Fano 3-folds, Lagrangian fibration, and a supersingular OG10 with Co_2 configuration

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§1 Preliminary

K3 surface $S \rightsquigarrow X = S^{[n]} \xrightarrow[\text{min. resolution}]{} \mathrm{Sym}^n S$

$\mathrm{Pic}\, S \oplus \mathbb{Z}\delta = \mathrm{Pic}\, X$

$\uparrow$ 1/2 of exceptional divisor class

= Hilbert scheme of n-pts

= moduli of ideal of colength n

$= M_S(v)$ with $v=(1, 0, 1-n)$

Isom. as lattice

intersection form $\longleftrightarrow$ Beauville form q s.t. $q^n \sim$ self-int #

Lagrangian fibration $\longrightarrow$ D with $q(D)=0$

$\exists\ X^{2n} \longrightarrow Y^n \quad \Leftarrow\!\!? \quad \exists\ D \ \ldots$

gen. fiber = Lagrangian A.V.

(Bayer-Macri)

$\boxed{\text{Basic Example}}$ $S \xrightarrow[\pi]{2:1} \mathbb{P}^2 \supset B = (6)$

$$\begin{array}{ccc} S^{[2]} & \dashrightarrow & \mathbb{P}^{2,*}, \\ \cup & & \\ \mathbb{P}^2 & \text{indetermanacy} & \nearrow \\ \text{flop} \downarrow & & \\ X & & \end{array}$$

$(p, q) \to \overline{\pi(p)\pi(q)}$ line

$$\begin{cases} D = h - \delta, \quad h = \pi^* \mathcal{O}(1) \\ \text{fiber} = \mathrm{Jac}^2\, \pi^{-1}(l) \end{cases}$$

§2 Examples related with Fano 3-folds

① Very general K3 of degree 8 $S = S_8 = (2, 2, 2) \subset \mathbb{P}^5$

$S^{[2]}$ $\oplus \mathbb{Z}\delta,\ q(\delta) = -2.$

$\mathrm{Pic}\, S = \mathbb{Z}h$

$(h^2) = 8$

Q. What is the Lagrangian fibration of $X = S^{[2]}$ for $D = h - 2\delta$ with $q(D) = 0$?

Answer. $\Phi_{|D|} : X \longrightarrow \mathbb{P}^{2,*}$ is Jacobian fibration over net $\langle Q_1, Q_2, Q_3 \rangle =: \mathbb{P}^2_S$ of quadrics defining $S = S_8$.

☺ In fact, Φ is O'Grady map. $\{p, q\} \in S \rightsquigarrow$ line $\overline{pq} \not\subset S_8$

$$\overline{pq} \cup S_8 \subset^{\exists !} V_4 = Q_1' \cap Q_2' \subset \mathbb{P}^5$$

$\Phi_{|D|}$ is the map $\{p, q\} \longmapsto \langle Q_1', Q_2' \rangle \in \mathbb{P}^{2,*}_S$ subpencil

fiber = Fano variety of lines in a fixed V_4
= Jacobian of curve of genus 2 q.e.d.

② Very general K3 of degree 18, g= 10 $S = S_{18} \subset \mathbb{P}^{10}$

Q. What is the Lagrangian fibration of $X = S^{[2]}$ for $D = h - 3\delta$ with $q(D) = 0$?

Key variety:
(of Borcea)

$$\Sigma^5_{18} = G_2/P_{adj} \subset \mathbb{P}(\mathfrak{g}) = \mathbb{P}^{13}$$

contact Fano manifold

$$[S_{18} \subset \mathbb{P}^{10}] = [\quad '' \quad] \cap H_1 \cap H_2 \cap H_3$$

Hint 1. ① and ② are similar.

Hint 2. S_8 in ① is also a linear section:

$$S_8 = [v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}] \cap H_1 \cap H_2 \cap H_3.$$

$$v_2(\mathbb{P}^5) = Sp(6)/P_{adj}., \quad \mathbb{P}^{20} = \mathbb{P}(sp(6)) = \mathbb{P}(S^2\mathbb{C}^6)$$

Moreover, a linear section of contact Fano 5-fold!

KEY: Rational homogeneous contact manifold

$$X = G\cdot[v] \subset \mathbb{P}(\mathfrak{g})$$

$\mathfrak{g}$: simple Lie algebra

h.w. vector

has a unique conic property: for every pair p, q $\in$ X in general position, unique conic C on X passing through p, q.

