Enriques surfaces: Fundamental polarizations and ~~their mirror~~

G/co/16(T).

Warwaraw

Enriques surface $\qquad p_g = q = 0,\ 2K_S \sim 0$

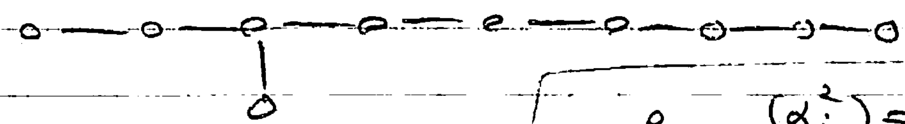
$S = X/\varepsilon \qquad$ K3 / free involution

# of moduli $= h^1(T_S) = 10$

$\mathrm{Pic}\, S \simeq H^2(S,\mathbb{Z})$ $\begin{cases} \mathbb{Z}/2 & \text{torsion part gen. by } K_S \\ H^2(S,\mathbb{Z})_f \cong \mathbb{Z}^{10} & \text{free part} \end{cases}$

$H^2(S,\mathbb{Z})_f$ is a lattice of rank 10. $\begin{cases} \text{Even, unimodular} \\ \text{sgn } (1,9) \end{cases}$

E_{10}



o $\quad (\alpha_i^2) = -2$, basis

$(\alpha_i . \alpha_j) = \begin{cases} 1 & \text{o—o} \\ 0 & \text{o} \quad \text{o} \end{cases}$

§1 Fundamental polarization

polarization $\Leftrightarrow$ projective model

$\Leftrightarrow$ numerically ample class $h \in \mathrm{Pic}\, S$

polarization $= \sum_{i=1}^{10} a_i$ (fund. quasi-polarization) $\quad a_i \in \mathbb{Z}_{\geq 0}$

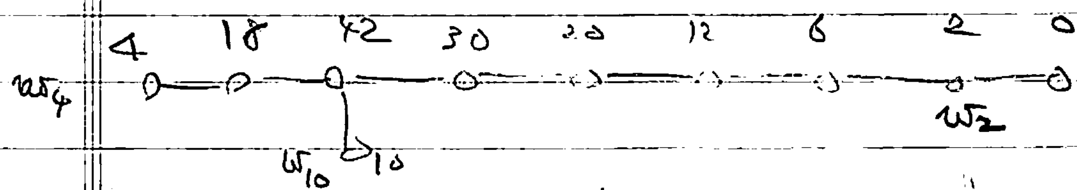
unique up to $O(H^2(S,\mathbb{Z})_f)$

Positive cone $\{ x \in H^2(X,\mathbb{R}) \mid (x^2) > 0 \} = \underset{\text{polarization}}{\underset{\cup}{C}} \cup (-C)$

Fundamental domain $\{ x \in C \mid (x.\alpha_i) \geq 0 \ \forall i \}$

fund. quasi-polarization $\Leftrightarrow (w.\alpha_j) \geq 0$ for all j except one, nonzero, primitive.

(ϖ^2) is as follows for 10 fundamental f. polarizations

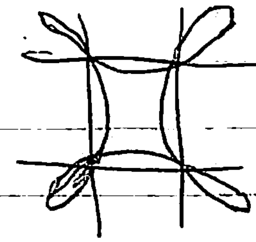


Notation ϖ_d fund. polarization ϖ with $(\varpi^2)=d>0$.

Minimal model for

ϖ_2-model $\quad S \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$

double product



# of parameters $5\times5-1-4\times3-2 = 24-14=10$

Branch = coord. quadrilateral + (4,4)-curve simple at coord. points.

ϖ_4-model $\quad X_8 : a_i(x)+b_i(y)=0 \quad i=1,2,3 \quad$ in $\mathbb{P}^5$ $(x_1\, x_2\, x_3\, y_1\, y_2\, y_3)$

# of parameters $=3\times(12-3)-(9+9-1) = 27-17=10$

$S = X_8/\varepsilon \qquad (x;y) \xleftrightarrow{\varepsilon} (x:-y)$

ϖ_{10} (Fano)-model $\quad S_{10}\subset\mathbb{P}^5$

defined by 10 cubic equations in generic case.

§2 Definition of "mirror"

By definition, $H^2(S,\mathbb{Z})_f \supset \mathbb{Z}\varpi_d \perp R_d$ — sublattice generated by 9 α_i's.

$\varpi_2\perp(E_8 + A_1)$, $\varpi_4\perp D_9$,

$\varpi_{10}\perp A_9$ etc.

Since $H^2(S,\mathbb{Z})_f$ is unimodular, $|\mathrm{disc}\, R_d| = d$.

Mirror of ϖ_d = Enriques surface S which contains $\mathbb{P}^1$'s whose classes form root system of type R_d.

Two formulation : $\mathbb{Z}/2$ & $\mathbb{Z}^{\omega}$.

