := \operatorname{Spec} K \times_{\overline{\mathcal{M}}_{g,[r]}^{\log}} \overline{\mathcal{M}}_{g,[r]+n}^{\log}$$

— where the (1-)morphism $\operatorname{Spec} K \to \overline{\mathcal{M}}_{g,[r]}^{\log}$ is the classifying morphism determined by the curve $X \to \operatorname{Spec} K$, and the (1-)morphism $\overline{\mathcal{M}}_{g,[r]+n}^{\log} \to \overline{\mathcal{M}}_{g,[r]}^{\log}$ is obtained by forgetting the ordered $n$ marked points of the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+n}^{\log}$. In the following, for simplicity, we shall write $\overline{X}^{\log}$ for $\overline{X}_1^{\log}$.

**Proposition 1.2.**

(i) *The* 0*-interior (cf. § 0) of the log scheme* $\overline{X}_n^{\log}$ *is naturally isomorphic to the n-th configuration space* $X_n$ *associated to* $X$.

(ii) *The log scheme* $\overline{X}_n^{\log}$ *is log regular and its underlying scheme is connected and regular.*

(iii) *The projection* $p_k^{\log} : \overline{X}_n^{\log} \to \overline{X}_{n-1}^{\log}$*, induced from the (1-)morphism* $\overline{\mathcal{M}}_{g,[r]+n}^{\log} \to \overline{\mathcal{M}}_{g,[r]+n-1}^{\log}$ *obtained by forgetting the k-th* ($k = 1, \cdots, n$) *ordered points of the tautological family of curves over* $\overline{\mathcal{M}}_{g,[r]+n}^{\log}$*, is log smooth (cf. § 0) and its underlying morphism of schemes is the natural projection* $p_k : X_n \twoheadrightarrow X_{n-1}$ *obtained by forgetting the k-th factor, and hence, is flat, geometrically connected, and geometrically reduced.*

*Proof.* See, for example, [4], Proposition 2.2. □

**Definition 1.3.**

We shall denote (cf. § 0) by

$$\Pi_{X_n} \quad (\text{resp.}, \Delta_{X_n})$$

the geometrically pro-$l$ étale fundamental group of $X_n$ (resp., $X_n \otimes_K \overline{K}$), and

$$\Pi_{\overline{X}_n^{\log}} \quad (\text{resp.}, \Pi_{\overline{X}^{\log \times n}})$$

the geometrically pro-$l$ log fundamental group of $\overline{X}_n^{\log}$ (resp., the fiber product $\overline{X}^{\log \times n}$ of $n$ copies of $\overline{X}^{\log}$ over $K$). Moreover, we shall denote (cf. § 0) by

$$\Pi^{\dagger}_{X_n}, \quad \Delta^{\dagger}_{X_n} (\cong \Delta_{X_n}), \quad \Pi^{\dagger}_{\overline{X}_n^{\log}}, \quad \Pi^{\dagger}_{\overline{X}^{\log \times n}}$$

respective restricted pro-$l$ fundamental groups. If we write

$$i_k : \Delta^k_{X_{n/n-1}} \hookrightarrow \Delta_{X_n}$$

for the kernel of the surjection $p_k^{\Delta} : \Delta_{X_n} \twoheadrightarrow \Delta_{X_{n-1}}$, where $p_k^{\Delta}$ denotes the morphism induced by the projection $p_k : X_n \twoheadrightarrow X_{n-1}$ obtained by forgetting the $k$-th factor, then we have exact sequences

$$1 \longrightarrow \Delta_{X_n} \longrightarrow \Pi^{(-)}_{X_n} \longrightarrow G^{(-)}_K \longrightarrow 1$$

$$1 \longrightarrow \Delta^k_{X_{n/n-1}} \xrightarrow{i_k} \Delta_{X_n} \xrightarrow{p_k} \Delta_{X_{n-1}} \longrightarrow 1$$

$$1 \longrightarrow \Delta^k_{X_{n/n-1}} \xrightarrow{i_k} \Pi^{(-)}_{X_n} \xrightarrow{p_k} \Pi^{(-)}_{X_{n-1}} \longrightarrow 1$$

— where the symbol $(-)$ denotes either the presence or absence of "$\dagger$", and when there is no fear of confusion, we shall write "$i_k$", "$p_k$" (by abuse of notation) for the morphisms induced by $i_k$, $p_k$, respectively.

Also, we have a square diagram

$$\begin{array}{ccccc}
\Pi^{(-)}_{X_{n-1}} & \xleftarrow{p_k} & \Pi^{(-)}_{X_n} & \longrightarrow & \overbrace{\Pi^{(-)}_X \times_{G^{(-)}_K} \cdots \times_{G^{(-)}_K} \Pi^{(-)}_X}^{n} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi^{(-)}_{\overline{X}^{\log}_{n-1}} & \xleftarrow{p_k^{\log}} & \Pi^{(-)}_{\overline{X}^{\log}_n} & \longrightarrow & \Pi^{(-)}_{\overline{X}^{\log \times n}}
\end{array}$$

— which can be made commutative without conjugate-indeterminacy by choosing compatible base points — arising from a natural commutative diagram

$$\begin{array}{ccccc}
X_{n-1} & \xleftarrow{p_k} & X_n & \longrightarrow & X^{n\times} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{X}^{\log}_n & \xleftarrow{p_k^{\log}} & \overline{X}^{\log}_n & \longrightarrow & \overline{X}^{\log \times n}.
\end{array}$$

Then, it follows from Proposition 1.2 (i), (ii) together with the log purity theorem (cf. [6], [9]) that the two vertical homomorphisms are isomorphisms. In the following, we shall identify $\Pi^{(-)}_{X_n}$ with $\Pi^{(-)}_{\overline{X}^{\log}_n}$, $\Pi^{(-)}_{\overline{X}^{\log \times n}}$ with $\overbrace{\Pi^{(-)}_X \times_{G^{(-)}_K} \cdots \times_{G^{(-)}_K} \Pi^{(-)}_X}^{n}$ and the surjection $p_k : \Pi_{X_n} \to \Pi_{X_{n-1}}$ with the surjection $p_k : \Pi^{(-)}_{\overline{X}^{\log}_n} \to \Pi^{(-)}_{\overline{X}^{\log}_{n-1}}$ by means of these specific isomorphisms.

**Proposition 1.4.**

(i) $\Delta^k_{X_{n/n-1}}$ *may be naturally identified with the maximal pro-$l$ quotient of the étale fundamental group of a geometric fiber of the projection morphism* $p_k : X_n \to X_{n-1}$.

(ii) *The images of the* $i_k : \Delta^k_{X_{n/n-1}} \to \Delta_{X_n}$, *where* $k = 1, \cdots, n$, *generate* $\Delta_{X_n}$.

(iii) *The profinite groups* $\Delta_{X_n}$, $\Delta^k_{X_{n/n-1}}$, $\Pi^{\dagger}_{X_n}$, $\Pi^{\dagger}_{X^{\times n}}$ *are slim (i.e., every open subgroup of each profinite group is center-free).*

*Proof.* Assertion (i) follows from [14], Proposition 2.2, or [18], Proposition 2.3. Assertions (ii) and (iii) follow from induction on $n$, together with the exact sequence

$$1 \longrightarrow \Delta^n_{X_{n/n-1}} \xrightarrow{i_n} \Delta_{X_n} \xrightarrow{p_n} \Delta_{X_{n-1}} \longrightarrow 1$$

displayed in Definition 1.3. Indeed, with regard to (ii), $\Delta^k_{X_{n/n-1}}$ maps to $\Delta^k_{X_{n-1/n-2}}$ (for $k = 1, \cdots n-1$) via $p_n : \Delta_{X_n} \to \Delta_{X_{n-1}}$, and it is verified that this map $\Delta^k_{X_{n/n-1}} \to \Delta^k_{X_{n-1/n-2}}$ is surjective by regarding it as the morphism induced by an open immersion between the hyperbolic curves that arise as geometric fibers of the projection morphisms involved. With regard to (iii), the slimness of $\Delta_X$ is well-known (cf., e.g., [10], Lemma 1.3.10); the slimness of $\Pi^{\dagger}_X$ follows from the fact that the character of $G^{\dagger}_K$ arising from the determinant of $\Delta^{\mathrm{ab}}_X$ coincides with some positive power of the cyclotomic character; the other statements follow from the fact that an extension of slim profinite groups is itself slim. □

Next, we recall from [11], § 3, the theory of the weight filtration of fundamental groups and the associated graded Lie algebra.

**Definition 1.5.**

Let $l$ be a prime number; $G$, $H$, $A$ topologically finitely generated pro-$l$ groups; $\phi : H \twoheadrightarrow A$ a (continuous) surjective homomorphism. Suppose further that $A$ is abelian, and that $G$ is an $l$-adic Lie group.

(i) We shall refer to as the *central filtration* $\{H(n)\}_{n\geq 1}$ *on* $H$ *with respect to the homorphism* $\phi$ the filtration defined as follows:

$$H(1) := H$$
$$H(2) := \mathrm{Ker}(\phi)$$
$$H(m) := \big\langle [H(m_1), H(m_2)] \;\big|\; m_1 + m_2 = m \big\rangle \text{ for } m \geq 3$$

— where $\langle N_i \mid i \in I \rangle$ is the group topologically generated by the $N_i$'s. In the following, for $a, b, n \in \mathbb{Z}$ such that $1 \leq a \leq b, n \geq 1$, we shall write

$$H(a/b) := H(a)/H(b)$$
$$\mathrm{Gr}(H) := \bigoplus_{m\geq 1} H(m/m+1)$$
$$\mathrm{Gr}(H)(a/b) := \bigoplus_{b>m\geq a} H(m/m+1)$$
$$\mathrm{Gr}_{\mathbb{Q}_l}(H) := \mathrm{Gr}(H) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$
$$\mathrm{Gr}_{\mathbb{Q}_l}(a/b) := \mathrm{Gr}(H)(a/b) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$H(a/\infty) := \varprojlim_{b>a} H(a/b)\ .$$

(ii) We shall denote by $\mathrm{Lie}(G)$ the Lie algebra over $\mathbb{Q}_l$ determined by the $l$-adic Lie group $G$. We shall say that $G$ is *nilpotent* if there exists a positive integer $m$ such that if we denote by $\{G(n)\}$ the central filtration with respect to the natural surjection $G \twoheadrightarrow G^{\mathrm{ab}}$ (cf. (i)), then $G(m)=\{1\}$. If $G$ is nilpotent, then $\mathrm{Lie}(G)$ is a nilpotent Lie algebra over $\mathbb{Q}_l$, hence determines a connected, unipotent linear algebraic group $\mathrm{Lin}(G)$, which we shall refer to as the *linear algebraic group associated to* $G$. In this situation, there exists a natural (continuous) homomorphism (with open image)

$$G \longrightarrow \mathrm{Lin}(G)(\mathbb{Q}_l)$$

(from $G$ to the $l$-adic Lie group determined by the $\mathbb{Q}_l$-valued points of $\mathrm{Lin}(G)$) which is uniquely determined (since $\mathrm{Lin}(G)$ is connected and unipotent) by the condition that it induce the identity morphism on the associated Lie algebras.

In the situation of (i), if $1 \leq a \in \mathbb{Z}$, then we shall write

$$\mathrm{Lie}(H(a/\infty)) := \varprojlim_{b>a} \mathrm{Lie}(H(a/b))$$

$$\mathrm{Lin}(H(a/\infty)) := \varprojlim_{b>a} \mathrm{Lin}(H(a/b))$$

— where we note that each $H(a/b)$ is a nilpotent $l$-adic Lie group.

**Definition 1.6.**

For $n \in \mathbb{Z}_{\geq 1}$, we shall denote by

$$\{\Delta_{X_n}(m)\}$$

the central filtration of $\Delta_{X_n}$ with respect to the natural surjection $\Delta_{X_n} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\overline{X}^{\times n}}$ (where $\overline{X}$ denotes the smooth compactification of $X$ (cf. § 0)), and refer to it as the *weight filtration* on $\Delta_{X_n}$.

**Proposition 1.7.**

*If we equip $\Delta^k_{X_{n/n-1}}$ with the central filtration induced from the identification given by Proposition 1.4 (i) and its weight filtration, then the sequence of morphisms of graded Lie algebras*

$$1 \longrightarrow \mathrm{Gr}(\Delta^k_{X_{n/n-1}}) \xrightarrow{i_k} \mathrm{Gr}(\Delta_{X_n}) \xrightarrow{p_k} \mathrm{Gr}(\Delta_{X_{n-1}}) \longrightarrow 1$$

*induced by the second displayed exact sequence of Definition 1.3 is exact.*

*Proof.* See [4], Proposition 4.1. □

Next, let us fix a section $\sigma : G_K \to \Pi_{X_n}$ of the surjection $\Pi_{X_n} \twoheadrightarrow G_K$ induced by the structure morphism of $X_n$. This section $\sigma$ determines natural conjugate actions of $G_K$ on $\Delta_{X_n}$, hence also on

$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b)$$
$$\mathrm{Lie}_{X_n}(a/b) := \mathrm{Lie}(\Delta_{X_n}(a/b))$$
$$\mathrm{Lin}_{X_n}(a/b) := \mathrm{Lin}(\Delta_{X_n}(a/b))(\mathbb{Q}_l)$$

for $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$.

**Proposition 1.8.**

*Let us assume that $K$ is a finite field whose cardinality we denote by $q_K$, and write $Fr \in G_K$ for the Frobenius element of $G_K$. Then:*

(i) *The eigenvalues of the action of $Fr$ on $\mathrm{Lie}_{X_n}(a/a+1)$ are algebraic numbers all of whose complex absolute values are equal to $q_K^{a/2}$ (i.e., weight $a$).*

(ii) *There is a unique $G_K$-equivariant isomorphism of Lie algebras*

$$\mathrm{Lie}_{X_n}(a/b) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b)$$

*which induces the identity isomorphism*

$$\mathrm{Lie}_{X_n}(c/c+1) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(c/c+1)$$

*for all $c \in \mathbb{Z}_{\geq 1}$ such that $a \leq c < b$.*

*Proof.* Assertion (i) follows from the "Riemann hypothesis for abelian varieties over finite fields" (cf., e.g., [15], p. 206). Assertion (ii) follows formally from assertion (i) by considering the eigenspaces with respect to the action of Fr. □

The following proposition is a special case of a result proven previously (cf. [17]). For simplicity, we discuss only the case used in the proofs of the present paper.

