

## On Fano 3-Folds with $B_2 \geq 2$

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This article is an introduction to the classification of Fano 3-folds, i.e., 3-dimensional smooth projective varieties with ample anticanonical bundles, whose second Betti numbers are not less than two. We shall show, with proof or by an example, the principle of "how to classify them" so that one will be able to do it. The complete classification will be published elsewhere.

After stating our main results in §1, we summarize the results of Iskovskih, Šokurov and Mori which are indispensable to our classification in §§ 2 and 3. § 5 is devoted to the classification of Fano 3-folds with  $B_2 = 2$ , especially, imprimitive ones. In § 6, we investigate the properties of Fano conic bundles which play an essential role in the classification of imprimitive Fano 3-folds with  $B_2 \geq 3$  (§ 7), in the proof of Theorem 1.6 (§ 8) and in the proof of Theorem 1.2. (§ 9).

### § 0. Del Pezzo surface

We review here the theory of del Pezzo surfaces because it is useful for understanding the outline of the classification of Fano 3-folds.

Let  $S$  be a del Pezzo surface, i.e., a smooth surface with negative canonical bundle. The positive integer  $(-K_S)^2$  is called the degree of  $S$ . By Serre duality, Kodaira's vanishing theorem and the Riemann-Roch theorem, we have

$$(0.1) \quad q = p_g = 0, \quad \rho(S) = B_2(S), \quad \dim | -K_S | = (-K_S)^2.$$

By Noether's formula  $c_1^2 + c_2 = 12\chi(\mathcal{O}_S)$ , We have

$$(0.2) \quad (-K_S)^2 + \rho(S) = 10.$$

In particular, we have

**Proposition 0.3.**  $(-K_S)^2 \leq 9$  and  $\rho(S) \leq 9$  for every del Pezzo surface  $S$ .

In the case of Fano 3-folds, we have  $(-K_X)^3 \leq 64$  and  $\rho(X) \leq 10$ . But

the proof is not so easy as the proposition above.

About the anticanonical system  $|-K_S|$ , we have

**Proposition 0.4.** (1) *If  $(-K_S)^2 \geq 2$ , then  $|-K_S|$  is free, i.e., free from fixed components and base points.*

(2) *If  $(-K_S)^2 \geq 3$ , then  $|-K_S|$  is very ample.*

In the case  $d = (-K_S)^2 \geq 3$ , the surface  $S_d$  of degree  $d$  embedded into  $P^d$  by the morphism  $\varphi$  attached to  $|-K_S|$  is called the anticanonical model of  $S$ .

If  $\rho(S) = 1$ , then by the Poincaré duality, there is an ample divisor  $D$  with  $(D^2) = 1$ . Since  $(-K_S)^2 = 9$ ,  $-K_S$  is numerically equivalent to  $3D$ . It is easy to see that  $\dim |D| = 2$  and the map associated to  $|D|$  is an isomorphism. Hence we have

**Proposition 0.5.** *If  $\rho(S) = 1$ , then  $S \cong P^2$ .*

(0.6) Extremal ray of a del Pezzo surface  $S$  (Theorem 2.1 [6])

Let  $NE(S) \subset (\{\text{divisors}\} / \cong) \otimes_{\mathbb{Z}} \mathbb{R}$  be the cone generated by effective divisors on  $S$  modulo numerical equivalence. This cone is a closed polyhedral cone. If  $\rho(S) \geq 2$ , then an irreducible reduced curve  $C$  such that the equivalence class  $[C]$  lies on the edge of  $NE(S)$  satisfies one of the following:

- 1)  $C$  is an exceptional curve of the first kind.
- 2)  $\rho(S) = 2$ ,  $S$  has a  $P^1$ -bundle structure and  $C$  is its fibre.

**Proposition 0.7.** *If  $S$  is minimal, then  $\rho(S) = 1$  or  $S \cong P^1 \times P^1$ .*

*Proof.* Assume that  $\rho(S) > 1$ . Since  $S$  has no exceptional curve of the first kind,  $\rho(S) = 2$  and  $S$  has a  $P^1$ -bundle structure by (0.6). Since  $NE(S)$  has two edge,  $S$  has two  $P^1$ -bundle structures. Hence  $S$  is isomorphic to  $P^1 \times P^1$ . q.e.d.

**Proposition 0.8.** *Let  $E$  be an exceptional curve of the first kind on  $S$  and  $\alpha: S \rightarrow S'$  the blowing down of  $E$ . Then  $S'$  is a del Pezzo surface.*

By these propositions, a del Pezzo surface  $S$  satisfies one of the following:

	$d = (-K_S)^2$	$\rho(S)$	$S$
1)	9	1	$P^2$
2)	8	2	$P^1 \times P^1$
3)	8	2	$F_1$
4)	$1 \leq d \leq 7$	$10 - d$	the blow-up of $P^2$ at $9 - d$ points in "general position".

The meaning of “general position” in 4) is as follows.

**Proposition 0.9.** ([5]) *The blow-up of  $P^2$  at  $n$  points  $x_1, \dots, x_n$  ( $n \leq 8$ ) is a del Pezzo surface if and only if no 3 of them lie on a line, no 6 on a conic, and for  $n=8$  all eight do not lie on a cubic which is singular at one of  $x_1, \dots, x_8$ .*

By these propositions, we can conclude that the del Pezzo surfaces have exactly 10 deformation types.

§ 1. Main results

Our final result on Fano 3-folds is the following:

**Theorem 1.1.** *There are exactly 87 types of Fano 3-folds with  $B_2 \geq 2$  up to deformations. (See [7] for the description of the 87 types.)*

**Theorem 1.2.** *A Fano 3-fold with  $B_2 \geq 6$  is isomorphic to  $P^1 \times S_{11-B_2}$ , where  $S_d$  is a del Pezzo surface of degree  $d$ . In particular, every Fano 3-fold has  $B_2 \leq 10$ .*

The number  $N(b)$  of Fano 3-folds with  $B_2=b$  up to deformations is as follows:

$b$	2	3	4	5	6	7	8	9	10	$\geq 11$
$N(b)$	36	31	12	3	1	1	1	1	1	0

We refer the reader to [4] for Fano 3-folds with  $B_2=1$  which are called Fano 3-folds of the first species.

**Definition 1.3.** A Fano 3-fold is *imprimitive* if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve. A Fano 3-fold is *primitive* if it is not imprimitive.

**Example 1.4.** A Fano 3-fold with  $B_2=1$  is primitive.  $P^1 \times P^2$ ,  $P^1 \times P^1 \times P^1$  and a divisor  $W_0 \subset P^2 \times P^2$  of bidegree (1, 1) are primitive Fano 3-folds. If a Fano 3-fold is a double cover of another primitive Fano 3-fold with ample branch locus, then it is also primitive.

**Example 1.5.** Let  $C$  be a disjoint union of  $n$  smooth irreducible curves in  $P^3$  and assume that  $C$  is a scheme theoretic intersection of cubics. Then the blow-up  $X$  of  $P^3$  along  $C$  is an imprimitive Fano 3-fold with  $B_2=n+1$ .

*Proof.* It suffices to show that  $-K_X$  is ample. Let  $\alpha: X \rightarrow P^3$  be the blowing up and  $D=\alpha^{-1}(C)$  the exceptional divisor. Then we have  $-K_X$

$\sim \alpha^*(-K_{P^3}) - D \sim 4\alpha^*H - D$  for a plane  $H$  in  $P^3$ . The linear system  $|3\alpha^*H - D|$  is free by our assumption and  $|\alpha^*H|$  is also free. Hence  $| -K_X | = |(3\alpha^*H - D) + \alpha^*H|$  is free. By Proposition 4.6. [2] it suffices to show that  $(-K_X \cdot Z) > 0$  for every irreducible reduced curve  $Z$  on  $X$ .

Case in which  $\alpha(Z)$  is a point.  $Z$  is an exceptional line and  $(D \cdot Z) = -1$ . Hence  $(-K_X \cdot Z) = (4\alpha^*H \cdot Z) - (D \cdot Z) = 4(H \cdot \alpha_*Z) + 1 = 1$ .

Case in which  $\alpha(Z)$  is not a point. Since  $|3\alpha^*H - D|$  is free and  $H$  is ample, we have

$$(-K_X \cdot Z) = (\alpha^*H \cdot Z) + (3\alpha^*H - D \cdot Z) \geq (H \cdot \alpha_*Z) > 0 \quad \text{q.e.d.}$$

The following is the first step of our classification and will be proved in § 8.

**Theorem 1.6.** *Let  $X$  be a primitive Fano 3-fold. Then we have*

- (1)  $B_2(X) \leq 3$ ,
- (2) if  $B_2(X) = 2$ , then  $X$  is a conic bundle over  $P^2$  and
- (3) if  $B_2(X) = 3$ , then  $X$  is a conic bundle over  $P^1 \times P^1$  and has either a divisor  $D \cong P^1 \times P^1$  such that  $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$  or another conic bundle structure over  $P^1 \times P^1$ .

By (3) of the theorem, it is not hard, though we omit it here, to see that a primitive Fano 3-fold with  $B_2 = 3$  satisfies one of the following:

- (a)  $(-K_X)^3 = 12$ .  $X$  is a double cover of  $P^1 \times P^1 \times P^1$  whose branch locus is a smooth divisor of tridegree  $(2, 2, 2)$ .
- (b)  $(-K_X)^3 = 14$ .  $X$  is a smooth member of  $|L^{\otimes 3} \otimes \mathcal{O}_{P^1 \times P^1}(2, 3)|$  on the  $P^2$ -bundle  $P(\mathcal{O} \oplus \mathcal{O}(-1, -1)^{\otimes 2})$  over  $P^1 \times P^1$  such that  $X \cap Y$  is irreducible, where  $L$  is the tautological line bundle and  $Y$  is the unique member of  $|L|$ .
- (c)  $(-K_X)^3 = 48$ .  $X$  is isomorphic to  $P^1 \times P^1 \times P^1$ .
- (d)  $(-K_X)^3 = 52$ .  $X$  is isomorphic to the  $P^1$ -bundle  $P(\mathcal{O} \oplus \mathcal{O}(1, 1))$  over  $P^1 \times P^1$ .