$= C_{p,q}$

Pf. Put p=[u], q=[v]. Then u and v generate a 3-dim'l Lie subalgebra $\mathfrak{a}$. C is the intersection with X and the 2-plane $\mathbb{P}(\mathfrak{a})$. //

Answer. $S_{18} \ni C_{p,q}$ is contained in a unique Fano 3-fold $V_{18} = \Sigma^5_{18} \cap H_1' \cap H_2'$.

The Lagrangian fibration $X = S^{[2]} \longrightarrow \mathbb{P}^2$

send $\{p, p'\}$ to $\mathbb{P}^1_{p,p'} = \langle H_1', H_2' \rangle$, the subpencil of $\mathbb{P}^2_S := \langle H_1, H_2, H_3 \rangle$ defining $S = S_{18}$.

fiber = Fano variety of conics in a fixed V
= Int-Jac of V = Jacobian of curve of genus 2
(Kuznetsov et al)

REMARK: $\exists$ Conjecture. "contact Fano $\overset{?}{\to}$ homogeneous," which includes (Hartshorne's conj.=) Mori's theorem as special case.

Side Problem. Does a contact Fano manifold satisfy a unique conic property? (Here "conic" means a curve of degree 2 with respect to the contact line bundle, whose (dim. - 1)/2-th power is anti-canonical.)

③ Very general K3 of degree 16, g= 9 (Omitted)

§3 Leech-K3 analogue of del Pezzo surfaces

Similarity between $I_{1,9}$ and $II_{1,25}$

Hyperbolic lattice, unimodular, odd and even

Both have beautiful fundamental domains & str. of orthog. grps

$$\star\ O(I_{1,9},\mathbb{Z}) \sim \begin{pmatrix}\text{gp generated}\\ \text{by } (-1)\\ \text{reflections}\end{pmatrix} \cdot \underbrace{\mathbb{Z}^8 \cdot W(E_8)}_{W(\tilde{E}_8)}$$

Coxeter group

Symmetry of a fund. domain

(Conway '80's)

$$\star\star\ O(II_{1,25},\mathbb{Z}) \sim \begin{pmatrix}\text{gp generated}\\ \text{by } (-2)\\ \text{reflections}\end{pmatrix} \cdot \mathbb{Z}^{24} \cdot Co_0$$

Leech lattice

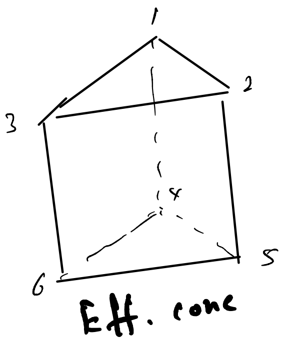
O(Leech)

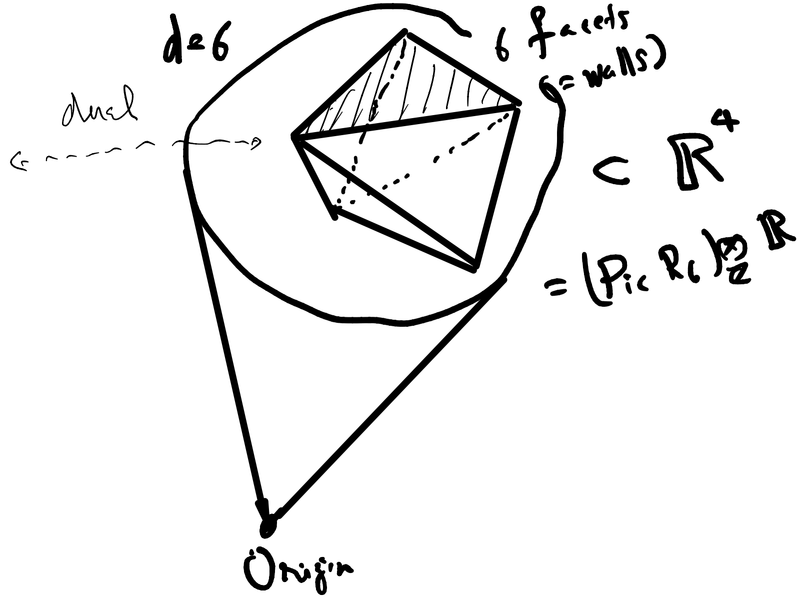
CAG (classical algebraic geometry)

$I_{1,9}$ contains Pic R_d, the Picard lattice of a del Pezzo surface of degree d, as the orthogonal complement of the sum of d copies of <-1>.