**Proposition 1.9.**

*For $n = 1, 2$, the graded Lie algebra $\mathrm{Gr}(\Delta_{X_n})$ has the following presentation.*

(i) *The case $n = 1$ (i.e., $X_n = X$):*

*generators* $(1 \leq j \leq r,\ 1 \leq i \leq g)$

$\bullet_1$ $\zeta_j \in \Delta_X(2/3)$

$\bullet_2$ $\alpha_i,\ \beta_i \in \Delta_X(1/2)$

*relation*

$\bullet_1$ $\sum_{j=1}^{r} \zeta_j + \sum_{i=1}^{g} [\alpha_i, \beta_i] = 0$

*— where $\zeta_j$ $(j = 1, 2, \cdots, r)$ topologically generates the inertia subgroup in $\Delta_X$ (well-defined up to conjugacy) associated to the $j$-th cusp [relative to some ordering of the cusps of $X \times_K \overline{K}$].*

(ii) *The case $n = 2$:*

*generators ($1 \leq j \leq r,\ 1 \leq i \leq g,\ k = 1, 2$)*

$\bullet_1$ $\zeta \in \Delta_{X_2}(2/3)$

$\bullet_2$ $\zeta_j^k \in \Delta_{X_{2/1}}^k(2/3)$

$\bullet_3$ $\alpha_i^k,\ \beta_i^k \in \Delta_{X_{2/1}}^k(1/2)$

*relations ($1 \leq j, j' \leq r,\ j \neq j',\ 1 \leq i, i' \leq g,\ i \neq i',\ \{k, k'\} = \{1, 2\}$)*

$\bullet_1$ $\zeta + \sum_{j=1}^{r} \zeta_j^k + \sum_{i=1}^{g} [\alpha_i^k, \beta_i^k] = 0$

$\bullet_2$ $[\alpha_i^k, \zeta_j^{k'}] = [\beta_i^k, \zeta_j^{k'}] = [\zeta_j^k, \zeta_{j'}^{k'}] = 0$

$\bullet_3$ $[\alpha_i^k, \alpha_{i'}^{k'}] = [\alpha_i^k, \beta_{i'}^{k'}] = [\beta_i^k, \beta_{i'}^{k'}] = 0$

$\bullet_4$ $[\alpha_i^1, \alpha_i^2] = [\beta_i^1, \beta_i^2] = 0$

$\bullet_5$ $[\alpha_i^1, \beta_i^2] = \zeta$

*— where $\zeta$ topologically generates the image in $\Delta_{X_2}(2/3)$ of the inertia subgroup in $\Delta_{X_2}$ (well-defined up to conjugacy) associated to the diagonal divisor of $X \times_K X$, and $\zeta_j^k$ generates the image in $\Delta_{X_{2/1}}^k(2/3)$ of the inertia subgroup in $\Delta_{X_{2/1}}^k$ asssociated to the $j$-th cusp [relative to some ordering of the cusps of $X \times_K \overline{K}$] of the $k$-th factor of $X_2$.*

## 2. Switching morphism on configuration spaces

We continue to use the notation of Section 1. In this section, we consider various automorphisms induced by the automorphism of $\overline{X}_2^{\log}$ determined by switching the two factors of $X$. The group-theoretic uniqueness of such induced switching morphisms between fundamental groups (Proposition 2.5) plays a key role in the proof of Theorem A.

We denote by

$$\mathbb{D}^{\log}$$

the log scheme obtained by equipping the diagonal divisor $\overline{X} \subseteq \overline{X}_2$ (which is the restriction of the (1-)morphism $\overline{\mathcal{M}}_{g,[r]+1} \to \overline{\mathcal{M}}_{g,[r]+2}$ obtained by gluing the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+1}^{\log}$ to a trivial family of tripods along the final ordered marked section) with the log structure pulled back from $\overline{X}_2^{\log}$. Thus, if we write $d : \mathbb{D}^{\log} \to \overline{X}_2^{\log}$ for the natural diagonal embedding, then it follows immediately from the definitions that $p_1 \circ d = p_2 \circ d : \mathbb{D}^{\log} \to \overline{X}^{\log}$ is a morphism of type $\mathbb{N}$ (cf. [2]), i.e., the underlying morphism of schemes is an isomorphism, and the relative characteristic sheaf (cf. § 0 ) is locally constant with stalk isomorphic to $\mathbb{N}$.

Observe that the (1-)automorphism on $\overline{\mathcal{M}}_{g,[r]+2}^{\log}$ over $\overline{\mathcal{M}}_{g,[r]}^{\log}$ given by switching the two ordered marked points of the tautological family of curves over $\overline{\mathcal{M}}_{g,[r]+2}^{\log}$ induces automorphisms $s$, $\overline{s}$, and $s_D$, which fit into a commutative diagram as follows:

$$\begin{array}{ccccc} \mathbb{D}^{\log} & \xrightarrow{d} & \overline{X}_2^{\log} & \xrightarrow{p=(p_1,p_2)} & \overline{X}^{\log} \times_K \overline{X}^{\log} \\ {\scriptstyle \underline{s}}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle \overline{s}}\downarrow \\ \mathbb{D}^{\log} & \xrightarrow{d} & \overline{X}_2^{\log} & \xrightarrow{p=(p_1,p_2)} & \overline{X}^{\log} \times_K \overline{X}^{\log}. \end{array} \qquad (*)^X$$

**Lemma 2.1.**

*In the notation of the above situation,*

(i) $\overline{s}$ *is the morphism determined by switching the two factors.*

(ii) $\underline{s}$ *is the identity morphism on the underlying scheme; on the sheaf of monoids defining the log structure of* $\mathbb{D}^{\log}$*, for any étale local section* $\theta$ *of* $\mathcal{M}_{\mathbb{D}}$ *such that "*$\theta = 0$*" defines the diagonal divisor* $\overline{X} \subseteq \overline{X}_2$*,*

$$\underline{s}(\theta) = -\theta \ .$$

*Proof.* Recall that $\overline{X}_2$ is obtained by blowing-up $\overline{X} \times_K \overline{X}$ along the intersection of the diagonal divisor and the pull-backs of the cusps via $p_1, p_2 : \overline{X}_2 \to \overline{X}$. Thus, one verifies easily that assertions (i) and (ii) follow immediately from the fact that the ring homomorphism corresponding to $\overline{s}$ in an affine neighborhood of any diagonal point may be expressed as

$$A \otimes_K A \longrightarrow A \otimes_K A$$

$$\sum_j a_j \otimes a'_j \mapsto \sum_j a'_j \otimes a_j \ ,$$

hence maps $\theta$ to $-\theta$ for any local section $\theta$ such that "$\theta = 0$" defines the diagonal divisor $\overline{X} \subseteq \overline{X} \times_K \overline{X}$. $\square$

**Remark 2.1.1.**

Lemma 2.1 (ii) can be interpreted as the assertion that the automorphism induced by $\underline{s}$ on the sheaf of monoids $\mathcal{M}_{\mathbb{D}}$ defining the log structure of $D_X^{\log}$ may be expressed, relative to the étale local splitting of $\mathcal{M}_{\mathbb{D}} \twoheadrightarrow \mathcal{M}_{\mathbb{D}}/\mathcal{O}_X^* \cong \mathbb{N}$ corresponding to $\theta$, as

$$\begin{aligned} \mathbb{N} \oplus \mathcal{O}_X^* &\xrightarrow{\sim} \mathbb{N} \oplus \mathcal{O}_X^* \\ (m, v) &\longmapsto (m, (-1)^m v) \ . \end{aligned}$$

The above diagram $(*)^X$ induces a diagram of profinite groups as follows:

$$\begin{array}{ccccc} \Pi_{\mathbb{D}^{\log}} & \xrightarrow{[d^{\Pi}]} & \Pi_{X_2} & \xrightarrow{[p^{\Pi}]} & \Pi_X \times_{G_K} \Pi_X \\ {\scriptstyle [\underline{s}^{\Pi}]}\downarrow\wr & & {\scriptstyle [s^{\Pi}]}\downarrow\wr & & {\scriptstyle [\overline{s}^{\Pi}]}\downarrow\wr \\ \Pi_{\mathbb{D}^{\log}} & \xrightarrow{[d^{\Pi}]} & \Pi_{X_2} & \xrightarrow{[p^{\Pi}]} & \Pi_X \times_{G_K} \Pi_X \ . \end{array} \qquad (*)^{\Pi}$$

Note that the arrows in the diagram $(*)^{\Pi}$ are only defined (i.e., in the absence of appropriate choices of basepoints of respective log schemes) up to conjugacy.

Next, we observe that since the subgroups of the conjugacy class of subgroups determined by the image of $[d^{\Pi}]$ may be naturally regarded as decomposition groups associated to the diagonal divisor of $\overline{X}_2$, any *choice* of a *specific homomorphism* $d^{\Pi} : \Pi_{\mathbb{D}^{\log}} \to \Pi_{\overline{X}_2^{\log}}$ (i.e., among its various conjugates) determines a *specific decomposition group*

$$D_X \subseteq \Pi_{\overline{X}_2^{\log}}$$

— where we write $\underline{d}^{\Pi} : D_X \hookrightarrow \Pi_{\overline{X}_2^{\log}}$ for the natural inclusion — associated to the diagonal divisor (i.e., among its various $\Pi_{\overline{X}_2^{\log}}$-conjugates), as well as a *specific inertia subgroup*

$$I_X \subseteq \Pi_{\overline{X}_2^{\log}}$$

associated the diagonal divisor (i.e., among its various $\Pi_{\overline{X}_2^{\log}}$-conjugates). Here, we recall that $I_X$ is canonically isomorphic to $\mathbb{Z}_l(1)$.

**Definition 2.2.**

Let $x^{\log} \to \overline{X}^{\log}$ be a strict morphism (cf. [6], 1.2) such that the underlying scheme of $x^{\log}$ is equal to $\mathrm{Spec}(K)$. We shall write

$$\overline{X}_x^{\log} := \overline{X}_2^{\log} \times_{\overline{X}^{\log}} x^{\log},$$

$$\tilde{x}^{\log} := \mathbb{D}^{\log} \times_{\overline{X}^{\log}} x^{\log},$$

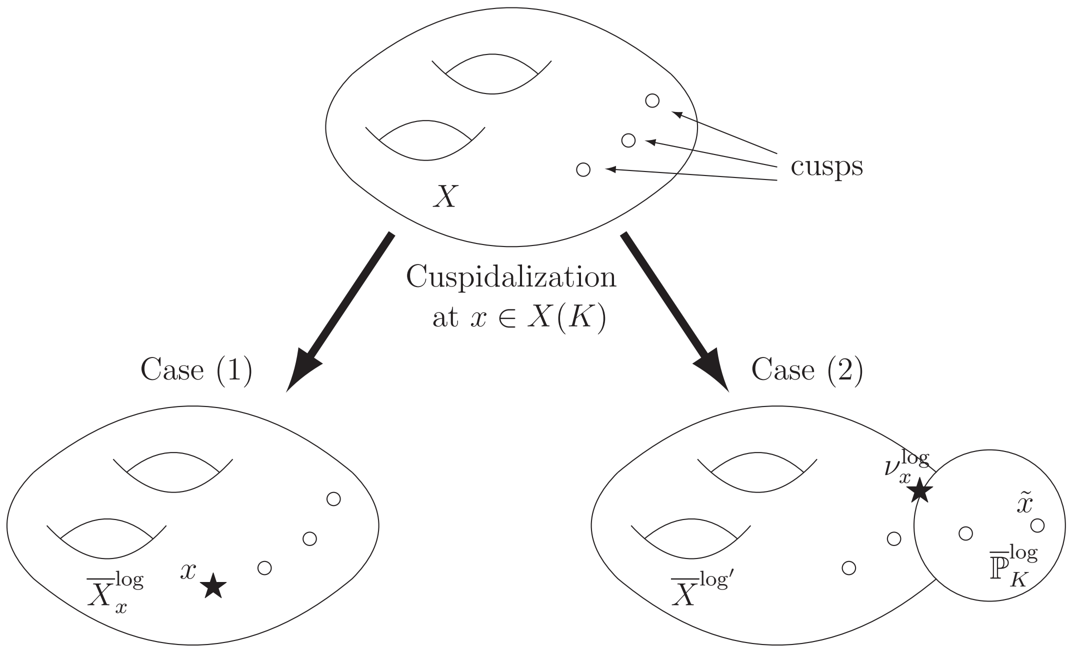
$$G_{K^{\log}}^{(-)} := \Pi_{x^{\log}}^{(-)}$$

— where the morphism $\overline{X}_2^{\log} \to \overline{X}^{\log}$ (resp., $\mathbb{D}^{\log} \to \overline{X}^{\log}$) in the fiber product defining $\overline{X}_x^{\log}$ (resp., $\tilde{x}^{\log}$) is $p_1$ (resp., $p_1 \circ d = p_2 \circ d$), and the symbol "$(-)$" denotes either the presence or absence of "$\dagger$" — and refer to $\overline{X}_x^{\log}$ (resp., $\tilde{x}^{\log}$)

as the *cuspidalization of* $X$ *at* $x$ (resp., *diagonal cusp of* $\overline{X}_x^{\log}$). We note that both the log structure of $x^{\log}$ and the underlying scheme of $\overline{X}_x^{\log}$ depend on the choice of $x \in \overline{X}$:

(1) *The Case* $x \in X$:
In this case, $x = x^{\log}$, i.e., the log structure of $x^{\log}$ is trivial. As we discussed in Section 1, the underlying scheme of $\overline{X}_x^{\log}$ is naturally isomorphic to $\overline{X}$; this isomorphism maps $\tilde{x}$ to $x$ and the interior of $\overline{X}_x^{\log}$ onto $X \setminus \{x\}$.

(2) *The Case* $x \in \overline{X} \setminus X$:
In this case, the log structure of $x^{\log}$ has a chart modeled on $\mathbb{N}$, which determines a local uniformizer of $X$ at $x$. The scheme $\overline{X}_x$ consists of precisely two irreducible components, one of which maps to the point $x \in X$ (resp., maps isomorphically to $\overline{X}$) via $\overline{X}_x^{\log} \xrightarrow{p_2 \circ i_1} \overline{X}^{\log}$; denote this irreducible component by $\overline{\mathbb{P}}_K$ (resp., $\overline{X}$, via a slight abuse of notation). Thus, $\overline{X}$, $\overline{\mathbb{P}}_K$ are joined at a single *node* $\nu_x$. Let us refer to $\overline{X}$ (resp., $\overline{\mathbb{P}}_K$, $\nu_x$) as the *major cuspidal component* (resp., the *minor cuspidal component*, the *nexus*) at $x$, and denote by $\overline{X}^{\log'}$, $\overline{\mathbb{P}}_K^{\log}$, $\nu_x^{\log}$ the log schemes obtained by equipping $\overline{X}$, $\overline{\mathbb{P}}_K$, $\nu_x$ with the respective log structures pulled back from $\overline{X}_x^{\log}$ (cf. [13], Definition 1.4). Note that the 1-interior of $\overline{X}^{\log'}$ (resp., $\overline{\mathbb{P}}_K^{\log}$) is isomorphic to $X$ (resp., is a *tripod*).




