Coble 6-folds and K3 surfaces

4/4/23(T)

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Abstract: The moduli space of polarized K3 surfaces is of general type except for finitely many genus. As an example of exception I explain the unirationality for genus 13 using the Coble quartic in P7, which is the moduli space N+ of 2-bundles on a plane quartic curve. The generic K3 is described explicitly in the moduli space N-, the Hecke partner of N+.

§1 Background

$$\mathcal{F}_g := \left\{ \begin{array}{l} \text{(quasi-)stable polarized K3 surfaces} \\ (S, h) \ \text{ with } \ (h^2) = 2g-2 \end{array} \right\} \Big/ \text{isom.}$$

$$\cong \mathcal{D}^{19} \Big/ \Gamma_{2g-2} \qquad \text{(type IV domain)/disc. grp.}$$

Gritsenko-Hulek-Sankaran (2007)

$\mathcal{F}_g$ is of general type for $g \geq 63$ (and for other 7 values)

Mini-history of opposite direction, unirationality:

1. Corollary of unirationality of moduli of prime Fano 3-folds X

   since general $S \in \mathcal{F}_g$ belongs to $|-K_X| = \mathbb{P}^{g+1}$,

   g = 2, 3, ..., 10 & 12.

2. g = 11: Follows from uniruledness by Mori-M.(1981) and

   unirationality of $\mathcal{M}_{11}$ by Chang-Ran (1984)

3 $g = 18, 20, 13, 16$ (M. 1992 ~ 2016)

4 $g = 14$ Nuer (2017)
$g = 22$ Lai (2017) uniruled
Farkas-Verra (2022)

$$\mathcal{F}_g \xrightarrow{\sim} \mathcal{H}_g \hookrightarrow \mathcal{D}^{20}/\Gamma$$ moduli of cubic 4-folds

Heegner divisor

Description of generic cubic 4-fold $\in \mathcal{H}_g$

$\Rightarrow$ unirat. of $\mathcal{H}_g$ and hence $\mathcal{F}_g$.

Today : New description of general $S \in \mathcal{F}_{13}$ using Coble 6-fold and it's Hecke partner

Advantage: This idea may work also in unknown case g=19.

§2 Coble quartics $\mathcal{C}_4 \subset \mathbb{P}^7$

Plane quartic C (⇄ non hyper-elliptic $C \in \mathcal{M}_3$)

$$C \xrightarrow[K_C]{\sim} C_4 \subset \mathbb{P}^2$$

$$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad \omega_1 = \frac{dx \wedge dy}{f_4(x,y)} \quad \text{etc.}$$

$\mathrm{Jac}\, C = \mathbb{C}^3/(\text{periods})$ as Part I.

$\cup$

Θ theta divisor $\cong \mathrm{Sym}^2 C$

$\Phi_{|2\Theta|} : \mathrm{Jac}\, C \longrightarrow \mathbb{P}^7$

$\searrow$ $\mathrm{Km}\, C$ $\nearrow$

$= \mathrm{Jac}\, C/\pm 1$

$\mathrm{Km}\, C$ is $\cap$ of 8 cubics.

$\exists!$ quartic hypersurface $\mathcal{C}_4 \subset \mathbb{P}^7$ s.t.

$\mathrm{Sing}\, \mathcal{C}_4 = \mathrm{Km}\, C$, called Coble ass. w. $C_4 \subset \mathbb{P}^2$.

Main Theorem (M.–Kanemitsu) $\exists$ $\mathcal{C}_4$- program

Input : $C = C_4 \subset \mathbb{P}^2$, $p_1, p_2 \in C$,

$f_1 \in \mathbb{P}^7 \hookleftarrow \mathcal{C}_4 \hookrightarrow \mathbb{P}^7 \ni f_2$

Output : $S = S_{C, p_1, p_2, f_1, f_2} \in \mathcal{F}_{13}$ general member

Corollary $\mathcal{F}_{13}$ is unirational

(# of parameters) $= 6+2+14 = 22 > 19 = \dim \mathcal{F}_{13}$

§3 Grassmannian method v.s. VBAC method

E vector bundle on a K3 surface S

rigid (or spherical) $\overset{\text{def.}}{\Longleftrightarrow}$ $H^i(sl\, E) = 0 \quad \forall i$

semi-rigid $\qquad h^i(sl\, E) = \begin{cases} 2 & i=1 \\ 0 & \text{otherwise} \end{cases}$

By R-R, rigid $\Longleftrightarrow (v(E)^2) = -2$

semi $\qquad\qquad\qquad\qquad = 0$

when E is stable, where $v(E) = (r(E), c_1(E), ch^2(E) + r(E)) \in \mathbb{Z} \oplus \mathrm{Pic}\, S \oplus \mathbb{Z}$, and

$(v^2) := (l^2) - 2r\Delta$ for $v = (r, l, \Delta)$.