Background for $\mathbb{Z}/2$-formulation

- S : "very" general $\Rightarrow$ preserving C — full level-2 subgp

$$\Rightarrow \operatorname{Aut} S \cong O^{+}(H^2(S,\mathbb{Z})_f)(2)$$

$$\cong \operatorname{Ker}[O^{+}(E_{10}) \longrightarrow O(E_{10} \bmod 2)]$$

So $H^2(S,\mathbb{Z})_f \otimes \mathbb{Z}/2$ is the space to look at.

- $H^2(S,\mathbb{Z})_f \otimes \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \quad \alpha \longmapsto (\alpha^2)/2$

is a quad. form, that is, $f(\alpha+\beta) = f(\alpha) + f(\beta) + (\alpha.\beta)$.

- Reflection by α with $f(\alpha)=1$ is defined

by $x \longmapsto x + (x.\alpha)\alpha$, and belongs to $O(f)$.

In particular, $\mathbb{P}^1$ on S defines a reflection.

S is a mirror of ω_d $\overset{\text{def}}{\Longleftrightarrow}$ Root system

generated by mod 2 classes of $\mathbb{P}^1$'s in $H^2(S,\mathbb{Z})\otimes\mathbb{Z}/2$

contains R_d as subsystem

{mirror Enriques of ω_d} is an irred. 1-dim'l

family (by Torelli type theorem).

Problem : describe 9 mirror families explicitly.

§3 Mirror of $W_2 \longleftrightarrow E_8 + A_1$

Obtained as quadratic twist of RES of type $\tilde{A}_7 + \tilde{A}_1$ in unique way.

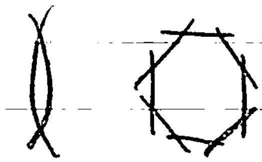
$$\begin{tikzcd} & X \arrow[dl, "2:1"', "\varepsilon"] \arrow[dr, "2:1"] & \\ S \arrow[d] & & R \arrow[d] \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

anti-symplectic lift of σ

double cover w. branch I_8 and a smooth fiber

RES

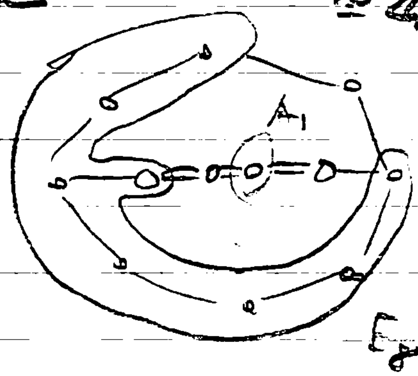
I_8



Fact: $\mathrm{Aut}\, S < +\infty$ (Nikulin-Kondo, type 1) $MW(R/\mathbb{P}^1) \simeq \mathbb{Z}/4 \ni \sigma$ order 2

$(\#$ of $\mathbb{P}^1$'s on $S) = 12$

The dual graph to



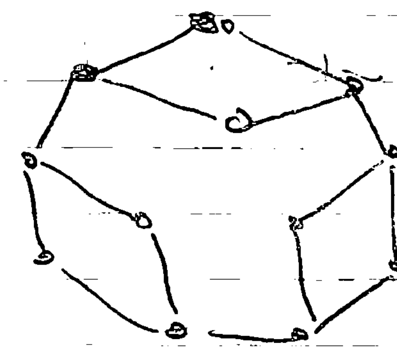
$\#$ of $\mathbb{P}^1$ classes in $H^2(S, \mathbb{Z})_f \otimes \mathbb{Z}/2$ is also 12.

§4 Mirror of $W_4 \longleftrightarrow D_9$

$\mathrm{Aut}\, S < +\infty$, $\#$ of $\mathbb{P}^1$ classes mod 2

$= \#$ of $\mathbb{P}^1$'s $= 12$

(Nikulin-Kondo, type 2)



(Ohashi) Special Enriques' sextic

$$\underset{\text{minimal resol.}}{S \to} \overline{S}_6 : \begin{array}{l} (yzt)^2 + (xzt)^2 + (xyt)^2 + (xyz)^2 \\ + xyzt(x^2 + y^2 + z^2 + kt^2) = 0 \end{array} \subset \mathbb{P}^3 \quad \text{for } k \in \mathbb{C} \text{ parameter}$$

singular along 6 edges of tetrahedron $xyzt = 0$.

§5 Mirror of Fano polarization $\mathfrak{W}_{10} \Longleftrightarrow A_9$

Most natural polarization is $\mathfrak{W}_{80}$ but I describe $\mathfrak{W}_9$-model.