(the two thick arrows in the picture do not represent morphisms of log schemes)

In the following, we fix *choices of specific* [i.e., in the sense that they are *not* subject to conjugacy indeterminacy!] homomorphisms

$$i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \longrightarrow \Pi_{X_2}, \qquad p_1^{\Pi} : \Pi_{X_2} \longrightarrow \Pi_X$$

induced by the morphisms of log schemes $i_1 : \overline{X}_x^{\log} \to \overline{X}_2^{\log}, p_1 : \overline{X}_2^{\log} \to \overline{X}^{\log}$ and a *choice of a specific* decomposition group

$$D_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}$$

associated to $\tilde{x}^{\log}$ of $\overline{X}_x^{\log}$ among the various conjugates of this subgroup. Write

$$\gamma : D_{\tilde{x}} \hookrightarrow \Pi_{\overline{X}_x^{\log}}$$

for the natural inclusion. These choices determine a homomorphism of profinite groups

$$f_X : \Pi_{\overline{X}_x^{\log}} \longrightarrow G_{K^{\log}},$$

arising from the structure morphism $\overline{X}_x^{\log} \to x^{\log}$, by taking $f_X := i_1^{\Pi} \circ p_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to G_{K^{\log}} \big(:= \mathrm{Im}(\Pi_{\overline{X}_x^{\log}} \overset{i_1^{\Pi}\circ p_1^{\Pi}}{\to} \Pi_X)\big)$, as well as a profinite group

$$I_{\tilde{x}} := \mathrm{Ker}(D_{\tilde{x}} \overset{\gamma}{\hookrightarrow} \Pi_{\overline{X}_x^{\log}} \overset{f}{\to} G_{K^{\log}})$$

— where we note that $I_{\tilde{x}}$ is naturally isomorphic to $\mathbb{Z}_l(1)$ (i.e., a *Tate twist* of $\mathbb{Z}_l$).

**Lemma 2.3.**

(i) *The subgroup $D_{\tilde{x}}(\overset{\gamma}{\hookrightarrow} \Pi_{\overline{X}_x^{\log}})$ is the normalizer of $I_{\tilde{x}}$ in $\Pi_{\overline{X}_x^{\log}}$.*

(ii) *For any choice of a specific decomposition group $\underline{d}^{\Pi} : D_X \hookrightarrow \Pi_{X_2}$ (i.e., among its various conjugates) associated to the diagonal divisor of $\overline{X}_2$ , the subgroup $D_X$ of $\Pi_{X_2}$ coincides with the normalizer of $I_X$ in $\Pi_{X_2}$.*

*Proof.* Note that we have commutative diagrams

$$\begin{tikzcd} I_{\tilde{x}} \arrow[r] \arrow[d, "i_1^{\Pi}\circ\gamma|_{I_{\tilde{x}}}"'] & D_{\tilde{x}} \arrow[d, "i_1^{\Pi}\circ\gamma"'] \\ \Delta^1_{X_{2/1}} \arrow[r, "i_1"] & \Pi_{X_2} \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & I_X \arrow[r] \arrow[d, "\underline{d}^{\Pi}|_{I_X}"'] & D_X \arrow[r, "p_1^{\Pi}\circ\underline{d}^{\Pi}"] \arrow[d, "\underline{d}^{\Pi}"'] & \Pi_X \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & \Delta^1_{X_{2/1}} \arrow[r, "i_1"] & \Pi_{X_2} \arrow[r, "p_1^{\Pi}"] & \Pi_X \arrow[r] & 1, \end{tikzcd}$$

— where the two horizontal sequences in the second diagram are exact, and both the first displayed diagram and the left-hand square in the second diagram are cartesian. Next, let us recall the well-known fact (cf., e.g., [16], (2.3.1)) that $I_{\tilde{x}}$ and $I_X$ coincide with their respective normalizers in $\Delta^1_{X_{2/1}}$. Thus, assertion

(ii) follows immediately from the surjectivity of $p_1^{\Pi} \circ \underline{d}^{\Pi}$. On the other hand, assertion (i) follows immediately from the observation that the images of $D_{\tilde{x}}$ and $\Pi_{\overline{X}_x^{\log}}$ via $p_1^{\Pi} \circ i_1^{\Pi}$ *coincide.* This observation is a consequence of the geometry of the corresponding morphisms of log schemes, which implies that these images coincide with a decomposition group $\subseteq \Pi_X$ associated to the point $x$. $\square$

**Lemma 2.4.**

*Under the determination discussed preceding Lemma 2.3, we can uniquely take a pair of specific homomorphisms*

$$\underline{d}^{\Pi} : D_X \longrightarrow \Pi_{X_2}, \qquad p_2^{\Pi} : \Pi_{X_2} \longrightarrow \Pi_X$$

*among the various conjugates of these subgroups, obtained by morphisms between log schemes $d : \mathbb{D}^{\log} \to \overline{X}_2^{\log}$, $p_2 : \overline{X}_2^{\log} \to X$, satisfying the following conditions:*

(1) *The image of the inertia subgroup $I_X$ (via $\underline{d}^{\Pi}$) coincides with the image of $I_{\tilde{x}}$ via $i_1^{\Pi}$.*

(2) *$D_X$ maps (via $(p_1^{\Pi}, p_2^{\Pi}) \circ \underline{d}^{\Pi} : D_X \to \Pi_X \times_{G_K} \Pi_X$) onto the image of the diagonal embedding $\Pi_X \hookrightarrow \Pi_X \times_{G_K} \Pi_X$.*

*Proof.* Since $D_X$ is the normalizer of $I_X$ in $\Pi_{\overline{X}_2^{\log}}$ by Lemma 2.3, it is enough to take $D_X$ as the normalizer of $I_{\tilde{x}}$ in $\Pi_{\overline{X}_2^{\log}}$, and take $p_2^{\Pi}$ so as to $p_1^{\Pi} \circ \underline{d}^{\Pi} = p_2^{\Pi} \circ \underline{d}^{\Pi}$. Uniqueness follows from the requred two conditions. $\square$

Now, before we continue the discussion, we shall give comments for Proposition 2.5.

(1) Recall that the natural surjection $D_{\tilde{x}} \twoheadrightarrow G_K$ (since $G_K$ is abelian, this map is uniquely determined without the discussion of base points) has a section. Indeed, fixing a choice of such a section is equivalent to extracting roots of any local uniformizers of the divisors $\overline{X}_x \subseteq \overline{X}_2$ and $\mathbb{D} \subseteq \overline{X}_2$ at $\tilde{x}$.

(2) We shall consider the restriction map $H^1(G_K, \mathbb{Z}_l(1)) \to H^1(G_K^{\dagger}, \mathbb{Z}_l(1))$ of cohomology groups induced by the natural inclusion $G_K^{\dagger} \hookrightarrow G_K$. Since $G_K^{\dagger}$ is the maximal pro-$l$ subgroup of $G_K$ and $I_{\tilde{x}}$ is isomorphic to $\mathbb{Z}_l(1)$ as $G_K$-module, $H^1(G_K, \mathbb{Z}_l(1))$ is isomorphic to $H^1(G_K^{\dagger}, \mathbb{Z}_l(1))$ and is isomorphic to the maximal pro-$l$ completion $(K^{\times})^{\wedge}$ of the multiplicative group $K^{\times}$ of $K$. Therefore, if we denote by $Z^1(G_K^{\dagger}, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) the set of (continuous) 1-cocycle maps of $G_K^{\dagger}$ (resp., $G_K$) with coefficients in $I_{\tilde{x}}$, then we can refer to any element of $Z^1(G_K^{\dagger}, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) belonging to the inverse image of $a \in (K^{\times})^{\wedge} \cong$

$H^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $\cong H^1(G_K^\dagger, I_{\tilde{x}})$ via the natural surjection as a *(continuous) 1-cocycle map representing the Kummer class a*

**Proposition 2.5.**

*Following the discussions and results in this section until now, we shall fix a choice of a section*

$$\sigma : G_K \longrightarrow D_{\tilde{x}}$$

*(hence also a choice of an induced morphism* $\sigma^\dagger : G_K^\dagger \longrightarrow D_{\tilde{x}}^\dagger$*) of the surjection* $D_{\tilde{x}} \to G_K$*.*

*Then, for any 1-cocycle map*

$$\delta : G_K^\dagger \to I_{\tilde{x}}$$

*representing the Kummer class* $-1 \in (K^\times)^\wedge$*, there exists a unique triple* $(\underline{s}_\delta^\dagger, s_\delta^\dagger, \overline{s}_\delta^\dagger)$ *of vertical automorphisms in the following diagram*

$$\begin{array}{ccccc} D_X^\dagger & \xrightarrow{\underline{d}^\dagger} & \Pi_{X_2}^\dagger & \xrightarrow{p^\dagger} & \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger \\ {\scriptstyle \underline{s}_\delta^\dagger}\downarrow & & {\scriptstyle s_\delta^\dagger}\downarrow & & {\scriptstyle \overline{s}_\delta^\dagger}\downarrow \\ D_X^\dagger & \xrightarrow{\underline{d}^\dagger} & \Pi_{X_2}^\dagger & \xrightarrow{p^\dagger} & \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger \end{array} \qquad (*)^\dagger$$

*— where* $p^\dagger : \Pi_{X_2}^\dagger \to \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger$ *(resp.,* $\underline{d}^\dagger : D_X^\dagger \to \Pi_{X_2}^\dagger$*) denotes the morphism induced by* $p_1^\Pi, p_2^\Pi$ *(resp.,* $\underline{d}^\Pi$*) determined in the preceding disucussions — which makes* $(*)^\dagger$ *commute and satisfies the following two conditions:*

(1) $\overline{s}_\delta^\dagger : \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger \xrightarrow{\sim} \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger$ *is the morphism obtained by switching the two factors.*

(2) *The continuous function from* $G_K^\dagger$ *to* $\Pi_{X_2^{\log}}^\dagger$ *defined by*

$$g \mapsto (s_\delta^\dagger \circ \sigma^\dagger)(g) \cdot \sigma^\dagger(g)^{-1}$$

*is valued in* $I_{\tilde{x}} \subseteq \Pi_{X_2}^\dagger$ *and coincides with* $\delta$*.*

*Proof.* We begin by proving the existence portion. By the surjectivity of $p^\dagger$, we can take $s_\delta^\dagger, \overline{s}_\delta^\dagger$ such that the right-hand square of the diagram $(*)^\dagger$ commutes and the condition (1) is satisfied. If we take arbitrary $\underline{s}_\delta^\dagger \in \mathrm{Aut}(D_X^\dagger)$ from the conjugacy class, commutativity of the rectangle in $(*)^\dagger$ up to conjugacy implies that there exists $\lambda \in \Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger$ such that $\overline{s}_\delta^\dagger \circ (p^\dagger \circ \underline{d}^\dagger) = \mathrm{Inn}(\lambda) \circ (p^\dagger \circ i^\dagger) \circ \underline{s}_\delta^\dagger$ (where $\mathrm{Inn}(\lambda)$ denotes the inner automorphism obtained by conjugating by $\lambda$). As obtained in Lemma 2.4, $p^\dagger \circ \underline{d}^\dagger$ maps $D_X^\dagger$ onto the diagonal subgroup of $\Pi_X^\dagger \times_{G_K^\dagger} \Pi_X^\dagger$, hence $\mathrm{Inn}(\lambda)$ preserves the diagonal subgroup. Since $\Pi_X^\dagger$ is center-free (by Lemma 1.4 (iii)), it is verified that $\lambda$ lies in the diagonal. By taking a lifting $\tilde{\lambda} \in D_X^\dagger$ of $\lambda$ and replacing $\underline{s}_\delta^\dagger$ by $\mathrm{Inn}(\tilde{\lambda}^{-1}) \circ \underline{s}_\delta^\dagger$, we can make the rectangle in $(*)^\dagger$ commute in the strict sense. Next, we observe that $s_\delta^\dagger \circ \underline{d}^\dagger = \mathrm{Inn}(\mu) \circ \underline{d}^\dagger \circ \underline{s}_\delta^\dagger$

for some $\mu \in \Pi^\dagger_{X_2}$. By the commutativity of the rectangle in $(*)^\dagger$, $\mu$ projects via $p^\dagger$ into the center of $\Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X$ (hence, to the unit element). Therefore, replacing $s^\dagger_\delta$ by $\mathrm{Inn}(\mu^{-1}) \circ s^\dagger_\delta$, we can make $(*)^\dagger$ commute.

Thus, we obtain an automorphism of an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_X & \longrightarrow & D^\dagger_X & \xrightarrow{p^\dagger_1 \circ \underline{d}^\dagger = p^\dagger_2 \circ \underline{d}^\dagger} & \Pi^\dagger_X & \longrightarrow & 1 \\ & & {\scriptstyle \mathrm{id}} \downarrow \wr & & {\scriptstyle \underline{s}^\dagger_\delta} \downarrow \wr & & {\scriptstyle \mathrm{id}} \downarrow \wr & & \\ 1 & \longrightarrow & I_X & \longrightarrow & D^\dagger_X & \xrightarrow{p^\dagger_1 \circ \underline{d}^\dagger = p^\dagger_2 \circ \underline{d}^\dagger} & \Pi^\dagger_X & \longrightarrow & 1 \end{array}$$

which consists of the identity morphisms of $\Pi^\dagger_X$ and $I_X$. Let $\mathbb{M} \subseteq \mathbb{Q}$ be the monoid of positive rational numbers with denominators of $l$-power, and $\mathcal{N}$ the global sections of the sheaf of monoids defining the log structure on a universal geometrically pro-$l$ két covering of $\mathbb{D}^{\log} \times_{\overline{X}^{\log}} x^{\log}$. Then, $\mathcal{N}$ forms a direct product splitting $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus \overline{K}^\times$, where (cf. the discussion (1) preceding this proposition) the first (resp., second) factor of direct product is due to extracting roots of a local uniformizer of the divisor $\overline{X}_x \subseteq \overline{X}_2$ (resp., $\mathbb{D} \subseteq \overline{X}_2$) at $\tilde{x}$ inside of a choice of $\sigma$. Then the automorphism $\underline{s}^\dagger_\delta$ of $D^\dagger_X$ may be characterized by the choice of a projective system $\{(-1)^{\frac{1}{l^m}}\}_{m \in \mathbb{Z}_{\geq 0}}$ of $l$-power roots of $-1$ (well-defined up to multiplications by projective systems of $l$-power roots of 1) in a way that the automorphism of $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus (\overline{K}^\times)$ is expressed as

$$\left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, k\right) \mapsto \left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, (-1)^{\frac{a_2}{l^{m_2}}} \cdot k\right)$$

(cf. Lemma 2.1 (ii))). We observe that for any element $g$ of $G^\dagger_K$, the automorphism of $\mathcal{N}$ corresponding to $\sigma^\dagger(g)$ (resp., $(s^\dagger \circ \sigma^\dagger_X)(g)) \in \Pi^\dagger_{X_2}$ is expressed as

$$\left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, k\right) \mapsto \left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, g(k)\right)$$

$$\Big(\text{resp.,}\quad \left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, k\right) \mapsto \left(\frac{a_1}{l^{m_1}}, \frac{a_2}{l^{m_2}}, \frac{g((-1)^{\frac{a_2}{l^{m_2}}})}{(-1)^{\frac{a_2}{l^{m_2}}}} \cdot g(k)\right)\ \Big)$$

Thus, $(s^\dagger \circ \sigma^\dagger_X)(g) \cdot \sigma^\dagger_X(g)^{-1}$ is valued in $I_X$ ($\cong I_{\tilde{x}}$ by Lemma 2.3) and the 1-cocycle $g \mapsto (s^\dagger \circ \sigma^\dagger_X)(g) \cdot \sigma^\dagger_X(g)^{-1}$ represents, by definition, the Kummer class $-1 \in (K^\times)^\wedge \cong H^1(G^\dagger_K, \mathbb{Z}_l(1))$. Therefore, after modifications of $\underline{s}^\dagger_\delta$, $s^\dagger_\delta$ by some $I_X$-inner automorphisms, condition (2) is satisfied. This completes the existence assertion.