The primitive Fano 3-folds with  $B_2 = 2$  cannot be classified only by (2) of Theorem 1.6. The following will be proved in § 5.

**Theorem 1.7.** *The primitive Fano 3-folds with  $B_2 = 2$  have the following 9 deformation types:*

	$(-K_X)^3$	$X$	types of extremal rays
1)	6	a double cover of $P^1 \times P^2$ whose branch locus is a divisor of bidegree $(2, 4)$	$C_1 - D_1$
2)	12	a double cover of $W_6$ , see 6), whose branch locus is a member of $  -K_{W_6}  $ or a smooth divisor on $P^2 \times P^2$ of bidegree $(2, 2)$	$C_1 - C_1$

3)	14	a double cover of $V_7$ , see 8), whose branch locus is a member of $ -K_{V_7} $	$C_1 - E_3$ or $E_4$
4)	24	a double cover of $P^1 \times P^2$ whose branch locus is a divisor of bidegree (2,2)	$C_1 - D_2$
5)	30	a smooth divisor on $P^2 \times P^2$ of bidegree (1,2)	$C_1 - C_2$
6)	48	$W_8$ , a smooth divisor on $P^2 \times P^2$ of bidegree (1,1)	$C_2 - C_2$
7)	54	$P^1 \times P^2$	$C_2 - D_3$
8)	56	$V_7 = P(\mathcal{O} \oplus \mathcal{O}(1))$	$C_2 - E_2$
9)	62	$P(\mathcal{O} \oplus \mathcal{O}(2))$	$C_2 - E_3$

§ 2. Known results

First we recall some known results on Fano 3-folds which are necessary for our classification.

O. Elementary facts. Let  $X$  be a Fano 3-fold. Then by Serre duality, the Riemann-Roch theorem and Kodaira's vanishing theorem, we have

- i)  $h^i(\mathcal{O}_X) = 0$  for every  $i > 0$ ,
- ii)  $(-K_X \cdot c_2(X)) = 24\chi(\mathcal{O}_X) = 24$ ,
- iii)  $\dim |-K_X| = (1/2)(-K_X)^3 + 2$  and
- iv)  $\text{Pic } X$  is torsion-free.

In particular, the Picard number  $\rho(X)$  is equal to the second Betti number  $B_2(X)$  and is a topological invariant of  $X$ .

I. Fano 3-folds with index  $\geq 2$  ([3]). The largest integer  $r$  which divides  $-K_X$  in  $\text{Pic } X$  is called the *index* of  $X$ . Fano 3-folds with index  $\geq 2$  are classified as follows:

index of $X$	$B_2(X)$	$(-K_X)^3$	$X$
4	1	64	$P^3$
3	1	54	a smooth quadric $Q$ in $P^4$
2	1	$8d$	$V_d, 1 \leq d \leq 5$
	2	48	$W_8$
	3	48	$P^1 \times P^1 \times P^1$
	2	56	$V_7$ , see Theorem 1.7,

where 1)  $V_1$  is a double cover of the Veronese cone  $W_4 \subset P^5$  whose branch locus is a smooth intersection of  $W_4$  and a cubic hypersurface not passing through the vertex of the cone,

2)  $V_2$  is a double cover of  $P^3$  whose branch locus is a smooth quartic hypersurface,

3)  $V_3$  is a smooth cubic hypersurface of  $P^4$ ,

- 4)  $V_4$  is a complete intersection of two quadrics in  $P^5$ ,  
 5)  $V_5$  is a complete intersection of a linear subspace  $P^6$  in  $P^9$  and the Grassmann variety  $\text{Grass}(P^4 \supset P^1)$  embedded in  $P^9$  by the Plücker embedding, and  
 6)  $W_6$  is a smooth divisor on  $P^2 \times P^2$  of bidegree (1,1).  $W_6$  is isomorphic to the  $P^1$ -bundle  $P(T_{P^2})$  over  $P^2$ .

II. Fano 3-folds whose anticanonical systems are not very ample ([3]). Let  $X$  be a Fano 3-fold of index 1. Then we have

1) the anticanonical system  $|-K_X|$  is free (i.e., free from fixed components and base points) except for the following two cases

(i)  $X$  is isomorphic to the blow-up of  $V_1$  along a smooth elliptic curve which is a complete intersection of two members of  $|-(1/2)K_{V_1}|$ . One has  $(-K_X)^3=4$ .

(ii)  $X$  is isomorphic to  $P^1 \times S_1$ , where  $S_1$  is a del Pezzo surface of degree 1. One has  $(-K_X)^3=6$ .

2) Assume that  $|-K_X|$  is free but not very ample. Such Fano 3-folds are called hyperelliptic and classified as follows:

$B_2(X)$	$(-K_X)^3$	$X$
1	2	a double cover of $P^3$ whose branch locus is a smooth sextic.
1	4	a double cover of a quadric hypersurface $Q \subset P^4$ whose branch locus is a smooth intersection of $Q$ and a quartic hypersurface
2	6	a double cover of $P^1 \times P^2$ whose branch locus is a smooth divisor on $P^1 \times P^2$ of bidegree (2,4)
2	8	the blow-up of $V_2$ along an elliptic curve which is a complete intersection of two members of $ -(1/2)K_{V_2} $
9	12	$P^1 \times S_2$ , where $S_2$ is a del Pezzo surface of degree 2.

III. Existence of lines ([8], [9]). Let  $X$  be a Fano 3-fold whose anticanonical system is very ample. Its anticanonical model, i.e., the image of  $X$  by the morphism attached to  $|-K_X|$ , is a subvariety of degree  $2g-2$  in  $P^{2g+1}$ , where  $g=(1/2)(-K_X)^3+1$ .

**Theorem (Šokurov).** *If the index of  $X$  is equal to 1 and if  $X \not\cong P^1 \times P^2$ , then there exists a 1-dimensional family of lines on the anticanonical model  $X_{2g-2} \subset P^{2g+1}$ .*

§ 3. Extremal rays of Fano 3-folds

We apply the theory in [6] to Fano 3-folds. Let  $X$  be a Fano 3-fold with  $B_2 \geq 2$ . Let  $N_z(X)$  (resp.  $NE_z(X)$ ) be the set of numerically equivalence classes of 1-cycles (resp. effective 1-cycles) on  $X$ . Let  $N(X) = N_z(X) \otimes_{\mathbb{Z}} \mathbb{R}$  and  $NE(X)$  the cone in  $N(X)$  generated by  $NE_z(X)$ . By Theorem 1.2 [6], we have

$$(3.1) \quad NE(X) \text{ is a closed polyhedral cone.}$$

Let  $R$  be an extremal ray of  $X$ , i.e., a half line which is an edge of the polyhedral cone  $NE(X)$ . There exists a morphism  $f: X \rightarrow Y$  to a normal projective variety such that (i)  $f_* \mathcal{O}_X \cong \mathcal{O}_Y$  and (ii) for any irreducible reduced curve  $C$  on  $X$ ,  $[C] \in R$  if and only if  $f(C)$  is a point. Such an  $f$  is unique up to an isomorphism, called the contraction of  $R$  and denoted by  $\text{cont}_R: X \rightarrow Y$  (Theorem 3.1 [6]).

Put  $\mu(R) = \min \{(-K_X \cdot Z) \mid Z \text{ is a rational curve such that } [Z] \in R\}$  and let  $l = l_R$  be a rational curve such that  $[l] \in R$  and  $(-K_X \cdot l) = \mu(R)$ . Then there exists an exact sequence

$$(3.2) \quad 0 \rightarrow \text{Pic } Y \xrightarrow{f^*} \text{Pic } X \xrightarrow{(\cdot l)} \mathbb{Z} \rightarrow 0,$$

where  $(\cdot l)(D) = (D \cdot l)$  for  $D \in \text{Pic } X$ . (Theorem 1.2 [6]. The surjectivity of  $(\cdot l)$  is a consequence of classification of  $R$  and Corollary 3.6 below and not true in general if  $X$  is not a Fano 3-fold.) In particular, we have  $\rho(X) = \rho(Y) + 1$ .

$R$  and  $f = \text{cont}_R$  are classified as follows:

Case  $\dim Y = 3$ : There exists an irreducible reduced divisor  $D$  of  $X$  such that  $f|_{X-D}$  is an isomorphism and  $\dim f(D) \leq 1$ . Such  $D$  is uniquely determined by  $R$  and called the exceptional divisor of  $R$ . Moreover  $f$  is the blowing-up of  $Y$  by the ideal defining  $f(D)$  (given the reduced structure).  $f$  and  $D$  satisfy one of the following ([6] Theorem 3.3 and Corollary 3.4)

type of $R$	$f$ and $D$	$\mu(R)$	$l$
$E_1$	$f(D)$ is a smooth curve, $Y$ is smooth and $f _D: D \rightarrow f(D)$ is a $P^1$ -bundle.	1	exceptional line
$E_2$	$f(D)$ is a point, $Y$ is smooth, $D \cong P^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_P(-1)$ .	2	line in $D \cong P^2$
$E_3$	$f(D)$ is an ordinary double point. $D \cong P^1 \times P^1$ , $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$ and $s \times P^1$ and $P^1 \times t$ are numerically equivalent for every $s, t \in P^1$ .	1	$s \times P^1$ or $P^1 \times t$ in $D$

$E_4$	$f(D)$ is a double point. $D$ is an irreducible reduced singular quadric surface in $P^3$ , $\mathcal{O}_D(D) \cong \mathcal{O}_D \otimes \mathcal{O}_P(-1)$ .	1	generator of the cone $D$
$E_5$	$f(D)$ is a quadruple point of $Y$ , $D \cong P^2$ , and $\mathcal{O}_D(D) \cong \mathcal{O}_P(-2)$ .	1	line in $D \cong P^2$

If  $R$  is of  $E_1$ -type, then either (1)  $Y$  is a Fano 3-fold or (2)  $D \cong P^1 \times P^1$  and  $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$  (Proposition 4.5).

