Kummer quartics and associated symplectic 6-folds

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Abstract: Kondo(1998) described the automorphism group of the generic Jacobean Kummer surface Km(C) using a Conway chamber in the nef cone. As a sample of higher dimensional analogue, we describe the binational automorphism group of a certain holomorphic symplectic 6-fold associated with Km(C) using the Borcherds(2000) reflection group. Bir-Aut is generated by 896 involutions and an extended extraspecial group $2^{1+8}.2$ whose center is Rapagnetta(2007)'s involution.

§1 Motivation/Introduction

General Study Bir-Aut(IHS) similarly to Aut(K3), replacing the nef cone Nef(K3) with the movable cone Mov(IHS)

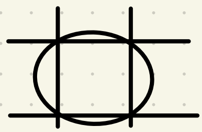
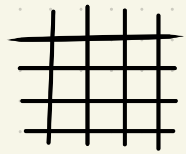
Peculiar (today) **1**. Higher dimensional analogue of Kondo(1998)'s description of Aut(generic Jacobian Kummer)
2. 5 reflection groups in $\mathbf{R}^{1,17}$

$L := II_{1,17}(2^{+2a})$, $a = 0, 1, 2, 3, 4$
even integral lattice of signature (1, 17), 2-elementary, and

$q_L : L^\vee/L = \mathbf{Z}/2\mathbf{Z}^{2a} \longrightarrow \mathbf{Q}/2\mathbf{Z}$ is even

$a = 0, 1, 2 \rightleftarrows L$ is realized as the Picard lattice of a K3

Table *today*

2a	0	2	4	6	8
L	unimodular U+E8+E8	U+E8+D8	U+D8+D8	U+**Kum**	U+**Bw**
realization	$S \longrightarrow \mathbf{P}1$ with $2\widetilde{E8}$	double $\mathbf{P}1\times\mathbf{P}1$	Km(E1×E2)	today	not (yet)
# of (-2)'s	19	20	24	64	0
and (-4)'s	0	2	24	896	∞
branch		 Barth-Peters surface			

Fact: Reflections generate a subgroup of finite index in $O_{\mathbb{Z}}(L)$ in each case.

Remark II_{1, 17}(2^{+2}), 2a=2, belongs to the Conway-Vinberg chain, on which I gave a talk in the last JES(2022).

Notation $U = (\mathbf{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$,

An, Dn, E6, E7, E8: negative definite root lattices, $\widetilde{An}$ etc.: affine

negative definite rank 16 $\begin{cases} \mathbf{Kum} \supset 16A1 & \text{Kummer lattice} \\ \mathbf{BW} & \text{Barnes-Wall lattice, (Leech lattice)} \end{cases}$ 2A-inv.

Borcherds(2000, 2a=6)

$$(*) \qquad O^{+}_{\mathbb{Z}}(II_{1,17}(2^{+6})) = \begin{pmatrix} \text{group generated by} \\ \text{960 reflections} \end{pmatrix} \rtimes G.L_4(2)$$

$G.L_4(2)$ = (symmetry of fundamental domain), #G = 2^10, $L_4(2) \cong \mathfrak{A}_8$

Today Give a geometric interpretation of (*), that is, describe Bir-Aut(X) for holomorphic symplectic 6-fold X=X(S, h) associated with (Picard general) Kummer quartic surface (S, h).

$$\begin{array}{l} S = \mathrm{Km}(C) \\ \downarrow \\ \mathrm{Jac}\, C/\pm 1 \xhookrightarrow[|2\Theta|]{} \mathbf{P}^3 \end{array}$$

$\rho(S)=17$, h: pull-back of $\mathcal{O}_{\mathbf{P}}(1)$

Kum is an overlattice of $16A_1 = \bigoplus_{i \in T_{(2)}} \mathbf{Z}e_i$, $(e_i^2) = -2$

$T = \mathbf{C}^2/\Gamma$, $\Gamma \cong \mathbf{Z}^4$, $T_{(2)}$: group of 2-torsions

$\mathbf{Kum} = \bigoplus_{i \in T_{(2)}} \mathbf{Z}e_i + \mathbf{Z}\left\{ \frac{1}{2}\sum_H e_i \;\middle|\; H \subset T_{(2)} \text{ subgroup of order } 8, 16 \right\} \subset \bigoplus_{i \in T_{(2)}} \mathbf{Q}e_i$

$\cup$

BW[*)] sublattice of index 2 and of Leech type, i.e., $\nexists$ (−2)-element

§2 Another background

$X_a = M_S(0, h, a)$, $a = 0, 1$, holomorphic symplectic 6-fold

$$\begin{cases} \text{birationally equivalent to } S^{[3]} & \text{when } a = 1 \\ \text{not equivalent} & \text{when } a = 0 \end{cases}$$

Birational double cover of $\widetilde{K}_{\mathrm{Jac}}(0, 2\Theta, 2a)$, which is OG6 (symplectic resolution of Albanese fibration

$M_{\mathrm{Jac}}(2v) \longrightarrow \mathrm{Jac} \times \widehat{\mathrm{Jac}}$, $v = (0, \Theta, a)$

of moduli of sheaves on Jac = Jac C. (Studied by Rapagnetta(2004, thesis), Mongardi-Rapagnetta-Sacca(2018), etc.)