Next we prove the uniqueness portion. If we take two triples $(\underline{s}^\dagger_1, s^\dagger_1, \overline{s}^\dagger_1)$, $(\underline{s}^\dagger_2, s^\dagger_2, \overline{s}^\dagger_2)$ so as to satisfy conditions (1) and (2), then $s^\dagger_1 \circ (s^\dagger_2)^{-1} = \mathrm{Inn}(\eta) \in \mathrm{Aut}(\Pi^\dagger_{X_2})$ for some $\eta \in \Pi^\dagger_{X_2}$ and we see that $\mathrm{Inn}(\eta)$ preserves the subgroup $D^\dagger_X \subseteq \Pi^\dagger_{X_2}$. Since $D^\dagger_X$ is normally terminal in $\Pi^\dagger_{X_2}$, it is verified that $\eta$ is in $\Pi^\dagger_{D^{\log}_X}$. Moreover, from condition (1) and the fact that $\Pi^\dagger_X$ is center-free, $\eta$ lies in $\mathrm{Ker}\big(D^\dagger_X \stackrel{p^\dagger \circ \underline{d}^\dagger}{\to} \Pi^\dagger_X \times_{G^\dagger_K} \Pi^\dagger_X\big)$, i.e., $\eta \in I_X$. On the other hand, as the section

$\sigma^\dagger$ acts on $I_X$ via the cyclotomic character, which is a faithful action, condition (2) implies that $\eta$ is the unit element, i.e., that $s_1^\dagger = s_2^\dagger$, hence $(\underline{s}_1^\dagger, s_1^\dagger, \overline{s}_1^\dagger) = (\underline{s}_2^\dagger, s_2^\dagger, \overline{s}_2^\dagger)$. □

**Remark 2.5.1.**
In the case $l \neq 2$, $-1$ coincides with the unit element 1 in $(K^\times)^\wedge$. Then, in the statement of Proposition 2.5, by taking a 1-cocycle map $\delta$ to be trivial, we may obtain $s_\delta^\dagger$ satisfying $s_\delta^\dagger \circ \sigma^\dagger = \sigma^\dagger$.

## 3. The proof of Theorem A

We begin with a review of the notation and the setup. Let $l$ be a prime number, $K$ a finite field in which $l$ is invertible, and $\overline{K}$ a separable closure of $K$. We shall denote by $G_K$ the Galois group of $\overline{K}$ over $K$. Next, let $X$ be a hyperbolic curve over $K$ of type $(g,r)$, $x^{\log}$ a strict $K$-rational log point of $\overline{X}^{\log} := \overline{X}_1^{\log}$, and write $\overline{X}_x^{\log} := \overline{X}_2^{\log} \times_{\overline{X}^{\log}} x^{\log}$, $\tilde{x}^{\log} := \mathbb{D}^{\log} \times_{\overline{X}_1^{\log}} x^{\log}$, $G_{K^{\log}} := \Pi_{x^{\log}}$. In addition, we assume that we have fixed choices of specific homomorphisms

$$i_1^\Pi : \Pi_{\overline{X}_x^{\log}} \longrightarrow \Pi_{X_2}, \qquad p_1^\Pi : \Pi_{X_2} \longrightarrow \Pi_X$$

(they determine a structure morphism $f_X : \Pi_{\overline{X}_x^{\log}} \to G_{K^{\log}}$ by taking $f_X = i_1^\Pi \circ p_1^\Pi : \Pi_{\overline{X}_x^{\log}} \to G_{K^{\log}} \big(:= \mathrm{Im}(\Pi_{\overline{X}_x^{\log}} \stackrel{i_1^\Pi \circ p_1^\Pi}{\to} \Pi_X)\big)\big)$, within in the respective conjugacy classes determined by these homomorphisms, a choice of specific decomposition group $D_{\tilde{x}} \subseteq \Pi_{\overline{X}_x^{\log}}$ associated to $\tilde{x}$, where we write

$$\gamma_x : D_{\tilde{x}} \longrightarrow \Pi_{\overline{X}_x^{\log}}$$

for the natural inclusion, and a choice of a section

$$\sigma_X : G_K \longrightarrow D_{\tilde{x}}$$

of the composite surjection $D_{\tilde{x}} \stackrel{\gamma_x}{\to} \Pi_{\overline{X}_x^{\log}} \stackrel{f_X}{\to} G_{K^{\log}} \to G_K$.

**Definition 3.1.**

(i) The section $\sigma_X$ determines, by composing with the morphism $D_{\tilde{x}} \stackrel{\gamma_x}{\to} \Pi_{\overline{X}_x^{\log}}$ (resp., $D_{\tilde{x}} \stackrel{i_1^\Pi \circ \gamma_x}{\longrightarrow} \Pi_{X_2}$, $D_{\tilde{x}} \stackrel{p^\Pi \circ i_1^\Pi \circ \gamma_x}{\longrightarrow} \Pi_{X^{\times 2}}$), a natural conjugate $G_K$-action on $\Delta^1_{X_{2/1}} \cong \mathrm{Ker}(\Pi_{\overline{X}_x^{\log}} \stackrel{f_X}{\twoheadrightarrow} G_{K^{\log}})$ (resp., on $\Delta_{X_2}$, on $\Delta_{X^{\times 2}}$), hence also on

$$\mathrm{Gr}^1_{X_{2/1}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}}),$$

$$\big(\text{resp., } \mathrm{Gr}_{X_2} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2}),\ \mathrm{Gr}_{X^{\times 2}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X^{\times 2}})\big),$$

$$\mathrm{Lie}^1_{X_{2/1}} := \mathrm{Lie}(\Delta^1_{X_{2/1}}(1/\infty)),$$
$$\big(\text{resp., } \mathrm{Lie}_{X_2} := \mathrm{Lie}(\Delta_{X_2}(1/\infty)),\ \mathrm{Lie}_{X^{\times 2}} := \mathrm{Lie}(\Delta_{X^{\times 2}}(1/\infty))\big),$$

$$\mathrm{Lin}^1_{X_{2/1}} := \mathrm{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(\mathbb{Q}_l).$$
$$\big(\text{resp., } \mathrm{Lin}_{X_2} := \mathrm{Lin}(\Delta_{X_2}(1/\infty))(\mathbb{Q}_l),\ \mathrm{Lin}_{X^{\times 2}} := \mathrm{Lin}(\Delta_{X^{\times 2}}(1/\infty))(\mathbb{Q}_l)\big).$$

In the following, we regard these objects as to be equipped with the $G_K$-actions in this sense. From the discussion in Definition 1.5 (ii), we have the following commutative diagram consisting of $G_K$-equivariant morphisms

$$\begin{array}{ccccc}
\Delta^1_{X_{2/1}} & \xrightarrow{i_1} & \Delta_{X_2} & \xrightarrow{p} & \Delta_{X^{\times 2}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Lin}^1_{X_{2/1}} & \xrightarrow{i_1^{\mathrm{Lin}}} & \mathrm{Lin}_{X_2} & \xrightarrow{p^{\mathrm{Lin}}} & \mathrm{Lin}_{X^{\times n}}
\end{array}$$

and topological groups equipped with the additional $G_K$-action structures

$$\Delta^{\mathrm{Lie}}_{X_2} := \Delta_{X^{\times 2}} \times_{\mathrm{Lin}_{X^{\times 2}}} \mathrm{Lin}_{X_2}, \quad \Pi^{\mathrm{Lie}}_{X_2} := \Delta^{\mathrm{Lie}}_{X_2} \rtimes G_K$$

as well as $G_K$-equivariant homomorphisms of topological groups

$$\mathrm{Int}^{\Delta}_X : \Delta_{X_2} \to \Delta^{\mathrm{Lie}}_{X_2}, \quad \mathrm{Int}^{\Pi}_X : \Pi_{X_2} \to \Pi^{\mathrm{Lie}}_{X_2}.$$

(ii) We shall fix a 1-cocycle map $\delta_X : G_K \to I_{\tilde{x}} := \mathrm{Ker}(D_{\tilde{x}} \twoheadrightarrow G_{K^{\log}})$ representing the Kummer class $-1$ (cf. the discussion preceding Proposition 2.5). Then, taking a product

$$\sigma_{\delta_X} : G_K \longrightarrow D_{\tilde{x}}; g \mapsto \delta_X(g) \cdot \sigma_X(g)$$

— which is a homomorphism of topological groups — gives a new section of the surjective homomorphism $D_{\tilde{x}} \twoheadrightarrow G_K$. In a similar way to (i), the section $\sigma_{\delta_X}$ determines a natural conjugate $G_K$-action on

$$\breve{\mathrm{Gr}}^1_{X_{2/1}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^1_{X_{2/1}}),$$
$$\big(\text{resp., } \breve{\mathrm{Gr}}_{X_2} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2}),\ \breve{\mathrm{Gr}}_{X^{\times 2}} := \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X^{\times 2}})\big),$$

$$\breve{\mathrm{Lie}}^1_{X_{2/1}} := \mathrm{Lie}(\Delta^1_{X_{2/1}}(1/\infty)),$$
$$\big(\text{resp., } \breve{\mathrm{Lie}}_{X_2} := \mathrm{Lie}(\Delta_{X_2}(1/\infty)),\ \breve{\mathrm{Lie}}_{X^{\times 2}} := \mathrm{Lie}(\Delta_{X^{\times 2}}(1/\infty))\big),$$

$$\breve{\mathrm{Lin}}^1_{X_{2/1}} := \mathrm{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(\mathbb{Q}_l).$$
$$\big(\text{resp., } \breve{\mathrm{Lin}}_{X_2} := \mathrm{Lin}(\Delta_{X_2}(1/\infty))(\mathbb{Q}_l),\ \breve{\mathrm{Lin}}_{X^{\times 2}} := \mathrm{Lin}(\Delta_{X^{\times 2}}(1/\infty))(\mathbb{Q}_l)\big).$$

— where, in the following, we regard these objects as to be equipped with the $G_K$-actions in this sense — as well as topological groups equipped with the additional $G_K$-action structures

$$\breve{\Delta}^{\mathrm{Lie}}_{X_2} := \Delta_{X^{\times 2}} \times_{\breve{\mathrm{Lin}}_{X^{\times 2}}} \breve{\mathrm{Lin}}_{X_2}, \quad \breve{\Pi}^{\mathrm{Lie}}_{X_2} := \breve{\Delta}^{\mathrm{Lie}}_{X_2} \rtimes G_K$$

Next, by taking account of $\delta_X$ and the setup of $i^{\Pi}_1$, $p^{\Pi}_1$, $\gamma_x$, $\sigma_X$ fixed above, we may take an automorphism $s^{\Pi}_{\delta_X} : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$, arising from the switching morphism $s_X : \overline{X}^{\log}_2 \to \overline{X}^{\log}_2$ inducing a uniquely determined morphism $s^{\dagger}_{\delta_X} : \Pi^{\dagger}_{X_2} \to \Pi^{\dagger}_{X_2}$ obtained in Section 2, and obtain $G_K$-equivariant homomorphisms of topological groups

$$s^{\Delta^{\mathrm{Lie}}}_{\delta_X} : \Delta^{\mathrm{Lie}}_{X_2} \xrightarrow{\sim} \breve{\Delta}^{\mathrm{Lie}}_{X_2}, \quad s^{\Pi^{\mathrm{Lie}}}_{\delta_X} : \Pi^{\mathrm{Lie}}_{X_2} \xrightarrow{\sim} \breve{\Pi}^{\mathrm{Lie}}_{X_2}$$

induced by $s^{\Pi}_{\delta_X}$.

**Lemma 3.2.**
*The $G_K$-action induced by $\sigma_{\delta_X}$ defined in Definition 3.1 (ii) on $\Delta_{X_2}$ (hence also on $\breve{\mathrm{Gr}}_{X_2}$, $\breve{\mathrm{Lie}}_{X_2}$, $\breve{\mathrm{Lin}}_{X_2}$ and $\breve{\Delta}^{\mathrm{Lie}}_{X_2}$) coincides with the action defined in a way that*

$$G_K \longrightarrow \mathrm{Aut}(\Delta_{X_2}); g \mapsto \mathrm{Inn}(s^{\Pi}_{\delta_X} \circ i^{\Pi}_1 \circ \gamma_x \circ \sigma_X(g)).$$

*Proof.* It follows immediately from the definition of $G_K$-action induced by $\sigma_{\delta_X}$. □

**Lemma 3.3.**
$\mathrm{Int}^{\Delta}_X$ *and* $\mathrm{Int}^{\Pi}_X$ *are injective.*

*Proof.* It is enough to verify that $\Delta_{X_2} \to \mathrm{Lin}_{X_2}$ is injective. But it follows from the discussion in Definition 1.5 (ii) and the fact that $\bigcap_{m\geq 1} \Delta_X(m) = 1$ (cf. [17], Corollary 2.6). □

**Definition 3.4.**

(i) Let us fix a choice of each inertia subgroup $I_j \subseteq \Delta^1_{X_{2/1}} \cong \mathrm{Ker}(\Pi_{\overline{X}^{\log}_x} \overset{f_X}{\twoheadrightarrow} G_{K^{\log}})$ $(j = 1, 2, \cdots, r)$ associated to the $j$-th cusp (relative to some ordering of the cusps of $X \times_K \overline{K}$) among the various $\Delta^1_{X_{2/1}}$-conjugates of these subgroups. Then, we have a canonical isomorphisms

$$\eta_j : I_{\tilde{x}} \xrightarrow{\sim} I_j \quad (j = 1, 2, \cdots, r)$$

For $n = 1, 2$ we shall denote by $\mathcal{V}^n$ the completion with respect to the filtration topology of the free Lie algebra generated by

$$V^n := I_{\tilde{x}} \oplus \Big(\bigoplus_{j=1}^{r} I_j \oplus \Delta^{\mathrm{ab}}_{\overline{X}}\Big)^{\oplus n}$$

equipped with a natural grading (hence also a filtration) by taking $I_{\tilde{x}}, I_j$ to be of weight 2, $\Delta_{\overline{X}}^{\mathrm{ab}}$ to be of weight 1.