$R$  is called of  $E_{1,a}$ -type in the case (1) and of  $E_{1,b}$ -type in the case (2). If  $R$  is of  $E_{1,b}$ -type, then a horizontal section of  $f|_D: D \rightarrow C$  is not numerically equivalent to a fibre of  $f|_D$  (because  $Y$  is projective), and belongs to another extremal ray of  $E_{1,b}$ -type. If  $B_2(X)=2$ , then  $Y$  is always a Fano 3-fold because  $Y$  is projective and  $\rho(Y)=1$ . Moreover,  $Y$  is of index  $\geq 2$  (Proposition 4.10).

In the case  $R$  is of  $E_2$ -type,  $Y$  is a Fano 3-fold. The proof is similar to and easier than Proposition 4.5.

If the ray  $R$  is of type  $E_2, E_3, E_4$  or  $E_5$ , then the exceptional divisor has the following property:

(3.3) Both  $\mathcal{O}_D(D)$  and  $\omega_D$  are negative, i.e., their inverses are ample. Every curve in  $D$  can move in  $D$ .

(3.4)  $D$  is mapped to a point by every morphism  $g$  from  $X$  onto a curve if there is any.

*Proof.* If  $R$  is of type  $E_2, E_4$  or  $E_5$ , then  $D$  has no nontrivial morphism onto a curve. If  $R$  is of type  $E_3$ , then  $D \cong P^1 \times P^1$ . Hence either  $g(s \times P^1)$  or  $g(P^1 \times t)$  is a point, or equivalently, the intersection number  $(s \times P^1 \cdot g^{-1}(\text{point}))$  or  $(P^1 \times t \cdot g^{-1}(\text{point}))$  is zero. Since  $s \times P^1$  and  $P^1 \times t$  are numerically equivalent for every  $s, t \in P^1$  by definition, both  $g(s \times P^1)$  and  $g(P^1 \times t)$  are points. It follows that  $g(D)$  is a point. q.e.d.

Case  $\dim Y=2$ :  $Y$  is a smooth projective surface,  $f: X \rightarrow Y$  is a conic bundle and  $f^{-1}(C)$  is irreducible for every irreducible curve  $C$  on  $Y$  (Proposition 6.3). We have the following two cases:

type of $R$	$f$	$\mu(R)$	$l$
$C_1$	$f$ has a singular fibre	1	an irreducible component of a reducible fibre or a reduced part of a multiple fibre
$C_2$	$f$ is a $P^1$ -bundle	2	a fibre

**Proposition 3.5.**  $Y$  is rational.

*Proof.* Since  $q(X)=0$  and  $f$  is surjective,  $q(Y)=0$ . By the formula



$-4K_Y \cong f_*(-K_X)^2 + A_f$  (Proposition 6.2, (4)),  $(K_Y \cdot A)$  is negative for every ample divisor  $A$  on  $Y$ . Hence all the plurigenera  $P_m(Y)$  vanish. Therefore  $Y$  is rational by Castelnuovo's criterion. q.e.d.

**Corollary 3.6.** *If  $R$  is of  $C_2$ -type, then  $f$  is locally trivial for Zariski topology.*

*Case  $\dim Y = 1$ :*  $Y$  is a smooth curve and  $\rho(X) = \rho(Y) + 1 = 2$ . Every fibre of  $f$  is irreducible and reduced and the generic fibre  $X_\eta$  is a del Pezzo surface. We have the following three cases:

type of $R$	$f$	$\mu(R)$
$D_1$	$X_\eta$ is a del Pezzo surface of degree $d$ , $1 \leq d \leq 6$ .	1
$D_2$	$f$ is a quadric bundle, i.e., every fibre is isomorphic to a normal quadric surface in $P^3$ .	2
$D_3$	$f$ is a $P^2$ -bundle.	3

Since  $q(X) = 0$  and  $f$  is surjective,  $q(Y) = 0$ . It follows that  $Y \cong P^1$ .

#### § 4. Blowing-up and blowing-down of Fano 3-folds

Let  $f: X \rightarrow Y$  be the blowing-up of a smooth 3-dimensional variety  $Y$  along a smooth irreducible curve  $C$  on  $Y$ . We will keep the meaning of these symbols in this section. The following are easily verified:

(4.1)  $-K_X \sim f^*(-K_Y) - D$  for the exceptional divisor  $D$  of  $f$ .

(4.2)  $D \cong P(N_{C/Y}^*)$  and  $\mathcal{O}_D(-D)$  is the tautological line bundle, where  $N_{C/Y}$  is the normal bundle of  $C$  and  $N_{C/Y}^*$  is its dual vector bundle.

(4.3)  $(D^3) = -\deg N_{C/Y}, \quad (D^2 \cdot -K_X) = 2p_a(C) - 2,$   
 $(D \cdot (-K_X)^2) = (-K_Y \cdot C) + 2 - 2p_a(C)$  and  
 $(-K_X)^3 = (-K_Y)^3 - 2\{(-K_Y \cdot C) - p_a(C) + 1\}.$

**Lemma 4.4.** *Assume that  $X$  is a Fano 3-fold. Then we have*

- (1)  $(-K_Y \cdot C) > 2p_a(C) - 2,$
- (2)  $(-K_Y \cdot C) \geq 0$  if  $C$  is rational. The equality holds if and only if  $N_{C/Y} \cong \mathcal{O}(-1)^{\oplus 2}$  or equivalently  $D \cong P^1 \times P^1$  and  $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$ , and
- (3)  $(-K_Y \cdot C) > p_a(C) - 1$  and  $(-K_X)^3 < (-K_Y)^3.$

*Proof.* Since  $-K_X$  is ample,  $((-K_X)^2 \cdot D)$  is positive. Hence (1) follows from (4.3). Since  $-K_X$  is ample, so is  $\mathcal{O}_D(-K_X)$ . Since  $\mathcal{O}_D(-K_X)$  is a tautological line bundle of  $f|_D: D \rightarrow C$ , the direct image  $F = (f|_D)_* \mathcal{O}_D(-K_X)$  is ample. If  $C \cong P^1$ , every vector bundle on  $C$  is a direct

sum of line bundles. Hence  $((-K_X)^2 \cdot D) = \deg F \geq 2$  and the equality holds if and only if  $F \cong \mathcal{O}(1)^{\oplus 2}$ . (2) follows from this and (4.3) because  $F \cong N_{C/Y} \otimes \omega_C^{-1}$ . (3) is an easy consequence of (1), (2) and (4.3). q.e.d.

**Proposition 4.5.** *If  $X$  is a Fano 3-fold, then one of the following holds:*

- (1)  $Y$  is a Fano 3-fold, and
- (2)  $C \cong \mathbf{P}^1$  and  $N_{C/Y} \cong \mathcal{O}(-1)^{\oplus 2}$  or equivalently  $D \cong \mathbf{P}^1 \times \mathbf{P}^1$  and  $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$ . Even in the case (2),  $|-nK_Y|$  is free for some  $n > 0$ .

*Proof.* First we show

*Claim:*  $|-nK_Y|$  is free for some  $n > 0$ .

$|-mK_X|$  is free for some  $m > 0$ . Therefore by (4.1),  $|-mK_Y|$  has no fixed components or no base points outside of  $C$ . By Kodaira's vanishing theorem, the restriction map  $H^0(\mathcal{O}_Y(-K_Y)) \cong H^0(\mathcal{O}_X(-K_X + D)) \rightarrow H^0(\mathcal{O}_D(-K_X + D)) \cong H^0(\mathcal{O}_C(-K_Y))$  is surjective. Since  $h^0(\mathcal{O}_C(-K_Y)) \geq \chi(\mathcal{O}_C(-K_Y)) = (-K_Y \cdot C) - p_a(C) + 1 > 0$  by Lemma 4.4,  $|-K_Y|$  has a member not containing  $C$ . Hence  $|-mK_Y|$  has only a finite number of base points. By Zariski's Theorem,  $|-m'mK_Y|$  is free for some  $m' > 0$ .

$(-K_Y \cdot Z) > 0$  for every irreducible curve  $Z \neq C$  on  $Y$ , because  $(-K_Y \cdot Z) = (f^*(-K_Y) \cdot Z') = (-K_X \cdot Z') + (D \cdot Z') > 0$ , where  $Z'$  is the strict transform of  $Z$ . Hence, if  $(-K_Y \cdot C) > 0$ , then  $Y$  is a Fano 3-fold by Proposition 4.6, [2]. If  $(-K_Y \cdot C) = 0$ , then  $C$  and  $D$  satisfy (2) by Lemma 4.4. q.e.d.

**Corollary 4.6.** *Let  $V$  be the blow-up of a Fano 3-fold  $W$  along a disjoint union of two smooth curves  $C_1$  and  $C_2$  on  $W$ . If  $V$  is a Fano 3-fold, then the blow-up  $V_i$  of  $W$  along  $C_i$  is a Fano 3-fold for each  $i = 1, 2$ .*

Next we consider a necessary condition for  $X$  to be a Fano 3-fold. The following is used very often in our classification.

**Proposition 4.7.** *If  $X$  is a Fano 3-fold, then  $C$  does not meet any curve  $Z$  with  $(-K_Y \cdot Z) = 1$  in a zero-dimensional set.*

*Proof.* Assume that  $C$  meets a curve  $Z$  with  $(-K_Y \cdot Z) = 1$  in a zero dimensional set. Let  $Z'$  be the strict transform of  $Z$  by  $f$ . Then, by (4.1), we have  $(-K_X \cdot Z') = (f^*(-K_Y) \cdot Z') - (D \cdot Z') = (-K_Y \cdot f_*Z') - (D \cdot Z') = 1 - (D \cdot Z')$ . By our assumption  $(D \cdot Z')$  is positive. Hence  $(-K_X \cdot Z)$  is nonpositive and  $-K_X$  is not ample. q.e.d.

