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K3 surfaces of genus 17

n	1	2	≥ 3
$n(5-2n)$	3	2	negative

Introduction [GSH]

Unirationality

[1] Modular description of polarized K3 surfaces of odd genus

[2] Unirationality, expected dimension $= n(5-2n)$

Basic idea = "Duality" + "double moduli"

Albanese map is the classical example

§1 Proof of Thm 1

$g = 4n+3$ $\quad \hat{S} = M_S(2, h, 2n+1)$

Properties i) ~ iv)

§2 Proof of Thm 2

$g = (4m+1 \Rightarrow) 8n+1$ $\quad T = M_S(2, h; (2n+1)F)$

$\alpha \in Br(T)$ 2-torsion

Properties i) ~ iv) similar to §1 but twisted by α.

K3 surfaces of genus 17

4/12/11 (Tue)

(S, h) $\begin{cases} \text{K3 surface } S \begin{cases} K_S = 0 \\ q = 0 \end{cases} \\ \text{primitive polarization } h \end{cases}$

$(h^2) = 2g-2$ $\left(\underbrace{C \subset S,\ C \sim h}_{C \in |h| \overset{\text{def}}{\Longleftrightarrow}} \Rightarrow g(C) = g \right)$

moduli space $\mathcal{F}_g = \{(S,h)\}/\text{isom.}$ 19-dim. quasi-proj.

<u>Problem</u> Describe general (S,h) and $\mathcal{F}_g$

[GSH] $\mathcal{F}_g$ is of g.t. for $g \geq 63$.

<u>Theorem</u> $\mathcal{F}_g$ is unirational for $g \leq 13$ and $g = 16, 17, 18, 20$.

Today Answer for $g = (17$ and$)$ 17.

$\boxed{1}$ Modular description of K3's as BN loci

C : curve

$U_C(2) = \{\text{stable 2-bundle on } C\}/\text{isom.}$

$\downarrow$

$\operatorname{Pic} C \ni [L]$ $\quad SU_C(2, L) = \text{fiber at } [L]$

$= \{\text{2-bundle } E,\ \det E \cong L\}/\text{isom.}$

quasi-proj variety of

dim $3g(C) - 3$

BN locus of type III, $a \in \mathbb{Z}$

$SU_C(2, K, a) = \{h^0(E) \geq a+2\} \subset SU_C(2, K)$ $\quad K = K_C$ can. l.b.

$\uparrow$ expected codim $= \frac{(a+1)(a+2)}{2}$

type II, F 2-bundle on C

$SU_C(2, K; aF) = \{\dim \operatorname{Hom}(F, E) \geq a\}$

$\begin{pmatrix}\text{expected}\\ \text{codim}\end{pmatrix} = \frac{a(a-1)}{2}$ $\longrightarrow$ $\subset SU_C(2, K \otimes \det F)$

<u>Theorem 1</u> $g = 4n+3 \geq 11$. General $(S, h) \in \mathcal{F}_g$ is the BN locus $SU_C(2, K, 2n+1)$ of type III for a curve C of genus g. ($S \cong SU_C(2, K, 2n+1)$ and h is the pull-back of h_{det} on $U_C(2)$.)

<u>Theorem 2</u> $g = 8m+1$. General $(S, h) \in \mathcal{F}_g$ is the BN locus $SU_C(2, K; (2n+1)F)$ for a curve C of genus $2n+1$ and a 2-bundle F of odd degree on C.

$\boxed{2}$ Unirationality of $\mathcal{F}_{11}$ and $\mathcal{F}_{17}$.