(ii) If $X$ has *genus* $\geq 1$, then we shall write

$$M_X := \mathrm{Hom}_{\mathbb{Z}_l}(H^2(\Delta_{\overline{X}}, \mathbb{Z}_l), \mathbb{Z}_l) \ .$$

Note that $M_X$ is canonically isomorphic to $I_{\tilde{x}}$ as a $G_K$-module. The cup product on the group cohomology of $\Delta_{\overline{X}}$

$$\bigwedge^2 H^1(\Delta_{\overline{X}}, M_X) \longrightarrow H^2(\Delta_{\overline{X}}, M_X \otimes_{\mathbb{Z}_l} M_X)$$

determines an isomophism

$$\mathrm{Hom}(\Delta_{\overline{X}}^{\mathrm{ab}}, M_X) \xrightarrow{\sim} \Delta_{\overline{X}}^{\mathrm{ab}},$$

hence compositions of natural homomorphisms

$$\phi : I_{\tilde{x}} \xrightarrow{\sim} M_X \longrightarrow \bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}}, \quad \psi : \bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}} \longrightarrow M_X \xrightarrow{\sim} I_{\tilde{x}}$$

(iii) If $X$ has *genus* $\geq 1$ (resp., *genus* $= 0$), then we define $\mathcal{L}_X^n$ to be the quotient of $\mathcal{V}^n$ by the relations determined by the images of the following morphisms:

(1) When $n = 1$,

$$\bullet_1 \ I_{\tilde{x}} \longrightarrow \mathcal{V}^1(2/3) \ ; \ m \mapsto (\mathrm{id}_{I_{\tilde{x}}} + \textstyle\sum \eta_j + \phi)(m)$$

$$\big(\text{resp., } \bullet_1 \ I_{\tilde{x}} \longrightarrow \mathcal{V}^1(2/3) \ ; \ m \mapsto (\mathrm{id}_{I_{\tilde{x}}} + \textstyle\sum \eta_j)(m)\big).$$

(2) When $n = 2$ $(1 \leq i \leq g, \ 1 \leq j \leq r, \ \{k, k'\} = \{1, 2\})$,

$$\bullet_1 \ I_{\tilde{x}} \longrightarrow \mathcal{V}^2(2/3) \ ; \ m \mapsto m + i_k(\textstyle\sum \eta_j + \phi)(m)$$

$$\bullet_2 \ I_{\tilde{x}} \otimes_{\mathbb{Z}_l} \Delta_{\overline{X}}^{\mathrm{ab}} \longrightarrow \mathcal{V}^2(3/4) \ ; m \otimes a \mapsto [i_k \circ \eta_j(m), i_{k'}(a)]$$

$$\bullet_3 \ \textstyle\bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}} \longrightarrow \mathcal{V}^2(2/3) \ ; a \wedge a' \mapsto [i_k(a), i_{k'}(a')]$$

$$\bullet_{4,5} \ \textstyle\bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}} \longrightarrow \mathcal{V}^2(2/3) \ ; a \wedge a' \mapsto i_k(a) \wedge i_{k'}(a') - \psi(a \wedge a')$$

$$\big(\text{resp., } \bullet_1 \ I_{\tilde{x}} \longrightarrow \mathcal{V}^2(2/3) \ ; \ m \mapsto m + i_k(\textstyle\sum \eta_j)(m)\big)$$

— where "[ , ]" denotes the Lie bracket, and for $k = 1, 2, \ i_k : (\bigoplus I_j \oplus \bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}}) \hookrightarrow (\bigoplus I_j \oplus \bigwedge^2 \Delta_{\overline{X}}^{\mathrm{ab}})^{\oplus 2}$ denotes the inclusion into the $k$-th factor.

(iv) If we consider a $G_K$-action on $\mathcal{V}^n$ as that taking the natural action on each direct summand in $\mathcal{V}^n$, then the ideal generated by the relations defined in (iii) is invariant under this action. Hence, we may equip the graded Lie algebra

$$\mathcal{L}_X^1 \ \ (\text{resp.}, \mathcal{L}_X^2)$$

with the additional $G_K$-action structure and have a $G_K$-equivariant homomorphism

$$i_1^{\mathcal{L}} : \mathcal{L}_X^1 \longrightarrow \mathcal{L}_X^2$$

of graded Lie algebras given by

$$\begin{array}{ccc} I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big) & \longrightarrow & I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} \\ (a,b) & \mapsto & (a, i_1(b)) \end{array}$$

and a $G_K$-equivariant isomorphism

$$s_X^{\mathcal{L}} : \mathcal{L}_X^2 \xrightarrow{\sim} \mathcal{L}_X^2$$

of graded Lie algebras given by

$$\begin{array}{ccc} I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} & \longrightarrow & I_{\tilde{x}} \oplus \big(\bigoplus_{j=1}^{r} I_j \oplus \Delta_{\overline{X}}^{\mathrm{ab}}\big)^{\oplus 2} \\ (a,b_1,b_2) & \mapsto & (a, b_2, b_1) \end{array}$$

**Lemma 3.5.**
*Let $h_{\mathrm{Gr}}^1 : \mathcal{L}_X^1 \to \mathrm{Gr}_{X_{2/1}}^1$, $\breve{h}_{\mathrm{Gr}}^1 : \mathcal{L}_X^1 \to \breve{\mathrm{Gr}}_{X_{2/1}}^1$ be homomorphisms of graded Lie algebra given by the natural inclusions $\Delta_{\overline{X}}^{\mathrm{ab}} \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{2/1}}^1)(1/2)$, $I_{\tilde{x}} \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{2/1}}^1)(2/3)$ and $I_j \hookrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{2/1}}^1)(2/3)$.*
*Then $h_{\mathrm{Gr}}^1$ and $\breve{h}_{\mathrm{Gr}}^1$ are $G_K$-equivariant isomorphisms of graded Lie algebras.*

*Proof.* It is enough to verify the assertion in the case where $x$ is cusp of $X$. But since $\Delta_{\overline{\mathbb{P}}_X}$ is trivial, it follows that the natural inclusion $\Pi_{\overline{X}^{\log\prime}} \hookrightarrow \Pi_{\overline{X}_x^{\log}}$ (well-defined up to inner automorphisms) induces a $G_K$-equivarinant isomorphism $\Delta_{\overline{X}}^{\mathrm{ab}} \xrightarrow{\sim} \Delta_{\overline{X}_x}^{\mathrm{ab}}$. This completes the proof by Proposition 1.9 (i). $\square$

**Lemma 3.6.**
*Let*

$$i_1^{\mathrm{Lie}} : \mathrm{Lie}_{X_{2/1}}^1 \longrightarrow \mathrm{Lie}_{X_2}, \quad i_2^{\mathrm{Lie}} : \breve{\mathrm{Lie}}_{X_{2/1}}^1 \longrightarrow \breve{\mathrm{Lie}}_{X_2}, \quad s_{\delta_X}^{\mathrm{Lie}} : \mathrm{Lie}_{X_2} \xrightarrow{\sim} \breve{\mathrm{Lie}}_{X_2}$$

*be the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$, $i_1^{\Pi} : \Pi_{\overline{X}_x^{\log}} \to \Pi_{X_2}$ and $s_{\delta_X}^{\Pi} : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$ respectively.*
*Then there exist $G_K$-equivariant isomorphisms of graded Lie algebras*

$$h_X^1 : \mathcal{L}_X^1 \xrightarrow{\sim} \mathrm{Lie}_{X_{2/1}}^1, \quad \breve{h}_X^1 : \mathcal{L}_X^1 \xrightarrow{\sim} \breve{\mathrm{Lie}}_{X_{2/1}}^1$$

$$h_X^2 : \mathcal{L}_X^2 \xrightarrow{\sim} \mathrm{Lie}_{X_2}, \quad \breve{h}_X^2 : \mathcal{L}_X^2 \xrightarrow{\sim} \breve{\mathrm{Lie}}_{X_2}$$

*which fit into the following commutative diagrams consisting of $G_K$-equivariant mophisms*

$$\begin{array}{ccc} \mathcal{L}^1_X & \xrightarrow{i^{\mathcal{L}}_1} & \mathcal{L}^2_X \\ {\scriptstyle h^1_X}\downarrow\wr & & {\scriptstyle h^2_X}\downarrow\wr \\ \mathrm{Lie}^1_{X_{2/1}} & \xrightarrow{i^{\mathrm{Lie}}_1} & \mathrm{Lie}_{X_2} \end{array} \quad \begin{array}{ccc} \mathcal{L}^1_X & \xrightarrow{i^{\mathcal{L}}_1} & \mathcal{L}^2_X \\ {\scriptstyle \breve{h}^1_X}\downarrow\wr & & {\scriptstyle \breve{h}^2_X}\downarrow\wr \\ \breve{\mathrm{Lie}}^1_{X_{2/1}} & \xrightarrow{\breve{i}^{\mathrm{Lie}}_1} & \breve{\mathrm{Lie}}_{X_2} \end{array} \quad \begin{array}{ccc} \mathcal{L}^2_X & \xrightarrow{s^{\mathcal{L}}_{\delta_X}} & \mathcal{L}^2_X \\ {\scriptstyle h^2_X}\downarrow\wr & & {\scriptstyle \breve{h}^2_X}\downarrow\wr \\ \mathrm{Lie}_{X_2} & \xrightarrow{s^{\mathrm{Lie}}_{\delta_X}} & \breve{\mathrm{Lie}}_{X_2}. \end{array}$$

*Proof.* Let $h^2 : \mathcal{L}^2_X \to \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2})$ be the homomorphism given by

$$I_{\tilde{x}} \oplus \Big(\bigoplus_{j=1}^{r} I_j \oplus \Delta^{\mathrm{ab}}_{\overline{X}}\Big)^{\oplus 2} \longrightarrow \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_2})$$

$$(a, b_1, b_2) \quad \mapsto i^{\mathrm{Gr}}_1 \circ h^1_{\mathrm{Gr}}(a + b_1) + s^{\mathrm{Gr}}_{\delta_X} \circ i^{\mathrm{Gr}}_1 \circ h^1_{\mathrm{Gr}}(b_2).$$

It is verified that $h^2$ is an isomorphism of graded Lie algebras by Proposition 1.9 (ii) and $G_K$-equivariant when we regard it as a map $h^2_{\mathrm{Gr}} : \mathcal{L}^2_X \to \mathrm{Gr}_{X_2}$ as well as $\breve{h}^2_{\mathrm{Gr}} : \mathcal{L}^2_X \to \breve{\mathrm{Gr}}_{X_2}$. If we denote $i^{\mathrm{Gr}}_1 : \mathrm{Gr}^1_{X_{2/1}} \to \mathrm{Gr}_{X_2}$, $\breve{i}^{\mathrm{Gr}}_1 : \breve{\mathrm{Gr}}^1_{X_{2/1}} \to \breve{\mathrm{Gr}}_{X_2}$, $s^{\mathrm{Gr}}_{\delta_X} : \mathrm{Gr}_{X_2} \xrightarrow{\sim} \breve{\mathrm{Gr}}_{X_2}$ be the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i^{\Pi}_1 : \Pi_{\overline{X}^{\log}_x} \to \Pi_{X_2}$, $i^{\Pi}_1 : \Pi_{\overline{X}^{\log}_x} \to \Pi_{X_2}$ and $s^{\Pi}_{\delta_X} : \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_2}$ respectively, then we have the following $G_K$-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{L}^1_X & \xrightarrow{i^{\mathcal{L}}_1} & \mathcal{L}^2_X \\ {\scriptstyle h^1_{\mathrm{Gr}}}\downarrow\wr & & {\scriptstyle h^2_{\mathrm{Gr}}}\downarrow\wr \\ \mathrm{Gr}^1_{X_{2/1}} & \xrightarrow{i^{\mathrm{Gr}}_1} & \mathrm{Gr}_{X_2} \end{array} \quad \begin{array}{ccc} \mathcal{L}^1_X & \xrightarrow{i^{\mathcal{L}}_1} & \mathcal{L}^2_X \\ {\scriptstyle \breve{h}^1_{\mathrm{Gr}}}\downarrow\wr & & {\scriptstyle \breve{h}^2_{\mathrm{Gr}}}\downarrow\wr \\ \breve{\mathrm{Gr}}^1_{X_{2/1}} & \xrightarrow{\breve{i}^{\mathrm{Gr}}_1} & \breve{\mathrm{Gr}}_{X_2} \end{array} \quad \begin{array}{ccc} \mathcal{L}^2_X & \xrightarrow{s^{\mathcal{L}}_{\delta_X}} & \mathcal{L}^2_X \\ {\scriptstyle h^2_{\mathrm{Gr}}}\downarrow\wr & & {\scriptstyle \breve{h}^2_{\mathrm{Gr}}}\downarrow\wr \\ \mathrm{Gr}_{X_2} & \xrightarrow{s^{\mathrm{Gr}}_{\delta_X}} & \breve{\mathrm{Gr}}_{X_2}. \end{array}$$

On the other hand, by taking account of the compatibility of $G_K$-actions (cf. Lemma 3.2) together with Proposition 1.8, we have the following $G_K$-equivariant commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^1_{X_{2/1}} & \xrightarrow{i^{\mathrm{Gr}}_1} & \mathrm{Gr}_{X_2} \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Lie}^1_{X_{2/1}} & \xrightarrow{i^{\mathrm{Lie}}_1} & \mathrm{Lie}_{X_2} \end{array} \quad \begin{array}{ccc} \breve{\mathrm{Gr}}^1_{X_{2/1}} & \xrightarrow{\breve{i}^{\mathrm{Gr}}_1} & \breve{\mathrm{Gr}}_{X_2} \\ \downarrow\wr & & \downarrow\wr \\ \breve{\mathrm{Lie}}^1_{X_{2/1}} & \xrightarrow{\breve{i}^{\mathrm{Lie}}_1} & \breve{\mathrm{Lie}}_{X_2} \end{array} \quad \begin{array}{ccc} \mathrm{Gr}_{X_2} & \xrightarrow{s^{\mathrm{Gr}}_{\delta_X}} & \breve{\mathrm{Gr}}_{X_2}. \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Lie}_{X_2} & \xrightarrow{s^{\mathrm{Lie}}_{\delta_X}} & \breve{\mathrm{Lie}}_{X_2}. \end{array}$$

By composing the vertical arrows in these commutative diagrams, we obtain the required isomorphisms. □

Now, let $L$ be a finite field of cardinality prime to $l$, $Y$ a hyperbolic curve over $L$ of type $(g', r')$, $y^{\log}$ a strict $L$-rational log point of $\overline{Y}^{\log} := \overline{Y}^{\log}_1$; we shall use similar notation for objects obtained from $Y$ (e.g., $Y_2$, $\overline{Y}^{\log}_y$, $\Pi_{Y_2}$, $\Pi_{\overline{Y}^{\log}_y}$, $G_{L^{\log}}$, etc.) to the notation used for objects obtained from $X$. Moreover, we shall fix a choice of homomorphisms

$$i_1^{\Pi}:\Pi_{\overline{Y}_y^{\log}}\longrightarrow\Pi_{Y_2},\qquad p_1^{\Pi}:\Pi_{Y_2}\longrightarrow\Pi_Y$$

arising from the morphisms $\overline{Y}_y^{\log}\xrightarrow{i_1}Y_2$, $Y_2\xrightarrow{p_1}Y$ within in the respective conjugacy classes determined by these homomorphisms.