**Corollary 4.8.** *Assume that  $Y$  is isomorphic to the blow-up of  $Y'$ . If  $X$  is a Fano 3-fold, then either  $C$  is disjoint from the exceptional divisor of  $Y|Y'$  or  $C$  is an exceptional line of  $Y|Y'$  (i.e., an irreducible reduced curve on  $Y$  which is mapped to a point of  $Y'$ ).*

**Proposition 4.9.** *Assume that  $C \cong P^1$  and  $(-K_Y \cdot C) = 1$ . If  $X$  is a Fano 3-fold, then  $N_{C/Y} \cong \mathcal{O}_P \oplus \mathcal{O}_P(-1)$ .*

*Proof.* Since  $\deg N_{C/Y} = (-K_Y \cdot C) + \deg K_C = -1$ ,  $N_{C/Y} \cong \mathcal{O}_P(n) \oplus \mathcal{O}_P(-1-n)$  for some  $n \geq 0$ . Let  $s$  be the section of  $P^1$ -bundle  $D = P(N_{C/Y}^*) \rightarrow C$  corresponding to the exact sequence

$$0 \rightarrow \mathcal{O}(n+1) \rightarrow N_{C/Y}^* \rightarrow \mathcal{O}(-n) \rightarrow 0.$$

By (4.1) and (4.2), we have  $(-K_X \cdot s) = (f^*(-K_Y) \cdot s) - (D \cdot s) = (-K_Y \cdot f_*s) + (\mathcal{O}_D(-D) \cdot s)_D = 1 - n$ . Therefore if  $X$  is a Fano 3-fold, we have  $n = 0$ .  
q.e.d.

The following proposition shows that, in order to classify imprimitive Fano 3-folds, it is not necessary to consider the blowing-up of Fano 3-folds of the first species. Its proof is heavily due to Šokurov's result.

**Proposition 4.10.** *If  $Y$  is a Fano 3-fold of the first species, i.e., with  $B_2 = 1$  and of index 1, then  $X$  is not a Fano 3-fold.*

*Proof.* Case 1.  $|-K_Y|$  is not very ample. By II of § 2,  $(-K_Y)^3 = 2$  or 4. If  $X$  were a Fano 3-fold,  $(-K_X)^3$  would be equal to 2 by (4.3). Since  $X$  is of index 1 and  $B_2(X) \geq 2$ ,  $|-K_X|$  would be very ample through II of § 2 and hence  $X \cong P^3$  by the anticanonical map, which is a contradiction.

Case 2.  $|-K_Y|$  is very ample. By III of § 2,  $Y$  has a 1-dimensional family of rational curves  $Z$  with  $(-K_Y \cdot Z) = 1$ . Let  $S$  be the union of such  $Z$ 's. Since  $\rho(Y) = 1$ ,  $S$  is ample. Hence  $C$  meets a rational curve  $Z$  with  $(-K_Y \cdot Z) = 1$ . If  $C \cap Z$  is 0-dimensional, then  $X$  is not a Fano 3-fold by Proposition 4.7. If  $C = Z$  and  $N_{Z/Y} \not\cong \mathcal{O} \oplus \mathcal{O}(-1)$ , then  $X$  is not a Fano 3-fold by Proposition 4.9. If  $C = Z$  and  $N_{Z/Y} \cong \mathcal{O} \oplus \mathcal{O}(-1)$ , then we can show that  $C$  meets other  $d+1$  rational curves  $Z'$  (counted with multiplicity) with  $(-K_Y \cdot Z') = 1$ , where  $d$  is the degree of the surface which is the union of all the deformations of  $Z$ . Hence  $X$  is not a Fano 3-fold by Proposition 4.7.

### § 5. Classification of Fano 3-folds with $B_2 = 2$

Let  $X$  be a Fano 3-fold with  $B_2 = 2$ . Since  $N(X) \cong \mathbb{R}^2$ ,  $NE(X)$  has just two edges, i.e., extremal rays  $R_1$  and  $R_2$  by (3.1). Set  $f_i = \text{cont}_{R_i}: X \rightarrow Y_i$ ,  $\mu_i = \mu(R_i)$  and  $l_i = l_{R_i}$  for  $i = 1, 2$ . Let  $L_i$  be an ample generator of  $\text{Pic } Y_i$  and put  $H_i = f_i^* L_i$ . The proof of the following is one of the essential parts of the classification of Fano 3-folds with  $B_2 = 2$ .

**Theorem 5.1.** (1) *Pic*  $X$  is the direct sum of  $f_1^* \text{Pic } Y_1$  and  $f_2^* \text{Pic } Y_2$ .  $\{H_1, H_2\}$  is a basis of *Pic*  $X$  and  $\{l_2, l_1\}$  is the dual basis of  $N_Z(X)$ . Moreover,  $-K_X \cong \mu_2 H_1 + \mu_1 H_2$ .

(2) If  $R_2$  is of type  $E_2, E_3, E_4$  or  $E_5$ , then  $(D_2 \cdot l_1) = 1$ , where  $D_2$  is the exceptional divisor of  $R_2$ .

*Proof of (1)* (mainly in the case  $X$  is primitive).

If  $R_1$  or  $R_2$  is of  $D_3$ -type, then  $X$  is a Fano  $P^2$ -bundle over  $P^1$  and we know that  $X$  is isomorphic to  $P^1 \times P^2$  or the blow-up of  $P^3$  along a line. For these  $X$ , (1) is easily checked. Hence we may assume that neither  $R_1$  nor  $R_2$  is of  $D_3$ -type.

By (3.2), we have two exact sequences

$$0 \longrightarrow \text{Pic } Y_i \xrightarrow{f_i} \text{Pic } X \xrightarrow{(\cdot l_i)} Z \longrightarrow 0 \quad i=1, 2.$$

By the definition of  $f_i = \text{cont}_{R_i}$ , it is obvious that  $f_1^* \text{Pic } Y_1 \cap f_2^* \text{Pic } Y_2 = \{0\}$ . Let  $a$  be the order of the quotient group  $\text{Pic } X / \sum_{i=1,2} f_i^* \text{Pic } Y_i$ . By the above exact sequences, both  $(H_1 \cdot l_2)$  and  $(H_2 \cdot l_1)$  are equal to  $a$ . Since  $(H_i \cdot l_i) = 0$  and  $(-K_X \cdot l_i) = \mu_i$  for  $i=1, 2$ , we have

$$(5.2) \quad a(-K_X) \cong \mu_2 H_1 + \mu_1 H_2.$$

Hence it suffices to show that  $a=1$ . We will deduce it from the equality

$$(5.3) \quad 24a = a(-K_X \cdot c_2(X)) = \mu_2(H_1 \cdot c_2(X)) + \mu_1(H_2 \cdot c_2(X))$$

and an estimation of  $(c_2(X) \cdot H_i)$ .

By the Riemann-Roch Theorem, we have

**Lemma 5.4.** *Let  $D$  be an effective divisor on  $X$ , then  $(c_2(X) \cdot D) = 6\chi(\mathcal{O}_D) + 6\chi(\mathcal{O}_D(D)) - 2(D^3) - ((-K_X)^2 \cdot D)$ .*

By the lemma, we have

(5.5)	type of $R$	$E_1 = E_{1,a}$	$E_2$	$E_3$ or $E_4$	$E_5$
	$(c_2(X) \cdot H)$	$24/r + \text{deg } C$	$24/r$	$24/r$	$45/r$
	$C_1$	$C_2$	$D_1$	$D_2$	$D_3$
	$\text{deg } \Delta + 6$	6	$12 - (K_{X_r})^2$	4	3

where  $r$  is the largest number which divides  $-K_X + D$  (resp.  $-K_X + 2D$ ,  $2(-K_X) + D$ ),  $D$  being the exceptional divisor of  $R$ , in *Pic*  $X$  if  $R$  is of type  $E_1$  or  $E_3$  or  $E_4$  (resp.  $E_2, E_3$ ),  $C$  is the center of the blowing-up  $f$  and  $\text{deg } C$  is  $(L \cdot C)$  if  $R$  is of  $E_1$ -type,  $\Delta$  is the discriminant locus  $\{y \in Y | X_y \text{ is}$

not smooth} of  $f$  if  $R$  is of  $C_1$ -type, where  $Y \cong \mathbb{P}^2$  by Proposition 3.7, in the case  $R$  is of type  $C_1$  or  $C_2$ , and  $X_r$  is the generic fibre of  $f$  if  $f$  is of  $D_1$ -type.

For example, in the case  $R$  is of type  $E_3$  or  $E_4$ , applying the lemma to the exceptional divisor  $D$ , we have  $(c_2(X) \cdot D) = 6 + 0 - 4 - 2 = 0$ . Since  $-K_X + D \approx rH$ ,  $(c_2(X) \cdot H)$  is equal to  $(1/r)\{(c_2(X) \cdot -K_X) + (c_2(X) \cdot D)\} = 24/r$ .