Expected dimension

<u>Th1.</u> $\rho_1 = 3g-3 - \frac{(2n+3)(2n+4)}{2} = 12n+6 - (2n+3)(n+2)$

$= 12n+\cancel{6} - 2n^2 - 7n - \cancel{6} = n(5-2n)$

<u>Th.2</u> $\rho_2 = 3g-3 - \frac{(2m+1)2m}{2} = 6m - m(2m+1)$

$\rho_i = 2 \iff n = 2$ $\quad g = \begin{cases} 11 & \text{for Th 1.} \\ 17 & 2. \end{cases}$

$\rho_i = 3 \iff n = 1$

$\rho_i < 0$ if $n \geq 3$

<u>Theorem 3.</u> Expected dim is attained if $C \in \mathcal{M}_{11}$ is general (resp. if $C \in \mathcal{M}_5$ and F is general.)

$$\begin{array}{ccc} \mathcal{M}_{11} & \overset{\text{dominant}}{\dashrightarrow} & \mathcal{F}_{11} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ C & \longmapsto & SU_C(2, K, 5) \end{array}$$

$\mathcal{M}_{11}$ is unirational (Chang-Ran)

Hence $\mathcal{F}_{11}$ is unirational.

$$\begin{array}{ccc} \mathcal{D}_5 : \coprod_{C \in \mathcal{M}_5} SU_C^-(2) & \dashrightarrow & \mathcal{F}_{17} \\ \rotatebox{90}{$\in$} & & \\ (C, F) & \longmapsto & SU_C(2, K; 5F) \end{array}$$

$\mathcal{D}_5$ is unirational (easy)

Hence $\mathcal{F}_{17}$ is unirational.

<u>Remark</u> $n=1 \Rightarrow$ Two BN loci are Fano 3-folds of genus 7 and 9.

Basic idea in "Duality and double models"

Classical case $\quad X$ smooth proj variety

$$\mathrm{Pic}\, X = \{\text{line bundles on } X\}/\text{iso.}$$

$$\cup$$

$\mathrm{Pic}^0 X \quad$ neutral component, A.V.

$\mathcal{P}$ normalized Poincaré bundle on $X \times \mathrm{Pic}^0 X$

$$X \xrightarrow{\alpha} \mathrm{Pic}^0\, \mathrm{Pic}^0 X, \quad x \longmapsto \mathcal{P}\big|_{x \times \mathrm{Pic}^0 X}$$

(restricting to opposite fiber)

α is the Albanese map, i.e.,

any morphism to A.V. factors thru α.

by the duality theorem $\quad \widehat{\widehat{A.V}} \simeq A.V.$

(Replace l. b's by 2-bundles!)
$\quad U(1) \qquad SU(2)$

§ Proof of Theorem 1 (as preparation for Theorem 2)

(S, h) ~~general~~ polarized K3 surface of genus $g = 4n+3$

$$v := (2, h, 2n+1) \in \mathbb{Z} \oplus H^2(S, \mathbb{Z}) \oplus \mathbb{Z} =: \widetilde{H}(S, \mathbb{Z})$$

$$(v^2) = 0$$

$$\hat{S} := M_S(v) = \left\{ \begin{array}{l} \text{stable bundle } E \text{ with } v(E) = v \\ \text{i.e., rk } 2,\ c_1(E) = h \text{ \& } \chi(E) = 2 + (2n+1) \end{array} \right\} \Big/ \text{isom.}$$

is again a K3 surface and carries a polarization $\hat{h}$ of the same genus. Moreover, we have

i) $\exists$ universal family $\mathcal{E}$ on $S \times \hat{S}$ with $c_1(\mathcal{E}) = \pi_1^* h + \pi_2^* \hat{h}$,

ii) $\mathcal{E}$ gives an equivalence of derived categories

$$D(S) \xrightarrow{\sim} D(\hat{S}) \quad \left\{ \begin{array}{l} E \longmapsto (\text{sky-scraper})[2] \\ (\text{sky-scraper}) \longmapsto \hat{E} \text{ with } v(\hat{E}) = (2, \hat{h}, 2n+1) \end{array} \right.$$

$$h \longleftrightarrow \hat{h}$$

iii) $S \xrightarrow{\sim} \hat{\hat{S}}$ by $\alpha \longmapsto \mathcal{E}\big|_{S \times \alpha}$ (opposite restriction)

iv) $C \subset \hat{S}$, $C \sim \hat{h}$ smooth curve in $|\hat{h}|$
$g(C) = 11$