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*) **BW** is also constructed from the Reed-Muller code of length 16 (cf. [Conway-Sloane]).

$$\begin{array}{ccl} \xi & \in & M_S(0, h, a) \quad \text{torsion sheaf on } S \\ \downarrow & & \downarrow \\ \operatorname{Supp}\xi & \in & |h| \cong \mathbf{P}^3 \ni D, \quad p_a(D) = 3 \end{array}$$

This morphism is a Lagrangian fibration. Generic fiber is $\operatorname{Pic}^a D$ (Abelian 3-fold).

$(\operatorname{Pic} X,$ Beauville form$) \cong (0, h, a)^{\perp}$ in $\mathbf{Z} \oplus \operatorname{Pic} S \oplus \mathbf{Z}$

$$= \begin{cases} \operatorname{Pic} S + \langle -4 \rangle & \text{when } a = 1 \\ U + (h^{\perp} \text{ in } \operatorname{Pic} S) = U + \mathbf{Kum} & \text{when } a = 0 \end{cases}$$

$X := X_0 = M_S(0, h, 0)$ from now on

Advantage of X: Fourier-M. transform with Kernel $\mathcal{P}$ Poincaré line bdl

descends to $S \times S = S \times M_S(2, -\frac{1}{2}\sum_1^{16} e_i, -2)$

$\downarrow$

Jac × Jac

$\Phi_1 = \Phi_{|H_1|} : X \longrightarrow \mathbf{P}^{3,*} \quad S, \qquad \xi \mapsto \operatorname{Supp}\xi$

interchanged by FM transf.

$\Phi_2 = \Phi_{|H_2|} : X \longrightarrow \mathbf{P}^3 \quad \hat{S} = S, \quad \xi \mapsto \operatorname{Supp}\hat{\xi}$

$\operatorname{Pic} X \supset U(2) + \mathbf{BW}$

$\langle H_1, H_2 \rangle$

$\uparrow \qquad \uparrow$

Rapagnetta's symplectic birational involution $\longrightarrow$ $+1 \qquad -1$

covering involution of $K3^{[3]} \to OG6$

$\otimes\, \mathcal{O}_S(-\frac{1}{2}\sum_1^{16} e_i)$

§3 Main Theorem

960 reflections = 64 (-2) refl's and 896 (-4) refl's

◇ (-2)'s : $\exists$ effective irreducible divisors $D \subset X$ with Beauville norm $(D^2)=-2$

Kummer $(16_6 - 16_6)$ configuration

16 (-2)**P**1's $\subset S$ nodes, $\Phi_1 : X \longrightarrow$ **P**^3,* has $16\widetilde{A1}$

16 (-2)**P**1's $\subset S$ tropes, $\Phi_2 : X \longrightarrow$ **P**^3 has $16\widetilde{A1}$

64 (-2) effective divisors = 32 $\widetilde{A1}$ 

◇ (-4)'s: $\exists$ (-4)-divisor classes with divisibility 2, #=896=32×28 corresponding to birational involutions of X

Main Theorem (S, h): Picard general Kummer, $X = M_S(0, h, 0)$

(1) Bir-Aut(X) = (group generated by 896 involutions) $\rtimes$ G, where G is Borcherds' group of order 2^10.

(2) (Geom. interpretation of G) G is the semi-direct product $H.C_2$ of an extraspecial group, i.e., finite analogue of Heisenberg group,

$1 \longrightarrow$ Rapagnetta's inv. $\rightarrow H \rightarrow (\mathrm{Jac} \times \widehat{\mathrm{Jac}})_{(2)} \longrightarrow 0$

by C_2 generated by FM transf. (cf. Mongardi-Wandel(2017)).

(3) $\exists$ 28 curves of genus 2 $C=C_1, C_2, \ldots, C_{28}$ such that $X \cong M_{S_i}(0, h_i, 0)$ for i=1, ..., 28 (by Torelli type theorem). Each C_i gives 32 birational involutions of X. Hence we have 896 involutions.

References

§1 Motivation/Introduction

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§2 Another background

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§3 Main Theorem

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