If  $R$  is of type  $C_1$ , then  $Y \cong \mathbb{P}^2$  and by the formula  $\Delta \approx -4K_{P_2} - f_*(-K_X)^2$  (Proposition 6.2), we have  $\deg \Delta < 12$ . It follows that

$$(5.6) \quad 7 \leq (c_2(X) \cdot H) \leq 17 \quad \text{if } R \text{ is of } C_1\text{-type.}$$

If  $R$  is of type  $D_1$ , then  $1 \leq (K_{X_r})^2 \leq 6$  by the classification of extremal rays. Hence we have

$$(5.7) \quad 6 \leq (c_2(X) \cdot H) \leq 11 \quad \text{if } R \text{ is of } D_1\text{-type.}$$

We consider the case in which  $X$  is primitive. By Theorem 1.6, at least one of  $R_1$  and  $R_2$  (say  $R_1$ ) is of type  $C_1$  or  $C_2$ .  $R_2$  is not  $E_1$ -type since  $X$  is primitive. Hence by (5.5), (5.6) and (5.7), we have

$$(5.8) \quad (c_2(X) \cdot H_2) \leq 24 \quad \text{or} \quad = 45.$$

Case 1.  $R_1$  is of  $C_1$ -type

i)  $\mu_2 = 1$ .  $a(-K_X) \approx H_1 + H_2$ . If  $(c_2(X) \cdot H_2) = 45$ , then we have by (5.3) and (5.6),  $51 = 7 + 45 \leq 24a \leq 17 + 45 = 62$ , which is a contradiction. Hence  $(c_2(X) \cdot H_2) \leq 24$  by (5.8) and we have  $24a = (c_2(X) \cdot H_1) + (c_2(X) \cdot H_2) \leq 17 + 24 = 41$ . It follows that  $a = 1$ .

ii)  $\mu_2 = 2$ .  $a(-K_X) \approx 2H_1 + H_2$ . If  $a$  were even, then  $H_2$  would be divisible by 2 in Pic  $X$ , which contradicts to our choice of  $H_2$ . Hence  $a$  is odd. Since  $R_2$  is of type  $E_2$  or  $C_2$  or  $D_2$  (see § 3),  $(c_2(X) \cdot H_2) \leq 24$  by (5.5). Hence we have  $24a \leq 2 \cdot 17 + 24 = 58$  by (5.3) and (5.6). It follows that  $a = 1$ .

Case 2.  $R_1$  is of  $C_2$ -type

i)  $\mu_2 = 1$ .  $a(-K_X) \approx H_1 + 2H_2$ . By the same reason as above,  $a$  is odd. By (5.3) and (5.5), we have  $12a = 3 + (c_2(X) \cdot H_2)$ . By (5.8), we have  $a = 1$ .

ii)  $\mu_2 = 2$ .  $a(-K_X) \approx 2H_1 + 2H_2$ . By (5.3),  $12a = 6 + (c_2(X) \cdot H_2)$ . In the case  $R_2$  is of  $E_2$ -type, from the equality  $12a = 6 + (24/r)$ , we obtain  $a = 1$  and  $r = 4$ . If  $R_2$  is not of  $E_2$ -type, then  $(c_2(X) \cdot H_2) \leq 6$  and we have  $a = 1$ .

In the case  $X$  is imprimitive, (1) can be proved in a way quite similar to the primitive case by Lemma 5.9 below. Since  $X$  is imprimitive,  $R_1$  or  $R_2$  is of  $E_1$ -type.

**Lemma 5.9.** *If  $R_1$  is of  $E_1$ -type, we have*

- (1)  $\deg C_1 \leq (r_1 - (\mu_2/a))^2 d_1$ , where  $C_1$  is the center of  $f_1 = \text{cont}_{R_1}$  and  $d_1 = (L_1^3)$ .
- (2)  $(c_2(X) \cdot H_1) \leq 31$ .
- (3)  $(c_2(X) \cdot H_1) \leq 25$  if  $a \leq 3$ ,  $\leq 23$  if  $a \leq 2$  and  $\leq 17$  if  $a = 1$ .
- (4)  $(c_2(X) \cdot H_1) \leq 20$  if  $a \leq 3$  and  $\mu_2 = 2$ .

*Proof.* (1) Since both  $H_1$  and  $H_2$  are numerically effective,  $(H_1 \cdot H_2^2) \geq 0$  (cf. [2]). On the other hand, by (4.2) and  $-K_X \approx r_1 H_1 - D_1$ , we have

$$\begin{aligned} 0 &\leq \mu_1^2 (H_1 \cdot H_2^2) = (H_1 \cdot (a(-K_X) - \mu_2 H_1)^2) \\ &= (H_1 \cdot ((ar_1 - \mu_2)H_1 - aD_1)^2) \\ &= (ar_1 - \mu_2)^2 (H_1^3) + a^2 (H_1 \cdot D_1^2). \end{aligned}$$

Since  $(H_1 \cdot D_1^2) = -\deg C_1$  and  $(H_1^3) = (L_1^3) = d_1$ , we have our assertion.

(2) By Proposition 4.10, the index  $r_1$  of the Fano 3-fold  $Y_1$  is greater than 1. Hence by I in § 2, the possibility of the pair  $(r_1, d_1)$  is limited to  $(4, 1)$ ,  $(3, 2)$  and  $(2, d_1)$ ,  $1 \leq d_1 \leq 5$ . Hence by (1), we have  $(c_2(X) \cdot H_1) = (24/r_1) + \deg C_1 \leq \max_{(r_1, d_1)} [(24/r_1) + (r_1 - (1/a)^2 d_1)] = 31$ , where [ ] is the Gauss symbol. This shows (2). (3) and (4) are proved in a similar way. q.e.d.

*Proof of (2) in Theorem 5.1.* Here we only prove (2) in the case  $X$  is primitive.

Case 1.  $R_1$  is of  $C_1$ -type. By (1) of Theorem 5.1,  $-K_X \approx \mu_2 H_1 + H_2$ . Since  $H_2 \cdot D_2 \approx 0$  and  $H_1^2 \approx 2l_1$ , we have  $2(l_1 \cdot D_2) = (H_1^2 \cdot D_2) = (1/\mu_2^2)((-K_X)^2 \cdot D_2) = 1$  if  $R_2$  is of type  $E_2$  or  $E_3$ , and  $= 2$  if  $R_2$  is of type  $E_3$  or  $E_4$ . Hence  $R_2$  is of type  $E_3$  or  $E_4$  and  $(l_1 \cdot D_2) = 1$ .

Case 2.  $R_1$  is of type  $C_2$ . By (1) of Theorem 5.1,  $-K_X \approx \mu_2 H_1 + 2H_2$ .  $R_2$  is not of type  $E_3$  or  $E_4$ , because  $24 = (c_2(X) \cdot H_1) + 2(c_2(X) \cdot H_2) = 6 + (48/r)$  has no integral solution. Hence  $R_2$  is of type  $E_2$  or  $E_3$  and we have  $(l_1 \cdot D_2) = (H_1^2 \cdot D_2) = (1/\mu_2^2)((-K_X)^2 \cdot D_2) = 1$ . q.e.d.

Now we classify the primitive Fano 3-folds with  $B_2 = 2$  by using Theorem 5.1. We denote by  $(*-**)$  the case in which the ray  $R_1$  is of  $*$ -type and  $R_2$  of  $**$ -type. We may assume that  $R_1$  is of  $C_1$  or  $C_2$ -type.

Case 1.  $R_2$  is also of  $C_1$  or  $C_2$ -type. Both  $f_1$  and  $f_2$  are conic bundles over  $P^2$ . Consider the morphism  $f = (f_1, f_2): X \rightarrow P^2 \times P^2$ . By the definition of  $f_*$ ,  $f$  is finite.

*Claim.*  $f_* X$  is a divisor of bidegree  $(2/\mu_2, 2/\mu_1)$ .

*Proof.* Put  $M_i = \pi_i^* \mathcal{O}_{P^1}(1)$  for  $i=1, 2$ , where  $\pi_i: P^2 \times P^2 \rightarrow P^2$  is the projection onto the  $i$ -th factor. Since  $f^* M_i \cong H_i$  and  $H_i^2 \cong (2/\mu_i)l_i$ , we have

$$\begin{aligned} (f_* X \cdot \text{point} \times \text{line}) &= (f_* X \cdot M_1^2 \cdot M_2) = (f^* M_1^2 \cdot f^* M_2) \\ &= (H_1^2 \cdot H_2) = \frac{2}{\mu_1} (l_1 \cdot H_2) = \frac{2}{\mu_1} \end{aligned}$$

by Theorem 5.1. It follows that  $f_* X \cong (2/\mu_2)M_1 + (2/\mu_1)M_2$ . q.e.d.

*Claim.* If the natural morphism  $\alpha: X \rightarrow f(X)$  is birational, then it is an isomorphism.

*Proof.* Assume that  $\alpha$  is birational. Since  $\alpha$  is finite, we have  $-K_X \sim \alpha^*(-K_{f(X)}) + \Delta$  by the residue formula, where  $\Delta$  is the zeros of the conductor ideal. Since  $f(X)$  is of bidegree  $(2/\mu_2, 2/\mu_1)$ , we have

$$\alpha^*(-K_{f(X)}) \cong f^* \left( 3 - \frac{2}{\mu_2} \right) M_1 + f^* \left( 3 - \frac{2}{\mu_1} \right) M_2 \cong \mu_2 H_1 + \mu_1 H_2.$$

Hence by Theorem 5.1,  $\Delta$  is empty, that is,  $X \rightarrow f(X)$  is an isomorphism. q.e.d.

$(C_1 - C_1)$   $f$  is generically one-to-one or two-to-one. In the former case,  $f$  is an embedding and the image is a divisor of bidegree  $(2, 2)$ . In the latter case,  $X$  is double cover of  $f(X)$ .  $f(X) \subset P^2 \times P^2$  is of bidegree  $(1, 1)$  and smooth because both  $\pi_1|_{f(X)}$  and  $\pi_2|_{f(X)}$  are equidimensional (type 2) in Theorem 1.7).

$(C_1 - C_2)$   $X$  is isomorphic to a smooth divisor on  $P^2 \times P^2$  of bidegree  $(1, 2)$  (type 5) in Theorem 1.7).

$(C_2 - C_2)$   $X$  is isomorphic to  $W_8$ , a smooth divisor on  $P^2 \times P^2$  of bidegree  $(1, 1)$  (type 6) in Theorem 1.7).

*Case 2.*  $R_2$  is of  $D_1, D_2$  or  $D_3$ -type. Consider the morphism  $f = (f_1, f_2): X \rightarrow P^2 \times P^1$ .  $f$  is finite and surjective. By Theorem 5.1, we have  $\deg f = (H_1^2 \cdot H_2) = (2/\mu_1)(l_1 \cdot H_2) = 2/\mu_1$ . Therefore  $X$  is a double cover of  $P^2 \times P^1$  or isomorphic to  $P^2 \times P^1$  (type 1), 4) and 7) in Theorem 1.7).