**Definition 3.7.**

Let $\beta$ be an isomorphism of profinite groups $\Pi_{\overline{X}_x^{\log}}\xrightarrow{\sim}\Pi_{\overline{Y}_y^{\log}}$ or $\Pi_X\xrightarrow{\sim}\Pi_Y$. Then the natural quotient $\Pi_{\overline{X}_x^{\log}}\twoheadrightarrow G_K$ (resp., $\Pi_{\overline{Y}_x^{\log}}\twoheadrightarrow G_L,\Pi_X\twoheadrightarrow G_K,\Pi_Y\twoheadrightarrow G_L$) arising from the structure morphism $\overline{X}_x^{\log}\rightarrow\mathrm{Spec}(K)$ (resp., $\overline{Y}_y^{\log}\rightarrow\mathrm{Spec}(L),X\rightarrow\mathrm{Spec}(K),Y\rightarrow\mathrm{Spec}(L)$) may be characterized group-theoretically (cf. [19], Proposition 3.3) as the (unique) maximal ($\hat{\mathbb{Z}}$-)free abelian quotient of $\Pi_{\overline{X}_x^{\log}}$ (resp., $\Pi_{\overline{Y}_y^{\log}}$, $\Pi_X$, $\Pi_Y$). Therefore $\beta$ induces an isomorphism $\beta_0:G_K\xrightarrow{\sim}G_L$.

Now we shall say that $\beta$ is *Frobenius-preserving* if the isomorphism $\beta_0:G_K\xrightarrow{\sim}G_L$ obtained as above preserves the Frobenius elements.

In the following, suppose further that we have given a Frobenius-preserving isomorphism $\alpha:\Pi_{\overline{X}_x^{\log}}\xrightarrow{\sim}\Pi_{\overline{Y}_y^{\log}}$ that maps $D_{\tilde{x}}(\overset{\gamma_x}{\hookrightarrow}\Pi_{\overline{X}_x^{\log}})$ onto a specific decomposition group $D_{\tilde{y}}$ of $\tilde{y}$. Then we may take a section $\sigma_Y:G_L\rightarrow D_{\tilde{y}}$ of the natural surjection $D_{\tilde{y}}\twoheadrightarrow G_L$ and a 1-cocycle map $\delta_Y:G_L\rightarrow I_{\tilde{y}}:=\mathrm{Ker}(D_{\tilde{y}}\twoheadrightarrow G_{L^{\log}})$ of $G_L$ with coefficients in $I_{\tilde{y}}$ so as to be compatible with $\sigma_X$, $\delta_X$ via isomorphisms $\alpha$ and $\alpha_0:G_K\xrightarrow{\sim}G_L$ induced by $\alpha$. By applying the preceding discussion and results in Section 2 to $Y$, we obtain objects $\Pi_{Y_2}^{\mathrm{Lie}}$, $\mathrm{Int}_Y^{\Pi}$, $s_{\delta_Y}^{\Pi^{\mathrm{Lie}}}$, etc.

**Proposition 3.8.**

*In the notation of the above situation, there exist a $G_K$-equivariant isomorphism $\alpha_2^{\Delta^{\mathrm{Lie}}}:\Delta_{X_2}^{\mathrm{Lie}}\xrightarrow{\sim}\Delta_{Y_2}^{\mathrm{Lie}}$ such that it is also $G_K$-equivariant when we regard it as a map $\breve{\Delta}_{X_2}^{\mathrm{Lie}}\xrightarrow{\sim}\breve{\Delta}_{Y_2}^{\mathrm{Lie}}$ and that if we denote by $\alpha_2^{\Pi^{\mathrm{Lie}}}:\Pi_{X_2}^{\mathrm{Lie}}\xrightarrow{\sim}\Pi_{Y_2}^{\mathrm{Lie}}$, $\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}:\breve{\Pi}_{X_2}^{\mathrm{Lie}}\xrightarrow{\sim}\breve{\Pi}_{Y_2}^{\mathrm{Lie}}$ the semi-direct products of $\alpha_2^{\Delta^{\mathrm{Lie}}}$ in these two ways, then these morphisms make the following diagrams commute*

$$\begin{tikzcd}
\Pi_{\overline{X}_x^{\log}} \arrow[r, "\mathrm{Int}_X^{\Pi}\circ i_1^{\Pi}"] \arrow[d, "\alpha", "\wr"'] & \Pi_{X_2}^{\mathrm{Lie}} \arrow[r] \arrow[d, "\alpha_2^{\Pi^{\mathrm{Lie}}}", "\wr"'] & \Pi_{X^{\times 2}} \arrow[d, "\overline{\alpha}\times\overline{\alpha}", "\wr"'] \\
\Pi_{\overline{Y}_y^{\log}} \arrow[r, "\mathrm{Int}_Y^{\Pi}\circ i_1^{\Pi}"] & \Pi_{Y_2}^{\mathrm{Lie}} \arrow[r] & \Pi_{Y^{\times 2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{X_2}^{\mathrm{Lie}} \arrow[r, "s_{\delta_X}^{\Pi^{\mathrm{Lie}}}"] \arrow[d, "\alpha_2^{\Pi^{\mathrm{Lie}}}"] & \breve{\Pi}_{X_2}^{\mathrm{Lie}} \arrow[d, "\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}"] \\
\Pi_{Y_2}^{\mathrm{Lie}} \arrow[r, "s_{\delta_Y}^{\Pi^{\mathrm{Lie}}}"] & \breve{\Pi}_{Y_2}^{\mathrm{Lie}}.
\end{tikzcd}$$

*Proof.* Since $\alpha$ is assumed to be Frobenius-preserving, it follows from [10], Lemma 1.3.9 and [12], Corollary 2.8 that $(g,r)=(g',r')$ and that $\alpha$ induces an isomorphism $\alpha^{\mathrm{cpt}} : \Delta^{\mathrm{ab}}_{\overline{X}} \xrightarrow{\sim} \Delta^{\mathrm{ab}}_{\overline{Y}}$ and a bijective correspondence between the respective sets of cusps of $\overline{X}^{\log}_x, \overline{Y}^{\log}_y$ as well as isomorphisms of the decompsition (inertia) groups of cusps corresponding via this bijection. By using these isomorphisms (together with constructions of $\mathcal{L}^1_X$, $\mathcal{L}^2_X$, $\mathcal{L}^1_Y$, $\mathcal{L}^2_Y$), Lemma 3.6 induces $G_K$-equivariant isomorphisms $\alpha^{\mathrm{Lie}} : \mathrm{Lie}^1_{X_{2/1}} \cong \mathrm{Lie}^1_{Y_{2/1}}$, $\breve{\alpha}^{\mathrm{Lie}} : \breve{\mathrm{Lie}}^1_{X_{2/1}} \cong \breve{\mathrm{Lie}}^1_{Y_{2/1}}$, $\alpha_2^{\mathrm{Lie}} : \mathrm{Lie}_{X_2} \cong \mathrm{Lie}_{Y_2}$ and $\breve{\alpha}_2^{\mathrm{Lie}} : \breve{\mathrm{Lie}}_{X_2} \cong \breve{\mathrm{Lie}}_{Y_2}$ such that $\alpha^{\mathrm{Lie}} = \breve{\alpha}^{\mathrm{Lie}}$ (by compatibility of $G_K$-actions induced by as a morphism of underlying graded Lie algebras and that these maps which make the following diagram commute in the sense of $G_K$-equivariant

$$\begin{tikzcd}
\mathrm{Lie}^1_{X_{2/1}} \arrow[r, "i_1^{\mathrm{Lie}}"] \arrow[d, "\alpha^{\mathrm{lie}}"', "\wr"] & \mathrm{Lie}_{X_2} \arrow[r, "s^{\mathrm{Lie}}_{\delta_X}"] \arrow[d, "\alpha_2^{\mathrm{Lie}}"', "\wr"] & \breve{\mathrm{Lie}}_{X_2} \arrow[d, "\breve{\alpha}_2^{\mathrm{Lie}}"', "\wr"] & \breve{\mathrm{Lie}}^1_{X_{2/1}} \arrow[l, "\breve{i}_1^{\mathrm{Lie}}"'] \arrow[d, "\breve{\alpha}^{\mathrm{lie}}"', "\wr"] \\
\mathrm{Lie}^1_{Y_{2/1}} \arrow[r, "i_1^{\mathrm{Lie}}"] & \mathrm{Lie}_{Y_2} \arrow[r, "s^{\mathrm{Lie}}_{\delta_Y}"] & \breve{\mathrm{Lie}}_{Y_2} & \breve{\mathrm{Lie}}^1_{Y_{2/1}}. \arrow[l, "\breve{i}_1^{\mathrm{Lie}}"']
\end{tikzcd}$$

Since $\sigma_X$, $\sigma_{\delta_X}$ have been taken to be compatible with $\sigma_Y$, $\sigma_{\delta_Y}$ via $\alpha$ respectively, we have $\alpha^{\mathrm{Lie}} = \breve{\alpha}^{\mathrm{Lie}}$, $i_1^{\mathrm{Lie}} = \breve{i}_1^{\mathrm{Lie}}$) as morphisms of underlying graded Lie algebras. On the other hand, $\mathrm{Lie}_{X_2}$ (resp., $\breve{\mathrm{Lie}}_{X_2}$) is, as a Lie algebra, generated by the images of $\mathrm{Lie}^1_{X_{2/1}} \xrightarrow{i_1^{\mathrm{Lie}}} \mathrm{Lie}_{X_2}$ (resp., $\breve{\mathrm{Lie}}^1_{X_{2/1}} \xrightarrow{\breve{i}_1^{\mathrm{Lie}}} \breve{\mathrm{Lie}}_{X_2}$) and the composition $\mathrm{Lie}^1_{X_{2/1}} \xrightarrow{i_1^{\mathrm{Lie}}} \mathrm{Lie}_{X_2} \xrightarrow{s^{\mathrm{Lie}}_{\delta_X}} \breve{\mathrm{Lie}}_{X_2} = \mathrm{Lie}_{X_2}$ (resp., $\breve{\mathrm{Lie}}^1_{X_{2/1}} \xrightarrow{\breve{i}_1^{\mathrm{Lie}}} \breve{\mathrm{Lie}}_{X_2} = \mathrm{Lie}_{X_2} \xrightarrow{s^{\mathrm{Lie}}_{\delta_X}} \breve{\mathrm{Lie}}_{X_2}$), hence we have $\alpha_2^{\mathrm{Lie}} = \breve{\alpha}_2^{\mathrm{Lie}}$. Therefore, it follows from the functoriality of $\mathrm{Lin}(-)$ that this diagram induces the required diagram.

□

One of main results of this paper, i.e., Theorem A, is as follows:

**Theorem 3.9.**

*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), $x$ a $K$-rational point of $\overline{X}$ (resp., $y$ an $L$-rational point of $\overline{Y}$), $X_2$ (resp., $Y_2$) the second configuration space associated to $X$ (resp., $Y$), $\overline{X}^{\log}_x$ (resp., $\overline{Y}^{\log}_y$) the cuspidalization of $X$ at $x$ (resp., of $Y$ at $y$) [cf. Definition 2.2], $D_{\tilde{x}} \overset{\gamma_x}{\hookrightarrow} \Pi_{\overline{X}^{\log}_x}$ (resp., $D_{\tilde{y}} \overset{\gamma_y}{\hookrightarrow} \Pi_{\overline{Y}^{\log}_y}$) the decomposition group of the diagonal cusp $\tilde{x}^{\log}$ (resp., $\tilde{y}^{\log}$).*

*Let*

$$\alpha : \Pi_{\overline{X}^{\log}_x} \xrightarrow{\sim} \Pi_{\overline{Y}^{\log}_y}$$

*be a Frobenius-preserving isomorphism [cf. Definition 3.5] which maps $D_{\tilde{x}}$ onto $D_{\tilde{y}}$. Here, we shall denote $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$, (resp., $D_x \overset{\overline{\gamma_x}}{\hookrightarrow} \Pi_X$, $D_y \overset{\overline{\gamma_y}}{\hookrightarrow} \Pi_Y$) by the isomorphism (resp., the decomposition group of $x$, the decomposition group of $y$) obtained by passing to the quotients $\Pi_{\overline{X}^{\log}_x} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}^{\log}_y} \twoheadrightarrow \Pi_Y$.*

*Then there exists a unique isomorphism*

$$\alpha_2 : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$$

*which is compatible with the natural switching automorphisms [cf. the discussion following Remark 2.1.1] up to an inner automorphism and fits into the following commutative square*

$$\begin{array}{ccc} \Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y \end{array} \qquad (**)$$

*which induce $\alpha$ by restricting $\alpha_2$ to the inverse images (via the vertical arrows) of $\overline{D}_x$ and $\overline{D}_y$.*

*Proof.* Let us take $s^{\Pi}_{\delta_X}$, $s^{\Pi}_{\delta_Y}$, $\alpha_2^{\Pi^{\mathrm{Lie}}}$, and $\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}$ as we obtained up to now. If we identify $(s^{\Pi}_X \circ i^{\Pi}_1)(\Delta^1_{X_{2/1}})$ with $i^{\Pi}_2(\Delta^2_{X_{2/1}})$ via the inclusion $\mathrm{Int}^{\Pi}_X : \Pi_{X_2} \hookrightarrow \Pi^{\mathrm{Lie}}_{X_2}$, then it follows from Proposition 1.4 (ii) that $i^{\Pi}_1(\Pi_{\overline{X}^{\log}_x})$ and $(s^{\Pi}_{\delta_X} \circ i^{\Pi}_1)(\Delta^1_{X_{2/1}}))$ generate $\Pi_{X_2}$ (similarly, $\Pi_{Y_2}$ is generated by $i^{\Pi}_1(\Pi_{\overline{Y}^{\log}_y})$ and $(s^{\Pi}_{\delta_Y} \circ i^{\Pi}_1)(\Delta^1_{Y_{2/1}}))$ ). Therefore, since the diagram

$$\begin{array}{ccccc} \Pi_{\overline{X}^{\log}_x} & \xrightarrow{i^{\Pi}_1} & \Pi^{\mathrm{Lie}}_{X_2} & \xrightarrow{s^{\Pi}_{\delta_X}} & \Pi^{\mathrm{Lie}}_{X_2} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha_2^{\Pi^{\mathrm{Lie}}}}\downarrow & & {\scriptstyle\breve{\alpha}_2^{\Pi^{\mathrm{Lie}}}}\downarrow \\ \Pi_{\overline{Y}^{\log}_y} & \xrightarrow{i^{\Pi}_1} & \Pi^{\mathrm{Lie}}_{Y_2} & \xrightarrow{s^{\Pi}_{\delta_Y}} & \Pi^{\mathrm{Lie}}_{Y_2} \end{array}$$

commute, $\alpha^{\Pi}_2$ maps $\Pi_{X_2}$ onto $\Pi_{Y_2}$. Thus, the restriction $\alpha_2$ of $\alpha^{\Pi}_2$ to $\Pi_{X_2}$ makes the diagram $(**)$ commutes.