The map $S = \hat{\hat{S}} = M_{\hat{S}}(2, \hat{h}, 2n+1) \longrightarrow SU_C(2, K)$

$$\hat{E} \longmapsto \hat{E}|_C$$

is an isomorphism onto the BN locus $SU_C(2, K, 2n+1)$ if $g \geq 11$. ($\Rightarrow$ Theorem 1) $h^0(\hat{E}) \geq \chi(\hat{E}) = 2 + (2n+1)$. Hence $h^0(\hat{E}|_C) \geq 2 + (2n+1)$

Rmk
If $g = 7$, $SU_C(2, K, 2n+1)$ is a Fano 3-fold of $g = 7$ and S is its hyperplane section

§ Proof of Theorem 2

(S, h) general polarized K3 surface of genus $4m+1$.

$v = (2, h, 2m) \in \widetilde{H}(S, \mathbb{Z}) \qquad (v^2) = 0$.

$T := M_S(v)$ is again a K3 surface

T carries a polarization $\hat{h}$ with $(\hat{h}^2) = (h^2) = 8m$

But $\hat{h} = 2l$, l: primitive, $(l^2) = 2m$

$\nexists$ universal family on $S \times T$. (New phenomenon)

i)+ii) $\exists$ universal family on $S \times (T, \alpha)$, where α is a 2 torsion element of the Brauer group of T. This universal family gives an equivalence $D(S) \xrightarrow{\sim} D(T, \alpha)$ of the derived categories of S and that of T twisted by α. ~~iii) S is the moduli space of semi-rigid 2-bundles on (T, α)~~ ~~(moduli of $\mathbb{P}^1$-bundles on T.)~~

Now assume further that $m = 2n$ (even) and put $v' = (2n+1, n h, 4n^2-2n+1)$. Then $\exists$ rigid (or exceptional) vector bundle G on S with $v(G) = v'$.

The equivalence $D(S) \xrightarrow{\sim} D(T, \alpha)$ maps G to a (2-bundle $\hat{G}$ on (T, α), or) $\mathbb{P}^1$-bundle whose Brauer class is α.

rank 2 vector bundle on (T, α)

(検算) $\chi(v, v') = 2(4n^2-2n+1) - n(h^2) + 4n(2n+1)$
$= 2(4n^2-2n+1) - 16n^2 + 4n(2n+1) = 2$ O.K.

$$D(S) \xrightarrow{\ \sim\ } D(T, \alpha)$$

$$\begin{pmatrix}\text{sky-}\\ \text{scraper}\end{pmatrix} \longleftrightarrow E$$

$$v = (2, h, 2n) \longleftrightarrow \begin{pmatrix}\text{sky-}\\ \text{scraper}\end{pmatrix}[2]$$

$$G \text{ with } v(G) = (n+1, \tfrac{n}{2} h, n^2-n+1) \longleftrightarrow \widehat{G}$$

$M_{(T,\alpha)}(\ldots)$

iii) (Duality) S is the moduli space of semi-rigid 2-bundles on (T, α) (or moduli of $\mathbb{P}^1$-bundles on T).

E

$$\dim \mathrm{Hom}(\widehat{G}, E) \geq \chi(\widehat{G}, E) = \chi(G, \text{sky-scraper}) = n+1$$

$$\dim H^0(\widehat{G}|_C)$$

The maps $M_{(T,\alpha)}(\ldots) \longrightarrow U_C(2)$, $E \longmapsto E|_C$,

is an isomorphism onto the BN-locus $SU_C(2, K; \widehat{G}|_C)$

if $g \geq 17$.

<u>Remark</u> If $g = 9$, then $SU_C(2, K; \widehat{G}|_C)$ is a Fano 3-fold of genus 9 and S as its hyperplane section.