*Case 3.*  $R_2$  is of  $E_2, E_3, E_4$  or  $E_5$ -type. By (2) of Theorem 5.1, the degree of  $f_1|_{D_2}: D_2 \rightarrow P^2$  is equal to  $(H_1^2 \cdot D_2) = (2/\mu_1)(l_1 \cdot D_2) = 2/\mu_1 = 1$  or 2. Hence if  $D_2 \cong P^2$ , then  $\mu_1 = 2$  and  $\deg f_1|_{D_2} = 1$ , that is, if  $R_2$  is of  $E_2$  or  $E_5$ -type, then  $f_1$  is a  $P^1$ -bundle and  $D_2$  is a section of  $f_1$ . Therefore  $X$  is isomorphic to  $P(\mathcal{O} \oplus \mathcal{O}(1))$  or  $P(\mathcal{O} \oplus \mathcal{O}(2))$  according as  $R_2$  is of  $E_2$ -type or  $E_5$ -type (type 8) or 9) in Theorem 1.7).

In the case  $R_2$  is of type  $E_3$  or  $E_4$ -type, then  $\mu_1 = 1$  and  $\deg f_1|_{D_2} = 2$ . Since  $\mathcal{O}_F(-K_X - D_2)$  is trivial for every fibre  $F$  of  $f_1$ , it is isomorphic to

$f_1^* \mathcal{O}_P(n)$  for some integer  $n$ . Since  $\mathcal{O}_{D_2}(-K_X - D_2) \cong \mathcal{O}_{D_2}(2)$ ,  $n=2$ . Since  $\mathcal{O}_{D_2}(-K_X) \cong \mathcal{O}_{D_2}(1)$ ,  $f_{1,*} \mathcal{O}_{D_2}(-K_X) \cong \mathcal{O}_P \oplus \mathcal{O}_P(1)$ . Hence by the exact sequence  $0 \rightarrow \mathcal{O}_X(-K_X - D_2) \rightarrow \mathcal{O}_X(-K_X) \rightarrow \mathcal{O}_{D_2}(-K_X) \rightarrow 0$ , we have  $0 \rightarrow \mathcal{O}_P(2) \rightarrow f_{1,*} \mathcal{O}_X(-K_X) \rightarrow \mathcal{O}_P \oplus \mathcal{O}_P(1) \rightarrow 0$ . Hence  $f_{1,*} \mathcal{O}_X(-K_X) = \mathcal{O}_P \oplus \mathcal{O}_P(1) \oplus \mathcal{O}_P(2)$ . Since  $f_2$  is conic bundle,  $X$  is a divisor of  $P(\mathcal{O}_P \oplus \mathcal{O}_P(1) \oplus \mathcal{O}_P(2))$ . Since  $-K_X \sim L|_X$ ,  $X$  is linearly equivalent to  $2L$ , where  $L$  is the tautological line bundle. Let  $S$  be the section of  $P(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$  corresponding to the exact sequence  $0 \rightarrow \mathcal{O}(1) \oplus \mathcal{O}(2) \rightarrow \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \rightarrow \mathcal{O} \rightarrow 0$ . Since  $L|_S$  is trivial and  $X$  is Fano,  $X \cap S = \emptyset$ .  $P(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) - S$  is isomorphic to the line bundle  $Z = \mathcal{V}(\mathcal{O}_{V_7}((-1/2)K_{V_7}))$  over  $V_7$ . It is easy to see the restriction of  $Z \rightarrow V_7$  to  $X$  is a double covering. (type 3) in Theorem 1.7

So we have proved that every primitive Fano 3-fold with  $B_2=2$  belongs to a class in Theorem 1.7. It is easily verified that every class in Theorem 1.7 is a family of Fano 3-folds parametrized by an irreducible variety. Moreover every two different classes cannot be deformed to each other because they have different values of  $(-K_X)^3$ . Therefore the primitive Fano 3-folds with  $B_2=2$  have exactly 9 deformation types as described in Theorem 1.7.

The imprimitive Fano 3-folds with  $B_2=2$  are also classified by using Theorem 5.1. For example, we consider the case  $(E_1 - E_1)$ . In this case,  $X$  is the graph of birational map between  $Y_1$  and  $Y_2$ . Let  $C_i$  be the center of the blowing-up  $f_i$ ,  $r_i$  the index of  $Y_i$  and put  $d_i = (L_i^3)$ .

$$(5.11) \quad \begin{aligned} \deg C_1 &= (r_1 - 1)^2 d_1 - (r_2 - 1) d_2 \\ (-K_X)^3 &= (3r_1 - 2) d_1 + (3r_2 - 2) d_2. \end{aligned}$$

*Proof.*  $-K_X \sim f_i^*(-K_{Y_i}) - D_i \sim r_i H_i - D_i$  for  $i=1, 2$ . On the other hand,  $-K_X \cong H_1 + H_2$  by Theorem 5.1. Hence  $D_1 \cong (r_1 - 1)H_1 - H_2$  and  $D_2 \cong (r_2 - 1)H_2 - H_1$ . Since  $(H_i^2 \cdot D_i) = 0$  for both  $i=1, 2$ , we have  $(H_1^2 \cdot H_2) = (r_1 - 1)(H_1^3) = (r_1 - 1)d_1$  and  $(H_1 \cdot H_2^2) = (r_2 - 1)d_2$ . Therefore

$$\begin{aligned} \deg C_1 &= -(H_1 \cdot D_1^2) = -(H_1 \cdot ((r_1 - 1)H_1 - H_2)^2) \\ &= -(r_1 - 1)^2 (H_1^3) + 2(r_1 - 1)(H_1^2 \cdot H_2) - (H_1 \cdot H_2^2) \\ &= -(r_1 - 1)^2 d_1 + 2(r_1 - 1)d_1 - (r_2 - 1)d_2 \\ &= (r_1 - 1)^2 d_1 - (r_2 - 1)d_2. \end{aligned}$$

The second equality is obtained in the same way. q.e.d.

By Proposition 4.10, both  $r_1$  and  $r_2$  are greater than 1. By (5.11),

$$\deg C_1 + \deg C_2 = (r_1 - 1)(r_1 - 2)d_1 + (r_2 - 1)(r_2 - 2)d_2.$$

Hence either  $r_1$  or  $r_2$  is greater than 2. By I of § 2, we can show that Fano



3-folds with  $B_2=2$  of type  $(E_1 - E_1)$  have the following 6 deformation types:

	$(-K_X)^3$	$Y_1$	$\deg C_1$	$p_a(C_1)$	$Y_2$	$\deg C_2$	$p_a(C_2)$
1)	20	$P^3$	6	3	$P^3$	6	3
2)	24	$P^3$	5	1	$Q$	5	1
3)	26	$P^3$	5	2	$V_4$	1	0
4)	28	$Q$	4	0	$Q$	4	0
5)	30	$P^3$	4	0	$V_5$	2	0
6)	34	$Q$	3	0	$V_5$	1	0

Here  $p_a(C_i)$  can be computed by (4.3).

As a corollary of the classification, we have

**Proposition 5.12.** *An imprimitive Fano 3-fold  $X$  with  $B_2=2$  satisfies one of the following:*

(a)  *$X$  is isomorphic to the blow-up of  $P^3$  along a smooth irreducible curve which is a scheme-theoretic intersection of cubics.*

(b)  *$X$  is isomorphic to the blow-up of  $Q \subset P^4$  along a smooth irreducible curve which is a scheme-theoretic intersection of members of  $|\mathcal{O}_Q(2)|$ .*

(c)  *$X$  is isomorphic to the blow-up of  $V_d$ ,  $1 \leq d \leq 5$ , along an elliptic or rational curve which is a scheme-theoretic intersection of members of  $|-(1/2)K_{V_d}|$ .*

### § 6. Fano conic bundle

First we recall some general properties of a conic bundle  $f: X \rightarrow S$ .

**Definition 6.1.** A morphism  $f: X \rightarrow S$  from a smooth variety  $X$  onto a smooth surface  $S$  is a *conic bundle* if every fibre is isomorphic to a conic, i.e., a scheme of zeros of a nonzero homogeneous form of degree 2 on  $P^2$ . The set  $\{s \in S \mid f^{-1}(s) \text{ is not smooth}\}$  is called the *discriminant locus* of  $f$  and denoted by  $\Delta_f$ .

**Proposition 6.2.** (1)  *$f$  is flat and  $f_*\omega_X^{-1}$  is a vector bundle of rank 3 and the natural map  $X \rightarrow P(f_*\omega_X^{-1})$  is an embedding. In particular,  $X$  is projective if  $S$  is projective.*

(2) *If  $\Delta_f$  is non-empty, then it is a curve with only ordinary double points and  $\text{Sing } \Delta_f = \{s \in S \mid f^{-1}(s) \text{ is non-reduced}\}$ .*

(3) *If a smooth rational curve  $C$  is a connected component of  $\Delta_f$ , then  $f^{-1}(C)$  is reducible.*

(4)  $\Delta_f \cong -f_*K_{X/S}^2 \cong -f_*K_X^2 - 4K_S$ .