Put $M_S(v) := \{\text{stable } E\text{'s w. } v(E) = v\} / \text{isom.}$

which is (smooth) $(v^2) + 2$ - dim'l.

$(v^2) = -2 \Rightarrow \# M_S(v) \leq 1$

$(v^2) = 0 \Rightarrow \overline{M_S(v)}$ is again a K3 surface.

G-method. Embedd S into Grassmannian by vector bundle E, usually (semi-) rigid, morphism $\Phi_{|E|} : S \longrightarrow G(H^0(E), r(E))$ and describe the image.

Example 1. (M. 2006) $g=13$, $E \in M_S(3, h, 4)$

semi-rigid

$\Phi_{|E|} : S \hookrightarrow G(7, 3)$ 12-dim'l

$h^0(E) = 3+4 = 7$

$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_G^{\oplus 7} \longrightarrow \mathcal{E} \longrightarrow 0$ univ. exact seq.

S is a complete intersection w. r. t.

$\bigwedge^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$. ($\Rightarrow$ unirationality of $\mathcal{F}_{13}$, 1st proof)

Advantage: This method describes the universal family $\mathcal{S}_g$ over $\mathcal{F}_g$ as orbit space, and hence shows its unirationality.

VBAC method

1. Choose a suitable $v = (r, h, s)$ with $(v^2) = 0$ ($\rightleftarrows$ $rs = g-1$) and consider moduli K3 $T := M(v)$, which has nat'l polarization $\hat{h}$ with $(\hat{h}^2) = (h^2)/a^2$ with $a = \mathrm{GCD}(r, s)$.

2. Take a curve $C \in |\hat{h}|$ on T. A universal family $\mathcal{E}$ exists on $S \times C$ by Tsen's theorem (even when $a > 1$). Rmk $\nexists$ univ. family on $S \times T$ when $a > 1$.

3. Consider the opposite moduli (or classification) map

$$S \dashrightarrow \mathcal{U}_C(r), \quad s \longmapsto \mathcal{E}|_{s\times C}$$

moduli of stable bundles on C

and describe the image of S in $\mathcal{U}_C(r)$.

Example 2 $(g=7)$ $v=(2,h,3)$, $a=1$,

$(T=M_S(v), \hat{h}) \in \mathcal{F}_7$. $C \in |\hat{h}| \Rightarrow S$ is $(-K)$-member of $V := SU_C(2, K; 5)$

$$= \left\{ \begin{array}{l} \text{stable 2-bundle } E \\ \text{w. } \det E = K_C,\ h^0(E) \\ \geq 5 \end{array} \right\} \Big/ \text{isom.}$$

V is a Fano 3-fold of genus 7

$$\begin{tikzcd}
V \arrow[ddrr, bend right=60, leftrightarrow, "\text{moduli to each other}"'] & & \text{Fano} \\
S \arrow[u, phantom, "\cup" rotate=90] \arrow[r, leftrightarrow] & T & \text{K3} \\
& C \arrow[u, phantom, "\cup"] & \text{curve}
\end{tikzcd}$$

Table of 3 methods

								Today
genus · · ·	6	7	8	9	10	11	12	13
G-method	(2,h,3)	(5,2h,5)	(2,h,4)	(3,h,3)	(2,h,5)	---	(3,h,4), (2,h,6)	(3,h,4) M.'04
VBAC method	-----	(2,h,3)	---	(2,h,4)	(3,h,3)	(2,h,5) M.'96	-------	(2,h,6) M.-Kanemitsu
cubic 4-fold $g=n^2+n+2$			n=2					

14	15	16	17	18	19	20	21	22 · · ·
Nuer'17	?	(2,h,8)& (3,h,5) M. '16	---	(3,h,6)& (2,h,9) M.'92	---	(4,h,5) M.'92	(3,h,7)& (4,h,5) ?	Lai'17 Farkas-Verra'22
	?	?	(2,h,8)	---	(3,h,6)	---		
n=3					?			n=4

○ indicates the existence of prime Fano 3-folds.