Next we consider the uniqueness. Let us take two maps $\beta$, $\beta' : \Pi_{X_2} \xrightarrow{\sim} \Pi_{Y_2}$ both of which make the diagram $(**)$ commutes. Then $\beta^{-1} \circ \beta'$ induces an automorphism of the exact sequence

$$1 \longrightarrow \Delta^1_{X_{2/1}} \xrightarrow{i^{\Pi}_1} \Pi_{X_2} \xrightarrow{p^{\Pi}_1} \Pi_X \longrightarrow 1$$

which consisits of the identities of $\Delta^1_{X_{2/1}}$ and $\Pi_X$. This implies that $\beta^{-1} \circ \beta'$ is the identity morphism (cf. § 0). □

**Corollary 3.10.**

*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), $x, x'$ $K$-rational points of $\overline{X}$ (resp., $y, y'$ $L$-rational points of $\overline{Y}$). Let*

$$\alpha : \Pi_{\overline{X}^{\log}_x} \longrightarrow \Pi_{\overline{Y}^{\log}_y}$$

*be a Frobenius-preserving isomorphism such that $\alpha$ (resp., the isomorphism $\overline{\alpha}$ : $\Pi_X \xrightarrow{\sim} \Pi_Y$ induced by passing to the quotients $\Pi_{\overline{X}^{\log}_x} \twoheadrightarrow \Pi_X$, $\Pi_{\overline{Y}^{\log}_y} \twoheadrightarrow \Pi_Y$) maps* *the decomposition group of the diagonal cusp $\tilde{x}$ (resp., $x'$) to the decomposition group of the diagonal cusp $\tilde{y}$ (resp., $y'$) up to conjugation.*

*Then there exists a unique Frobenius-preserving isomorphism*

$$\alpha' : \Pi_{\overline{X}^{\log}_{x'}} \longrightarrow \Pi_{\overline{Y}^{\log}_{y'}}$$

*well-defined up to an inner automorphism which induces $\overline{\alpha}$ by passing to the quotients and maps the decomposition group of the diagonal cusp $\tilde{x'}$ to the decomposition group of the diagonal cusp $\tilde{y'}$ up to conjugation.*

*Proof.* The existence assertion follows from Theorem 3.8 and the fact that if $D_{x'} \subseteq \Pi_X$, $D_{y'} \subseteq \Pi_Y$ denote the decomposition groups of $x'$, $y'$ respectively, then we have $\Pi_{\overline{X}^{\log}_{x'}} \cong D_{x'} \times_{\Pi_X} \Pi_{X_2}$, $\Pi_{\overline{Y}^{\log}_{y'}} \cong D_{y'} \times_{\Pi_Y} \Pi_{Y_2}$.

We consider the uniqueness assertion. Let us take two isomorphisms $\dot{\alpha}', \ddot{\alpha}' : \Pi_{\overline{X}^{\log}_{x'}} \overset{\sim}{\to} \Pi_{\overline{Y}^{\log}_{y'}}$ which induce $\overline{\alpha} : \Pi_X \overset{\sim}{\to} \Pi_Y$ by passing to the quotients. and write $\beta := (\dot{\alpha}')^{-1} \circ \ddot{\alpha}' \in \mathrm{Aut}(\Pi_{\overline{X}^{\log}_{x'}})$ which yields, from the existence assertion, $\beta_2 \in \mathrm{Aut}(\Pi_{X_2})$ reducing to the identity morphism of $\Pi_{X^{\times 2}}$ by passing to the natural quotient $\Pi_{X_2} \twoheadrightarrow \Pi_{X^{\times 2}}$. Then $\beta_2$ define an element $[\beta_2]$ of $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_2})$ and $[\beta_2]$ reduce to the unit element in $\mathrm{Out}(\Delta_X)$ by the definitions of $\dot{\alpha}'$, $\ddot{\alpha}'$. But $\mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2}) \to \mathrm{Out}(\Delta_Y)$ is injective (cf. [5] for the definition and results concerning to "$\mathrm{Out}^{\mathrm{FC}}$"), so we have $[\beta_2] = 1$. This completes the proof. □

**Remark 3.10.1.**
Any Frobenius-preserving isomorphism is quasi-point-theoretic (cf. [19], Corollary 2.10, Proposition 3.8 and [12], Corollary 2.8), i.e., induces a bijection between the set of decomposition groups of the points of $\overline{X}, \overline{Y}$. Therefore, in the statement of Corollary 3.10, a closed point $y'$ of $Y$ which corresponds to $x'$ via $\overline{\alpha}$ necessarily exists (but this choice is not unique).

## 4. Cuspidalization Problems for hyperbolic curves

In this last section, we apply of Theorem 3.9 to obtain group-theoretical constructions of the cuspidalization of a hyperbolic curve at a point infinitesimally close to a cusp (cf. Theorem 4.3), as well as of arithmetic fundamental groups of configuration spaces (cf. Theorem 4.4).

We maintain the notation of Section 3; moreover, until the end of Theorem 4.3, we shall assume that both $X$ and $Y$ are affine (i.e., $r, r' > 0$), and $x$, $y$ are split cusps of $X$, $Y$, respectively, i.e., $x \in \overline{X}(K) \setminus X(K)$, $y \in \overline{Y}(L) \setminus Y(L)$. As discussed in Definition 2.2, we obtain the major and minor cuspidal components

$\overline{X}^{\log'}$, $\overline{\mathbb{P}}_X^{\log}$ at $x$, together with the nexus $\nu_x^{\log}$ at $x$ as strict log closed subschemes of $\overline{X}_x^{\log}$ (cf. [6], 1.2), which determine subgroups well-defined up to conjugacy,

$$\Pi_{\overline{X}^{\log'}},\ \Pi_{\overline{\mathbb{P}}_X^{\log}},\ \Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$$

— which we shall refer to, respectively, as *major verticial*, *minor verticial*, and *nexus subgroup* (cf. [17], Definition 1.4).

**Lemma 4.1.**
*The composition of morphisms* $\overline{X}^{\log'} \to \overline{X}_x^{\log}$ *(resp.,* $\overline{\mathbb{P}}_X^{\log} \to \overline{X}_x^{\log}$*) and* $\overline{X}_x^{\log} \to \overline{X}^{\log} \times_K x^{\log}$ *(resp.,* $\overline{X}_x^{\log} \to \overline{\mathbb{P}}_X^{\log} \times_K x^{\log}$*) induces an isomorphism*

$$\Pi_{\overline{X}^{\log'}} \xrightarrow{\sim} \Pi_X \times_{G_K} G_{K^{\log}}$$

$$\big(\mathit{resp.,}\quad \Pi_{\overline{\mathbb{P}}_X^{\log}} \xrightarrow{\sim} \Pi_{\mathbb{P}_X} \times_{G_K} G_{K^{\log}}\quad\big).$$

*In particular, the major verticial subgroups may be thought of as defining sections of the projection* $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow \Pi_X \times_{G_K} G_{K^{\log}}$.

*Proof.* We shall only consider the *non-resp'd* portion due to a similar argument. We recall that the category of két coverings has the étale descent property and invariance for restriction from an henselian trait to its closed point (cf. [6]). Since $\overline{X}^{\log'} \to \overline{X}^{\log} \times_K x^{\log}$ is an isomorphism on $\overline{X} \setminus \{x\}$, it is enough to see isomorphicity of the morphism between the log inertia groups of $\nu_x^{\log}$ and $x^{\log} \times_K x^{\log}$, i.e., $\mathrm{Ker}(\Pi_{\nu_x^{\log}} \twoheadrightarrow G_K)$ and $\mathrm{Ker}(\Pi_{x^{\log}\times_K x^{\log}} \twoheadrightarrow G_K)$ (cf. [6], 4.7 for the terminology "log inertia subgroup"). If we fix a chart, modeled on $\mathbb{N}$, of $x^{\log}$ (i.e., roots of a local uniformizer at $x$ in $\overline{X}$), then we may give $x^{\log} \times_K x^{\log}$ (resp., $\nu_x^{\log}$) a chart of the form $\mathbb{N} \oplus \mathbb{N}$, where the first factor of direct sum is that pulled back from the ground log scheme $x^{\log}$ and the second factor is that pulled back from $x^{\log}$ as an exact closed subscheme of $\overline{X}^{\log}$ (resp., via $p_2 \circ i_1 : \overline{X}_x^{\log} \to \overline{X}^{\log}$). By using these splittings, we may express a chart of the homomorphism of monoids induced by the morphism $\nu_x^{\log} \to x^{\log} \times_K x^{\log}$ as

$$\begin{aligned} \mathbb{N} \oplus \mathbb{N} &\longrightarrow \mathbb{N} \oplus \mathbb{N} \\ (a, b) &\mapsto (a, a+b) \end{aligned}$$

Then, by applying the functor $\mathrm{Hom}((\ \_\ )^{\mathrm{gp}}, \mathbb{Z}_l(1))$ to this morphism of monoids, it is verified that the induced morphism of log inertia groups between $\nu_x^{\log}$ and $x^{\log} \times_K x^{\log}$ is an isomorphism. $\square$

**Lemma 4.2.**

(i) *Suppose that we fix a choice of a nexus subgroup* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *among its various* $\Pi_{\overline{X}_x^{\log}}$*-conjugates. Then there exists a unique pair of inclusions*

$$\Pi_{\overline{X}^{\log'}} \subseteq \Pi_{\overline{X}_x^{\log}}, \quad \Pi_{\overline{\mathbb{P}}_X^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$$

*(among their various* $\Pi_{\overline{X}_x^{\log}}$*-conjugates) both of which contain* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ .

(ii) *The compatible inclusions* $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$, $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{\mathbb{P}}_X^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ *obtained in (i) make a commutative square*

$$\begin{array}{ccc} \Pi_{\nu_x^{\log}} & \longrightarrow & \Pi_{\overline{\mathbb{P}}_X^{\log}} \\ \downarrow & & \downarrow \\ \Pi_{\overline{X}^{\log\prime}} & \longrightarrow & \Pi_{\overline{X}_x^{\log}} \end{array}$$

*which is co-cartesian in the category of extensions of* $G_{K^{\log}}$ *by pro-l groups.*

*Proof.* We consider assertion (i). If we fix a universal (geometrically pro-$l$) két covering $U^{\log} \to \overline{X}_x^{\log}$, then the fixed inclusion $\Pi_{\nu_x^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ corresponds, by definition, to a log geometric point $\tilde{\nu}_x^{\log}$ of $U^{\log}$ over $\nu_x^{\log}$. Therefore it is enough to take $\Pi_{\overline{X}^{\log\prime}} \subseteq \Pi_{\overline{X}_x^{\log}}$, $\Pi_{\overline{\mathbb{P}}_X^{\log}} \subseteq \Pi_{\overline{X}_x^{\log}}$ as those of corresponding to unique irreducible components of $U^{\log}$ over $\overline{X}^{\log\prime}$, $\overline{\mathbb{P}}_X^{\log}$ which contain $\tilde{\nu}_x$. Assertion (ii) follows from the construction of colimit and the "*van Kampen Theorem*" in algebraic topology. □

Now we consider Theorem B. Once we shall consider, for simplicity of the proof, a slightly weaker statement with respect to the case where the types $(g, r)$ of the hyperbolic curves satisfy that $(g, r) = (0, 3)$ as stated below. But by giving directly a proof of Theorem 4.4, which contain the statement of Theorem B, we also can conclude the same statement as the case $(g.r) \neq (0, 3)$.

**Theorem 4.3.**
*Let* $X$, $Y$ *be affine hyperbolic curves over a finite field* $K$, $L$, *respectively, of type* $(g, r) \neq (0, 3)$ *(resp., of type* $(g, r) = (0, 3)$*),* $x$ *a* $K$*-rational point of* $\overline{X} \setminus X$, $y$ *an* $L$*-rational point of* $\overline{Y} \setminus Y$. *Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

$$(\textit{resp.},\ \alpha' : \Pi_{X \times_K K'} \xrightarrow{\sim} \Pi_{Y \times_L L'})$$

*be a Frobenius-preserving isomorphism such that the decomposition groups of* $x$ *and* $y$ *(which are well-defined up to conjugacy) correspond via* $\alpha$ *(resp., where* $K'$, $L'$ *are the unique extensions of* $K$, $L$ *whose degrees of extensions are two).*
*Then there exists a unique isomorphism*

$$\tilde{\alpha} : \Pi_{\overline{X}_x^{\log}} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}}$$

$$(\textit{resp.},\ \tilde{\alpha}' : \Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{K'} \xrightarrow{\sim} \Pi_{\overline{Y}_y^{\log}} \times_{G_L} G_{L'})$$

*well-defined up to composition with an inner automorphism which maps the decomposition group (well-defined up to conjugacy) of* $\tilde{x}$ *in* $\overline{X}_x^{\log}$ *to that of* $\tilde{y}$

*in* $\overline{Y}_y^{\log}$ *and induces* $\alpha$ *(resp.,* $\alpha'$*) by passing to the quotients* $\Pi_{\overline{X}_x^{\log}} \twoheadrightarrow \Pi_X$*,* $\Pi_{\overline{Y}_y^{\log}} \twoheadrightarrow \Pi_Y$ *(resp.,* $\Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{K'} \twoheadrightarrow \Pi_{X\times_K K'}$*,* $\Pi_{\overline{Y}_y^{\log}} \times_{G_L} G_{L'} \twoheadrightarrow \Pi_{Y\times_L L'}$*).*

*Proof.* The proof of uniqueness is similar to that of Corollary 3.8.