*Proof.* For (1) and (2) see Chapter I [1]. Let  $C$  be a smooth rational curve which is a connected component of  $\Delta_f$ . By (2),  $f^{-1}(s)$  is a union of

two smooth rational curves, for every  $s \in C$ . Let  $\tilde{C} \subset \text{Hilb}_X$  be the parametrizing space of those rational curves. Since  $C \cong \mathbb{P}^1$  and  $\tilde{C}$  is isomorphic to an étale double cover of  $C$ ,  $\tilde{C}$  is disconnected. Hence  $f^{-1}(C)$  is reducible. This shows (3). Let  $C$  be a smooth curve on  $S$  intersecting  $\Delta_f$  transversally and such that the surface  $f^{-1}(C)$  is smooth. It is easy to see that rational equivalence classes of such curves generate  $\text{Pic } S$ .  $Y := f^{-1}(C)$  is isomorphic to the blow-up of a  $\mathbb{P}^1$ -bundle  $Y_0$  over  $C$ . Since  $(K_{Y_0/C})^2 \sim 0$ ,  $-K_{Y_0/C}^2$  is rationally equivalent to the sum of the singular points of fibres with all coefficients 1. It follows that  $-f_*K_{Y_0/C}^2 \sim \Delta_f \cdot C$ . Since  $\omega_{Y_0/C}$  is canonically isomorphic to  $\omega_{X/S}|_Y$ , we have  $\Delta_f \cdot C \sim -f_*K_{Y_0/C}^2 \sim -f_*(K_{X/S}^2 \cdot Y) \sim -f_*(K_{X/S}^2 \cdot f^*C) \sim -(f_*K_{X/S}^2) \cdot C$ . In particular,  $(\Delta_f \cdot C) = -(f_*K_{X/S}^2 \cdot C)$ , that is,  $\Delta_f \cong -f_*K_{X/S}^2$ . On the other hand, since  $f_*K_X \sim -2S$ , we have  $-f_*K_{X/S}^2 \sim -f_*(K_X - f^*K_S)^2 \sim -f_*K_X^2 - 4K_S$ , which shows (4). q.e.d.

**Proposition 6.3.** *Let  $f: X \rightarrow S$  be a conic bundle over a projective surface  $S$ . Then we have*

(1)  $\rho(X) - \rho(S) = 1$  if and only if  $f^{-1}(C)$  is irreducible for every irreducible curve  $C$  on  $S$ .

(2) Assume that  $f^{-1}(C)$  is reducible for an irreducible curve  $C$  on  $S$ . Then i)  $C$  is smooth, ii)  $f^{-1}(C)$  is a union of  $E_1$  and  $E_2$  such that  $f|_{E_i}: E_i \rightarrow C$  is a  $\mathbb{P}^1$ -bundle for  $i=1, 2$  and iii) there are a conic bundle  $g_i: Y_i \rightarrow S$  and a morphism  $\alpha_i: X \rightarrow Y_i$  which is the contraction of all fibres of  $f|_{E_i}$ , such that  $g_i \circ \alpha_i = f$  for both  $i=1, 2$ . Moreover  $\Delta_{E_1} = \Delta_{E_2}$ ,  $\rho(Y_1) = \rho(Y_2)$ ,  $\Delta_f = \Delta_{g_i}$ ,  $\coprod C$  and  $\rho(X) = \rho(Y_i) + 1$  for  $i=1, 2$ .

By an induction on  $\rho(X) - \rho(S)$ , we have

**Corollary 6.4.**  $\Delta_f$  has  $n$  connected components  $C_1, \dots, C_n$  such that  $C_i$  is smooth and  $f^{-1}(C_i)$  is reducible for  $i=1, \dots, n$ , where  $n = \rho(X) - \rho(S) - 1$ .

**Proposition 6.5.** *Let the situation be the same as in (2) of Proposition 6.3 and assume, in addition, that  $X$  is a Fano 3-fold. Then we have*

(1) If  $Y_1$  is not a Fano 3-fold, then  $C \cong \mathbb{P}^1$ ,  $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathcal{O}_{E_1}(E_1) \cong \mathcal{O}(-1, -1)$ .

(2) Either  $Y_1$  or  $Y_2$  is a Fano 3-fold.

*Proof.* (1) is an immediate consequence of Proposition 4.5. Assume that neither  $Y_1$  nor  $Y_2$  is a Fano 3-fold. Since every fibre of  $f$  is connected  $s = E_1 \cap E_2$  is not empty. Since  $\mathcal{O}_{E_1}(E_1)$  is negative by (1),  $(s, E_1) = (s \cdot E_1|_{E_1})_{E_1}$  is negative. On the other hand, since  $s$  moves in  $E_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $(s, E_1)$  is non-negative, which is a contradiction. q.e.d.

The following is another important property of Fano conic bundles.

**Proposition 6.6.** *Let  $f: X \rightarrow S$  be a conic bundle and assume that  $X$  is a Fano 3-fold. Let  $E$  be an irreducible reduced curve on  $S$  such that  $(E^2) < 0$  and  $f^{-1}(E)$  is irreducible. Then we have*

- (1)  $E$  is an exceptional curve of the first kind.
- (2)  $f|_{f^{-1}(E)}$  is a trivial  $P^1$ -bundle over  $E$ , and
- (3) there are a Fano conic bundle  $f': X' \rightarrow S'$  and  $A: X \rightarrow X'$ , the contraction of the horizontal fibres of  $f|_{f^{-1}(E)}$  such that  $f' \circ A = \alpha \circ f$ , where  $\alpha: S \rightarrow S'$  is the blow down of  $E$ .

*Proof.* Let  $Z$  be a curve on  $X$  such that  $f(Z) = E$ . By (3.1), there are irreducible reduced curves  $C_i$ ,  $1 \leq i \leq n$ , such that  $[C_i]$  is on an edge of  $NE(X)$  and that  $Z \cong \sum_{i=1}^n a_i C_i$  for some positive numbers  $a_i$ . Since  $(f_* Z \cdot E)$  is negative,  $(f_* C_i \cdot E)$  is negative for some  $i$ . It follows that there is an irreducible reduced curve  $C$  such that  $[C]$  belongs to an extremal ray  $R$  of  $X$  and  $f(C) = E$ . Since  $(C \cdot f^{-1}(E))$  is equal to  $(f_* C \cdot E)$  and negative,  $R$  is of type  $E_1, E_2, E_3, E_4$  or  $E_5$  and  $f^{-1}(E)$  is the exceptional divisor of  $R$  by the classification of extremal rays. Since  $f^{-1}(E)$  has a morphism  $\varphi = f|_{f^{-1}(E)}$  onto  $E \cong P^1$  which does not contract the curve  $C$  belonging to the ray  $R$ , and since  $C$  and the fibre of  $\varphi$  are not numerically equivalent,  $R$  is of  $E_1$ -type. Hence  $f^{-1}(E)$  is smooth and has a  $P^1$ -bundle structure  $\psi: f^{-1}(E) \rightarrow T$  over a smooth curve  $T$  which contracts  $C$  to a point. It is easy to see that the morphism  $(\varphi, \psi): f^{-1}(E) \rightarrow E \times T$  is an isomorphism and both  $E$  and  $T$  are rational, which shows (2). By (2),  $C$  is a section of  $\varphi$  and we have  $(E^2) = (f_* C \cdot E) = (C \cdot f^{-1}(E)) = -1$  from which (1) follows. Since  $N_{f^{-1}(E)/X}$  is isomorphic to  $\varphi^* N_{E/S}$  and is not negative, the  $X'$  in  $A = \text{cont}_R: X \rightarrow X'$ , is a Fano 3-fold by Proposition 4.5. It will be clear that the morphism  $f': X' \rightarrow S$  is well defined and is a conic bundle. q.e.d.

**Corollary 6.7.** *Let  $f: X \rightarrow S$  be a Fano conic bundle. Then we have*

- (1) every irreducible curve  $E$  on  $S$  with  $(E^2) < 0$  is an exceptional curve of the first kind, and
- (2) if  $E$  is an exceptional curve of the first kind, then  $\Delta_f$  is disjoint from  $E$  or contains  $E$  as a connected component.

*Proof.* (1) follows from (1) of Proposition 6.6, (2) of Proposition 6.3 and (2) of Proposition 6.5. If  $f^{-1}(E)$  is reducible then  $\Delta_f$  contains  $E$  as a connected component by Proposition 6.3. If  $f^{-1}(E)$  is irreducible, then  $\Delta_f$  is disjoint from  $E$  by Proposition 6.6. Hence we have (2). q.e.d.

For the classification of imprimitive Fano 3-folds with  $B_2 \geq 3$ , it is necessary to classify the curves  $C$  on a Fano conic bundle  $Y$  such that the blow-up of  $Y$  along  $C$  is a Fano 3-fold. Propositions 6.8 and 6.10 give strong necessary conditions on  $C \subset Y$ .

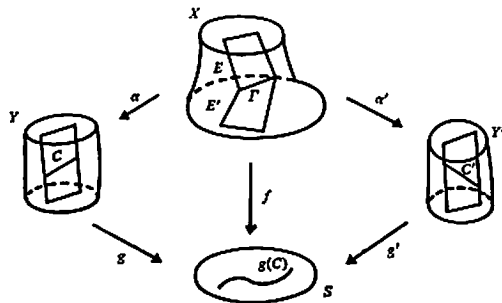
**Proposition 6.8.** *Let  $g: Y \rightarrow S$  be a Fano conic bundle and  $C$  a smooth irreducible curve on  $Y$ . Assume that the blow-up  $X$  of  $Y$  along  $C$  is a Fano 3-fold. Then we have*

(1)  $C$  does not meet any singular fibre of  $g$ ; and (2)  $C$  is either (i) a smooth fibre of  $g$  or (ii) a subsection of  $g$ , i.e.,  $g|_C$  is an embedding  $C \rightarrow S$ . In the case (i),  $X$  is a conic bundle over  $S'$ , the blow-up of  $S$  at  $g(C)$ . In the case (ii),  $f = g \circ \alpha$  is a conic bundle such that  $\Delta_f = \Delta_g \amalg g(C)$ , where  $\alpha: X \rightarrow Y$  is the blowing-up along  $C$ .