R_d = Bl-up of the plane at 9-d points in general position

The nef cone is finite polyhedral, and the walls are defined by lines.

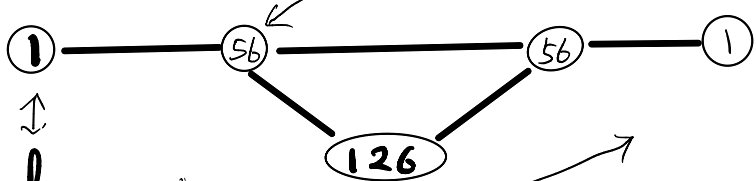




Remark (1) -K is the Weyl vector in the sense that (-K. l)=1 for all wall defining vectors l.
(2) The Weyl vector is isotropic for both and .

d	6	5	4	3	2	1	0
# of lines	6	10	16	27	56	240	∞

d=1, graph of line configuration ($\fallingdotseq E_8$ root system)
vertex ↔ line



Fixed line ℓ

Bertini involution of R_1

Intersection #

$(\ell.\ )$	-1	0	1	2	3

Task: Geometrize as possible as one can in the framework

negative definite Leech lattice

Extended Leech lattice

$$\mathrm{Pic}(\,*\,) \hookrightarrow II_{1,25} = \mathbb{Z} \oplus \Lambda \oplus \mathbb{Z}$$

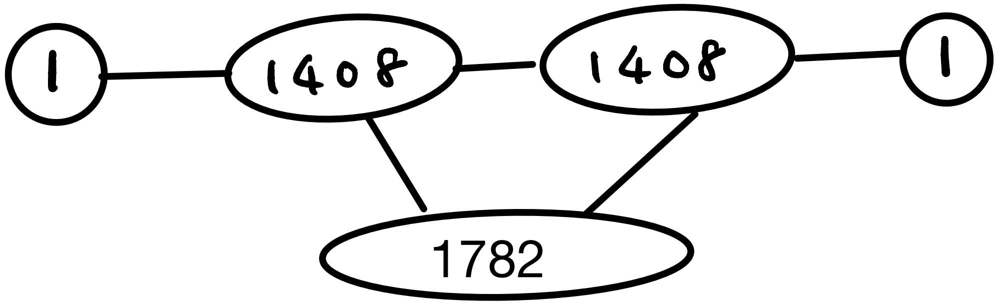
$(r, \quad x, \quad s)$ $\xrightarrow{\text{quad. form}}$ $(x^2)_\Lambda - 2rs$

where $*$ is a K3 surface, K3-like object/ category, etc.

Leech analogy of degree 1 del Pezzo is the double Conway graph

Γ_z $\begin{cases} \text{vertex } (1, x, -1), \text{ both } x \text{ and } z\text{-}x \text{ have min. norm } (\# = 4600) \\ \text{suitable adjacency by intersection number} \end{cases}$

in the orthogonal complement of sum of two copies of $\langle -2 \rangle$, one is generated by $(1, 0, 1)$ and the other by $(1, z, -1)$, for a fixed z of min. norm.



Bertini-like involution

$x \longleftrightarrow z\text{-}x$

The double Conway graph has symmetry of the 2nd Conway group Co_2 The vertex stabilizer group is the unitary group $U_6(\mathbb{F}_4)$.

Sub-task: Find a K3-like object of Picard number 24 which incarnates the double Conway graph.

§4. Relation between §2 and §3

§3 is partly inspired by the unfinished/untreated case of §2, namely the case of genus 8.

Case	①.	②.	③.	④
Sympl. Var.	S^[2]	S^[2]	S^[3]	OG10?
Fano	Quartic dP	g=10	g=9	g=8

Partial answer: The Leech analogy of degree 1 del Pezzo surface must be an OG10-like symplectic variety (with a Lagrangian fibration) related with the Fermat cubic 4-fold in characteristic 2.

Another supporting fact: the number 1782 in the graph is twice the number (=891) of 2-planes in the 4-fold.

I hope I will have another chance to discuss about this topic in near future.

Thank you for your attention, and the organizers, both Sho Tanimoto and Shigeyuki Kondo, for the wonderful conference.

References

(12/02/22)

§1

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§2

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§3

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§4

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