Now we consider the existence assertion. In the case $(g,r)=(0,3)$, $X\times_K K'$ is isomorphic to a tripod. Let $D'_x \overset{\iota}{\hookrightarrow} \Pi_{X\times_K K'}$ be a natural inclusion of a specific decomposition group $D'_x$ of $x$ in $\Pi_{X\times_K K'}$ and $\kappa_X : D'_x \times_{G_{K'}} \Pi_{X\times_K K'} \hookrightarrow \Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{K'}$ (resp., $\kappa_{\mathbb{P}} : \Pi_{X\times_K K'} \times_{G_{K'}} D'_x \hookrightarrow \Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{K'}$) the injective morphism induced by the isomorphism, obtained in Lemma 4.1, with respect to the major (resp., minor) verticial subgroup. If we fix an identification $D'_x \cong \mathbb{Z}_l(1) \rtimes G_K$, then the automorphism $\kappa$ of $(\mathbb{Z}_l(1)\times\mathbb{Z}_l(1))\rtimes G_{K'}$ given by $(s_1,s_2,t)\mapsto(s_1,s_1-s_2,t)$ coincides with $(\kappa_{\mathbb{P}})^{-1}\circ\kappa_X|_{D'_x\times_{G_{K'}}D'_x} : D'_x\times_{G_{K'}} D'_x \xrightarrow{\sim} D'_x\times_{G_{K'}} D'_x$. Hence the square

$$\begin{array}{ccc} D'_x \times_{G_{K'}} D'_x & \longrightarrow & \Pi_{X\times_K K'} \times_{G_{K'}} D'_x \\ \downarrow & & \downarrow{\scriptstyle \kappa_{\mathbb{P}}} \\ D'_x \times_{G_{K'}} \Pi_{X\times_K K'} & \xrightarrow{\kappa_X} & \Pi_{\overline{X}_x^{\log}} \times_{G_K} G_{K'} \end{array}$$

— where the left hand vertical arrow is the natural inclusion and the upper horizontal arrow is the composition of $\kappa$ and the natural inclusion — is cocartesian in the category of extensions of $D'_x$ by pro-$l$ groups by Lemma 4.2. Therefore, by comparing this diagram to that of $Y$, the proof is completed.

In the case $(g,r)\neq(0,3)$, by considering the number of topological generators of profinite groups, we take an open subgroup $\Pi_{\dot{Z}} \subseteq \Pi_X$ such that the corresponding étale covering $\dot{Z}$ of $X$ satisfies the following condition: $\dot{Z}$ is also étale over $x$ in their (smooth) compactification, and have at least two split points $z, z'$ over $x$ (where we shall choose $z$ so that a suitable choice of decomposition group $D_z \subseteq \Pi_{\dot{Z}}$ of $z$ coincides with $D_x$). Let $Z$ be the partial compactification of $\dot{Z}$ at $z'$, $\overline{Z}_z^{\log}$ the cuspidalization of $Z$ at $z$. Moreover, let $\overline{Z}^{\log'}$, $\overline{\mathbb{P}}_Z^{\log}$, $\nu_z^{\log}$ be the major, minor cuspidal component and the nexus of $\overline{Z}_z^{\log}$ at $z$. Then we have a sequence of inclusions

$$\Pi_{\nu_z^{\log}} \longrightarrow \Pi_{\mathbb{P}_Z} \times_{G_K} D_z \xrightarrow{\sim} \Pi_{\mathbb{P}_X} \times_{G_K} D_x \qquad (\star)$$

and a commutative diagram

$$\begin{array}{ccccccc} \Pi_X \times_{G_K} D_x & \longleftarrow & \Pi_{\dot{Z}} \times_{G_K} D_z & \longrightarrow & \Pi_Z \times_{G_K} D_z & \longleftarrow & \Pi_{\overline{Z}^{\log'}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Pi_X & \longleftarrow & \Pi_{\dot{Z}} & \longrightarrow & \Pi_Z & \longleftarrow & \Pi_{\overline{Z}_z^{\log}} \end{array}$$

— where the upper sequence induces a sequence of morphisms, by restricting to subgroups,

$$\Pi_X \times_{G_K} D_x \longleftarrow D_z \times_{G_K} D_z \xrightarrow{\sim} D_z \times_{G_K} D_z \xleftarrow{\sim} \Pi_{\nu_z^{\log}} \qquad (\star\star).$$

Hence, combining ($\star$) with ($\star\star$) yields the lower horizontal sequence of the commutative diagram

$$\begin{tikzcd} \Pi_{\overline{X}^{\log\prime}} \arrow[d, "\wr"] & \Pi_{\nu_x^{\log}} \arrow[l] \arrow[r] \arrow[d, "\wr"] & \Pi_{\overline{\mathbb{P}}_X^{\log}} \arrow[d, "\wr"] \\ \Pi_X \times_{G_K} D_x & \Pi_{\nu_z^{\log}} \arrow[l] \arrow[r] & \Pi_{\mathbb{P}_X} \times_{G_K} D_x \end{tikzcd}$$

Therefore, the colimit of the following diagram

$$\Pi_X \times_{G_K} D_x \longleftarrow \Pi_{\nu_z^{\log}} \longrightarrow \Pi_{\mathbb{P}_X} \times_{G_K} D_x$$

in the category of extensions of $D_x$ by pro-$l$ groups coincides (by Lemma 4.2) with $\Pi_{\overline{X}_x^{\log}}$. Therefore, by comparing this diagram to that of $Y$, the proof is completed. $\square$

Finally, we consider Theorem C, i.e., the cuspidalization problem for geometrically pro-$l$ fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields:

**Theorem 4.4.** *(cf.* [11], *Theorem 3.10,* [4], *Theorem 4.1)*
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let*

$$\alpha_1 : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists a unique isomorphism*

$$\alpha_n : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$$

*well-defined up to composition with an inner automorphism, which is compatible with the natural respective outer actions of the symmetric group on $n$ letters and makes the diagram*

$$\begin{tikzcd} \Pi_{X_{n+1}} \arrow[r, "\alpha_{n+1}"] \arrow[d, "p_i^{\Pi}"'] & \Pi_{Y_{n+1}} \arrow[d, "p_i^{\Pi}"] \\ \Pi_{X_n} \arrow[r, "\alpha_n"] & \Pi_{Y_n} \end{tikzcd}$$

*($i = 1, \cdots, n+1$) commute.*

*Proof.* If $n = 2$ and $X$ is proper, it follows from [11], Theorem 3.1. We shall consider the case where $n = 2$ and $X$ is affine. It follows from Theorem 3.8 and Proposition 4.3 that $\alpha_1$ induces an isomorphism $\alpha_0 : G_K \cong G_L$ and $\alpha_2' : \Pi_{X_2} \times_{G_K} G_{K'} \cong \Pi_{Y_2} \times_{G_L} G_{L'}$ — where $G_{K'} \subseteq G_K$ (resp., $G_{L'} \subseteq G_L$) denotes an open subgroup corresponding to some finite extension $K'$ of $K$ (resp., $L'$ of $L$) — as well as an isomorphism $\alpha_2^{\Delta} : \Delta_{X_2} \cong \Delta_{Y_2}$. Now let us denote by $\tau_X : G_K \to \mathrm{Out}(\Delta_{X_2})$ (resp., $\tau_Y : G_L \to \mathrm{Out}(\Delta_{Y_2})$) the morphism obtained naturally by lifting elements of $G_K$ (resp., $G_L$) via the surjection $\Pi_{X_2} \twoheadrightarrow G_K$ (resp., $\Pi_{Y_2} \twoheadrightarrow G_L$). Then $\alpha_2^{\Delta}$, $\alpha_0$ construct two morphisms

$$\tau_Y \circ \alpha_0,\ [\alpha_2^{\Delta}] \circ \tau_X : G_K \longrightarrow \mathrm{Out}(\Delta_{Y_2})$$

— where $[\alpha_2^\Delta]$ denotes the isomorphism $\mathrm{Out}(\Delta_{X_2}) \xrightarrow{\sim} \mathrm{Out}(\Delta_{Y_2})$ that sends each element $g \in \mathrm{Aut}(\Delta_{X_2})$ to $\alpha_2^\Delta \circ g \circ (\alpha_2^\Delta)^{-1} \in \mathrm{Aut}(\Delta_{Y_2})$ — which coincide after composing with the natural homomorphism $\mathrm{Out}(\Delta_{Y_2}) \to \mathrm{Out}(\Delta_Y)$ by definitions of $\alpha_0$, $\alpha_2^\Delta$. On the other hand, the images of $\tau_Y \circ \alpha_0$ and $[\alpha_2^\Delta] \circ \tau_X$ lie in $\mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2})$ and $\mathrm{Out}^{\mathrm{FC}}(\Delta_{Y_2}) \to \mathrm{Out}(\Delta_Y)$ is injective (cf. [5] for the definition and results concerning to "$\mathrm{Out}^{\mathrm{FC}}$"), so $\tau_Y \circ \alpha_0 = [\alpha_2^\Delta] \circ \tau_X$. Therefore, since $\Pi_{X_2} \cong \Delta_{X_2} \overset{\mathrm{out}}{\rtimes} G_K$ (resp., $\Pi_{Y_2} \cong \Delta_{Y_2} \overset{\mathrm{out}}{\rtimes} G_L$), we have an isomorphism $\Pi_{X_2} \cong \Pi_{Y_2}$, which satisfies the required uniqueness and compatibility from the construction. This completes the assertion in the case where $n = 2$ and $X$ is affine.

The assertion of the case $n \geq 3$ follows from a very similar argument as the above discussion. Indeed, we can apply an inductive argument on $n$ together with the natural extension

$$1 \longrightarrow \Delta_{(X \backslash \{x\})_{n-1}} \longrightarrow \Pi_{X_n} \longrightarrow \Pi_X \longrightarrow 1$$

(hence $\Pi_{X_n} \cong \Delta_{(X\backslash\{x\})_{n-1}} \overset{\mathrm{out}}{\rtimes} \Pi_X$) for an arbitrary $K$-rational point $x \in X$, and the fact that $\mathrm{Out}^{\mathrm{FC}}(\Delta_{(X\backslash\{x\})_{n-1}}) \to \mathrm{Out}^{\mathrm{FC}}(\Delta_{(X\backslash\{x\})_{n-2}})$ is injective. $\square$

The following corollary follows immediately from Theorem 4.4, together with the fact that any Frobenius-preserving isomorphism between hyperbolic curves over finite fields preserves the set of decomposition groups of closed points (as stated Remark 3.10.1).

**Corollary 4.5.** *(cf.* [4]*, Corollary 4.1)*
*Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), and $n \in \mathbb{Z}_{\geq 0}$. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

*be a Frobenius-preserving isomorphism, and $\{x_1, \cdots, x_n\}$ an ordered set of distinct $K$-rational points of $X$. Then there exists a ordered set $\{y_1, \cdots, y_n\}$ of distinct $L$-rational points of $Y$ and uniquely exists an isomorphism*

$$\tilde{\alpha} : \Pi_{X\backslash\{x_1,\cdots,x_n\}} \xrightarrow{\sim} \Pi_{Y\backslash\{y_1,\cdots,y_n\}}$$

*well-defined up to composition with an inner automorphism, which induces $\alpha$ by passing to quotients $\Pi_{X\backslash\{x_1,\cdots,x_n\}} \twoheadrightarrow \Pi_X$, $\Pi_{Y\backslash\{y_1,\cdots,y_n\}} \twoheadrightarrow \Pi_Y$.*

## References

[1] A. Grothendieck et al., *Revêtements étale et groupe fondamental,* Lecture Notes in Math., 224, Springer, Berlin, 1971.
[2] Y. Hoshi, The exactness of the log homotopy sequence, *Hiroshima Math. J.* 39 (2009), p. 61-121.
[3] Y. Hoshi, On the fundamental groups of log configuration schemes, *Math. J. Okayama Univ.* 51 (2009), p.1-26.
[4] Y. Hoshi, Absolute anabelian cuspidalizations of configuration spaces of proper hyperbolic curves over finite fields, *Publ. RIMS, Kyoto Univ.* 45 (2009), p. 661-744.

[5] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenarate outer representations, to appear in *Hiroshima Math. J*

[6] L. Illusie, An Overview of the Work of K. Fujiwara, K. Kato and C. Nakamura on Logarithmic Etale Cohomology, in Cohomologies p-adiques et applications arithmetiques (2), *Asterisque* 279 (2002), p. 271-322.

[7] K. Kato, Logarithmic structures of Fontaine-Illusie, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, p. 191-224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[8] F. F. Knudsen, The projectivity of the moduli space of stable curves. 2. The stacks $M_{g,r}$, *Math. Scand.* 52 (1983), p. 161-199.

[9] S. Mochizuki, Extending Families of Curves over Log Regular Schemes, *J. reine angew. Math.* 511 (1999), p. 319-423.

[10] S. Mochizuki, The absolute anabelian geometry of hyperbolic curves, in *Galois theory and modular forms,* p. 77-122, Dev. Math., 2, Kluwer Acad. Publ., Boston. MA, 2004.

[11] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* 47 (2007), p. 451-539.

[12] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math J.* 59 (2007), p. 455-479.

[13] S. Mochizuki, On the combinatorial cuspidalization of hyperbolic curves, *Osaka J Math.* 47, (2010), p. 651-715.

[14] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math.* J. 37 (2008), p. 75-131.

[15] D. Mumford, *Abelian varieties,*Tata Institute of Fundamental Reserch Studies in Mathematics, No.5 Published for the Tata Institute of Fundamental Reserch, Bombay; Oxford University Press, London, 1970.

[16] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines, *J. Reine Angew. Math.* 411 (1990), p. 205-216.

[17] H. Nakamura, N. Takao and R. Ueno, Some stability properties of Teichmuller modular function fields with pro-$l$ weight structures, *Math. Ann.* 302 (1995), p.197-213.

[18] J. Stix, A monodromy criterion for extending curves, *Int. Math.* Res. Not. 2005.

[19] A. Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* 109 (1997), p. 135-194.