§4 g=13, VBAC method

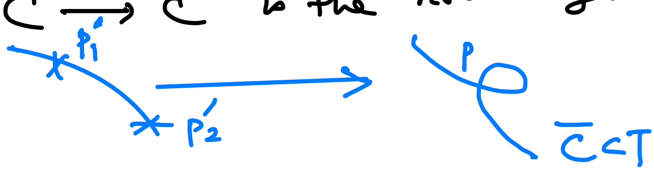
(S, h), $(h^2)=24$, h: primitive ($2 \nmid h$ in $\mathrm{Pic}\, S$), $v=(2, h, 6)$, $(v^2)=0$, $a=(2,6)=2$

$T := M_S(v)$ K3 of degree $(\hat{h})^2 = 24/4 = 6$

$T = (2) \cap (3) \subset \mathbb{P}^4$ if S is general.

Failure: Unlike (ideal) case of $g=7$ (Example 2), smooth $C \in |\hat{h}|$ of genus 4 does not work, not give a good description.

Key (of success): Take a 1-nodal $\bar{C} \in |\hat{h}|$ of $p_a = 4$ & $g=3$, and consider the pull-back $\mathcal{E}$ of the universal bundle $\bar{\mathcal{E}}$ on $S \times \bar{C}$, where $C \longrightarrow \bar{C}$ is the normalization and $g(C)=3$.



$\boxed{3}$ (improved form) Consider the opposite moduli map

$$S \longrightarrow \mathcal{U}_C(2), \quad s \longmapsto \mathcal{E}|_{s \times C}.$$

Then S has a nice description in the moduli. (Since C itself is a kind of moduli, $\mathcal{U}_C(2)$ is a "double moduli", or, a "modular hull".)

§5 Hecke partner of $\mathcal{C}_4$

$\xi \in \mathrm{Pic}\, C$

$$SU_C(2,\xi) := \left\{ E \;\middle|\; \begin{array}{l} \text{stable 2-bundle} \\ \text{w. det } \xi \end{array} \right\} \Big/ \text{isom.}$$

bdry = Km(C)

$\deg \xi$ even $\Rightarrow$ $\overline{SU_C(2,\xi)} \cong \mathcal{C}_4$ Coble quartic

$E \otimes \sqrt{\xi} \longleftrightarrow E \in SU_C(2, \mathcal{O}_C)$ ($\cong \mathcal{C}_4$)

$\deg \xi$ odd $\Rightarrow$ $SU_C(2,\xi)$ Fano 6 fold of index 2

$\exists$ univ. family $\mathcal{U}$ on $C \times SU_C(2,\xi)$

(Normalization: $\det \mathcal{U} \cong \xi \boxtimes \Xi_x$ where Ξ is positive generator of $\mathrm{Pic}\, SU_C(\;) \cong \mathbb{Z}$.)

$p \in C$ $\quad U_p := \mathcal{U}|_{p \times SU_C(2,\xi)}$ $\quad$ 1-dim'l family of 2-bundles

Fact: $H^0(U_p) \cong \mathbb{C}^8_p$ $\xrightarrow{\text{projectivisation}}$ ambient $\mathbb{P}^7_p$ of $\mathcal{C}_4$, or more precisely, of $SU_C(2, \xi(-p))$

Main Theorem $C = C_4$, $p_1, p_2 \in C$, $s_1 \in H^0(U_{p_1})$, $s_2 \in H^0(U_{p_2})$ are general

$\Longrightarrow$ $S := (s_1)_0 \cap (s_2)_0$ is a general member of $\mathcal{F}_{13}$ ($\Rightarrow$ unirat. of $\mathcal{F}_{13}$)

(Rmk: 4 points p_1', p_2', p_1 and p_2 are expected to lie on the same line in the ambient $\mathbb{P}^2$ of C.)

Rmk $g = 19$ Interchange the value of rank and g(C).

$$\left\{\begin{array}{l} g(C)=2 \\ SU_C(3) \end{array}\right. \qquad g=13 \qquad \left\{\begin{array}{l} g(C)=3 \\ SU_C(2) \end{array}\right.$$

Replace $\mathcal{C}_4^6 \subset \mathbb{P}^7 = |2\Theta|^\vee$ by

Coble cubic $\mathcal{C}_3^7 \subset \mathbb{P}^8 = |3\Theta|^\vee$ and its proj dual. ($\mathcal{C}_4$ is self dual.) General $S \in \mathcal{F}_{19}$ is expected to be obtained from the following data: $C \in \mathcal{M}_2$, $p_1, p_2 \in C$ and pair of points ξ_1, ξ_2 of the ambient of $\mathcal{C}_3$.

$$\xi_1 \in \mathbb{P}^8 \xleftarrow[p_1]{} \mathcal{C}_3 \xrightarrow[p_2]{} \mathbb{P}^8 \ni \xi_2$$