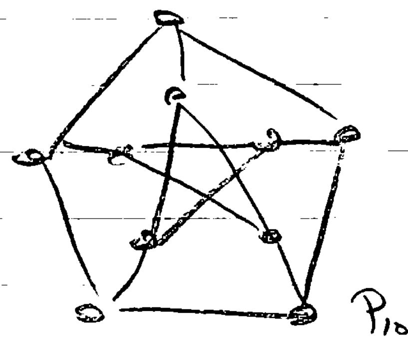
$$X_a \begin{cases} (y+az)^2 + x^2 = av^2 + w^2 \\ (z+ax)^2 + y^2 = aw^2 + u^2 \\ (x+ay)^2 + z^2 = au^2 + v^2 \end{cases} \quad \text{in } \mathbb{P}^5,$$

$$S_a = X_a/\varepsilon \qquad (xyz\,uvw) \xleftrightarrow{\varepsilon} (xyz\,-u\,-v\,-w)$$

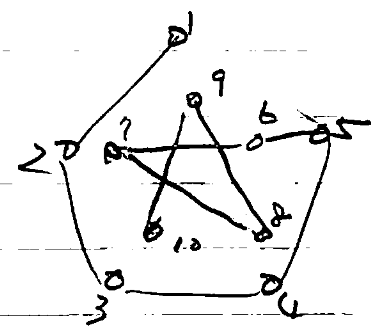
$$\mathbb{C} \ni a^3 \neq 0, -1, -\left(\frac{1\pm\sqrt{5}}{2}\right)^6$$ 1-dimensional family w. parameter a^3

(1) Observation. S_a is Enriques surface with 15 $\mathbb{P}^1$ of Petersen configuration, That is

$\exists$ 15 $\mathbb{P}^1$'s $E_{ij} \Longleftrightarrow$ edges of Petersen graph



$$(E_{ij} . E_{kl}) = \begin{cases} 1 & \text{2 edges have a common vertices} \\ 0 & \text{otherwise} \end{cases}$$



15 mod 2 classes $[E_{ij}] = e_i - e_j$ for $e_1, \dots, e_{10}$ and generates $A_9 \otimes \mathbb{Z}/2$ in $H^2(S, \mathbb{Z}) \otimes \mathbb{Z}/2$.

<u>Cor</u> $\{S_a\}_a$ is the mirror family of $\mathfrak{W}_{10}$.

I.e., dual graph $= L(P_{10})$ (line graph).

(2) Automorphism group

<u>Theorem</u> (1) $\mathrm{Aut}\, S_a = \begin{cases} \mathfrak{S}_5 & a^3 = 1 \quad \left(\begin{matrix}\text{Nikulin-} \\ \text{Kondo,} \\ \text{type 7}\end{matrix}\right) \\ \mathfrak{S}_4 \underset{\mathfrak{S}_4}{*} \mathfrak{S}_5 & a^3 \neq 1 \quad \text{amalgam} \end{cases}$

(2) # of $\mathbb{P}^1$'s in S_a $= \begin{cases} 15 + 5 = 20 & a^3 = 1 \\ \infty & a^3 \neq 1 \end{cases}$

(3) # of $\mathbb{P}^1$ classes in $H^2(S, \mathbb{Z})_p \otimes \mathbb{Z}/2$ $= \begin{cases} 15 + 1 = 16 & a^3 = 1 \\ 15 & a^3 \neq 1 \end{cases}$

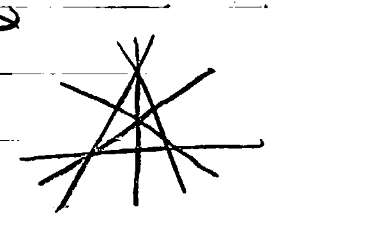
(Classes other than $e_i - e_j$ are not represented by $\mathbb{P}^1$ if $a^3 \neq 1$.)

Idea of proof. (When $a^3 = -1$, $-\left(\frac{1 \pm \sqrt{-3}}{2}\right)^{?}$, Coble (Horikawa) - degeneration happens. S_a is Coble surface with 10 or 1 $\frac{1}{4}(1,1)$ singular points.

Degeneration to S_{-1} is very useful.

- X_{-1} is Vinberg's most algebraic K3 surface $w^2 = xyz(x-y)(y-z)(z-x)$ of disc. 4
- ε is 2-elementary involution $w \leftrightarrow -w$
- $\mathrm{Aut}\, S_{-1}$ $\cong (C_2^{*5}) \rtimes \mathfrak{S}_5$ index 2 $\cup$ $\mathfrak{S}_4 * \mathfrak{S}_5$

$$\tilde{S} \xrightarrow{\text{b.u. 19 times } (19 = 4 + 15)} \ ; \quad \tilde{S} \xrightarrow{\text{blow-down } 10\ (-4)\ \mathbb{P}^1\text{'s}} S_{-1}$$



- ∃ effective divisors (passing the singularities) $\overline{E_{ij}}$ which are limits of E_{ij}.

Claim. $\mathfrak{S}_4 \times \mathfrak{W}_5$ acts on S_a

∵ $\mathfrak{S}_4 \times \mathfrak{W}_5$ is generated by the translation of genus-1 fibration on S_{-1}.

Every genus-1 fibration of S_{-1} deforms to S_a since it contains a reducible fibers consisting of some $\overline{E_{ij}}$'s. q.e.d.

$$\mathfrak{S}_4 \times \mathfrak{W}_5 = \mathrm{Aut}^{ss} S_a$$

semi-symplectic acting trivially on $H^0(2K_S)$

non-s.s. action is controlled by period.

Present status of 9 mirror families