*Proof.* (1) Assume that  $C$  meets a singular fibre. Then  $C$  meets  $l$ , an irreducible component of a reducible fibre or the reduced part of a multiple fibre. In both cases,  $(-K_Y \cdot l) = 1$ . Hence if  $C \neq l$ , then  $X$  is not a Fano 3-fold by Proposition 4.7. If  $C = l$  and  $l$  is an irreducible component of a reducible fibre,  $C$  meets another component of the reducible fibre and hence  $X$  is not a Fano 3-fold. If  $C = l$  and  $l$  is the reduced part of a multiple fibre, then  $N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$  and hence  $X$  is not a Fano 3-fold by Proposition 4.9.

(2) If  $g(C)$  is a point, then by (1),  $C$  is a smooth fibre of  $g$ . Assume that  $g(C)$  is not a point and  $X$  is a Fano 3-fold. Let  $Z$  be the proper transform of a fibre  $g^{-1}(s)$ . Then we have  $(D \cdot Z) = (\alpha^*(-K_Y) + K_X \cdot Z) = (-K_Y \cdot \alpha_* Z) - (-K_X \cdot Z) < (-K_Y \cdot g^{-1}(s)) = 2$ , where  $D$  is the exceptional divisor of  $\alpha$ . Hence, for every  $s \in S$ ,  $g^{-1}(s)$  is disjoint from  $C$  or intersect  $C$  transversally at one point. Therefore  $g|_C$  is an embedding. The latter part of (2) is almost clear. q.e.d.

If  $C$  is a subsection of a conic bundle  $g: Y \rightarrow S$  such that  $g(C) \cap \Delta_g = \emptyset$  and  $X$  is the blow-up of  $Y$  along  $C$ , then there are a conic bundle  $g': Y' \rightarrow S$  (the elementary transform of  $g$  along  $C$ ) and a morphism  $\alpha': X \rightarrow Y'$  such that  $\Delta_{g'} = \Delta_g$ ,  $\rho(Y) = \rho(Y')$ , satisfying the two conditions (a)  $g \circ \alpha = g' \circ \alpha'$  and (b)  $\alpha'$  is birational and an irreducible reduced curve  $Z$  on  $X$  is contracted to a point by  $\alpha'$  if and only if  $Z$  is the proper transform of a smooth fibre of  $g$  meeting  $C$  (Proposition 6.3).



$$(6.9) \quad (-K_{Y'})^3 = (-K_Y)^3 + 2(g(C)^2)_S - 4(-K_{Y/S} \cdot C).$$

*Proof.* Applying (4.3) to  $\alpha$  and  $\alpha'$ , we have

$$\begin{aligned} (-K_X)^3 &= (-K_Y)^3 - 2\{(-K_Y \cdot C) - p_a(C) + 1\} \\ &= (-K_{Y'})^3 - 2\{(-K_{Y'} \cdot C') - p_a(C') + 1\}, \end{aligned}$$

where  $C'$  is the center of the blowing-up  $\alpha': X \rightarrow Y'$ . Since  $C \cong C' \cong g(C)$  and  $(g^*K_S \cdot C) = (g'^*K_S \cdot C')$ , we have

$$(-K_{Y'})^3 = (-K_Y)^3 - 2(-K_{Y/S} \cdot C) + 2(-K_{Y'/S} \cdot C').$$

Since  $C$  and  $C'$  are subsections,  $(-K_{Y/S} \cdot C)$  and  $(-K_{Y'/S} \cdot C')$  are equal to the self-intersection numbers  $(C^2)_{g^{-1}g(C)}$  and  $(C'^2)_{g'^{-1}g'(C')}$ , respectively. Hence our assertion follows from:

*Claim.*  $(C^2)_{g^{-1}g(C)} + (C'^2)_{g'^{-1}g'(C')} = (g(C)^2)_S.$

Let  $E$  (resp.  $E'$ ) be the exceptional divisor of  $\alpha$  (resp.  $\alpha'$ ) and  $\Gamma$  the intersection of  $E$  and  $E'$ . Since  $E'$  is the strict transform of  $g^{-1}(C)$ ,  $(C^2)_{g^{-1}g(C)}$  is equal to  $(\Gamma^2)_{E'}$ . In the same way, we have  $(C'^2)_{g'^{-1}g'(C')} = (\Gamma^2)_E$ . On the other hand, we have  $(\Gamma^2)_E + (\Gamma^2)_{E'} = (E'^2 \cdot E) + (E^2 \cdot E') = (E + E' \cdot E \cdot E') = (f^*f_*\Gamma \cdot \Gamma) = (f_*\Gamma \cdot f_*\Gamma)_S = (g(C)^2)_S$ , which shows the claim. q.e.d.

**Proposition 6.10.** *Let  $g: Y \rightarrow S$  be a Fano conic bundle and  $C$  a smooth irreducible subsection of  $g$ . If the blow-up  $X$  of  $Y$  along  $C$  is a Fano 3-fold, then the elementary transform  $g': Y' \rightarrow S$  satisfies one of the following:*

- (1)  $Y'$  is a Fano 3-fold, and
- (2)  $C \cong P^1$ ,  $g|_{g^{-1}g(C)}$  is a trivial  $P^1$ -bundle and  $(-K_{Y'/S} \cdot C) = 2\{(g(C)^2)_S + 1\}$ .

*Proof.* Applying Proposition 4.5 for  $\alpha'$ , we have  $E' \cong P^1 \times P^1$  and  $(-K_{Y'} \cdot C') = 0$  if  $Y'$  is not a Fano 3-fold. Hence  $C \cong P^1$  and  $g^{-1}g(C)$  is a trivial  $P^1$ -bundle. By the claim in the proof of (6.9),  $(-K_{Y/S} \cdot C) + (-K_{Y'/S} \cdot C') = (g(C)^2)_S$ . Hence  $(-K_{Y/S} \cdot C) = (g(C)^2)_S - (-K_{Y'/S} \cdot C') = (g(C)^2)_S - (g'^*K_S \cdot C') - (-K_{Y'} \cdot C') = (g(C)^2)_S - (K_S \cdot g(C)) = 2\{(g(C)^2)_S + 1\}$ . q.e.d.

### § 7. Classification of imprimitive Fano 3-folds with $B_2 \geq 3$

We show how to classify the imprimitive Fano 3-folds with  $B_2 \geq 3$ .

**Proposition 7.1.** *An imprimitive Fano 3-fold  $X$  with  $B_2 = 3$  satisfies one of the following:*

- (1)  $X$  is isomorphic to the blow-up of a Fano conic bundle over  $P^2$ , and  
 (2)  $X$  is isomorphic to the blow-up of  $P^3$  or  $Q$  along a disjoint union of two smooth irreducible curves on it.

*Proof.* Since  $X$  is imprimitive,  $X$  is isomorphic to the blow-up of a Fano 3-fold  $Y$  with  $B_2=2$  along a smooth irreducible curve  $C$  on  $Y$ . If  $Y$  has a conic bundle structure, then  $X$  satisfies (1). Hence by Theorem 1.6, we may assume that  $Y$  is imprimitive. By Proposition 5.12 we have the following 3 cases:

*Case in which  $Y$  is a blow-up of  $P^3$ :* Let  $\alpha: Y \rightarrow P^3$  be the blow-up. If  $C$  is disjoint from the exceptional divisor of  $\alpha$ , then  $X$  satisfies (2). If  $C$  meets the exceptional divisor, then  $C$  is an exceptional line of  $\alpha$  by Corollary 4.8.  $X$  is isomorphic to the blow-up of  $V_7$  by the proper transform of the center of  $\alpha$  by  $\beta$ , where  $\beta: V_7 \rightarrow P^3$  is the blow-up at the point  $\alpha(C)$ . Since  $V_7$  is a  $P^1$ -bundle over  $P^2$ ,  $X$  satisfies (1).

*Case in which  $Y$  is a blow-up of  $Q$ :* Let  $\alpha: Y \rightarrow Q$  be the blow-up. If  $C$  is disjoint from the exceptional divisor of  $\alpha$ , then  $X$  satisfies (2). If  $C$  is an exceptional line of  $\alpha$ , then  $X$  is a blow-up of  $\tilde{Q}$  along a curve, where  $\tilde{Q}$  is the blow up of  $Q$  at  $\alpha(C)$ . Since  $\tilde{Q}$  is isomorphic to the blow-up of  $P^3$  along a conic,  $X$  satisfies (1) or (2) by the consideration in the above case.

*Case in which  $Y$  satisfies (c) in Proposition 5.12:* We show that no blow-up of  $Y$  along a smooth irreducible curve is a Fano 3-fold. Let  $\alpha: Y \rightarrow V_d$  be the blow-up.  $Y$  has a del Pezzo fibering  $f: Y \rightarrow P^1$ , and a one-dimensional family of curves  $l$  with  $(-K_Y \cdot l) = 1$  and contained in fibres of  $f$ . Let  $S$  be the union of such  $l$ 's.

Assume that  $C \cap S \neq \emptyset$ . Then  $C$  meets an  $l$  with the above property. If  $C \neq l$ , then we have  $(-K_X \cdot l') \leq 0$  for the proper transform  $l'$  and  $X$  is not a Fano 3-fold. If  $C = l$  and  $C$  is contained in a smooth fibre of  $f$ , then  $C$  meets another  $l'$  with  $(-K_Y \cdot l') = 1$  and contained in the same fibre because the degree of the del Pezzo surface is equal to  $d$  and  $\leq 5$ . Hence  $X$  is not a Fano 3-fold. In the case  $C = l$  and  $C$  is contained in a singular fibre, we have a deformation of  $C$  whose general members  $C'$  are contained in smooth fibres. Since the blow-up  $X'$  along  $C'$  is not a Fano 3-fold,  $X$  is not a Fano 3-fold because the ampleness of  $-K_X$  is an open condition.

Assume that  $C \cap S = \emptyset$ . Since  $S$  is not contained in the exceptional divisor of  $\alpha$  and  $\rho(V_d) = 1$ ,  $\alpha(S)$  is an ample divisor of  $V_d$ . Hence  $C$  is an exceptional line of  $\alpha$ .  $X$  is isomorphic to the blow-up of  $V'_d$  by the proper transform  $Z'$  of  $Z$  by  $\beta$ , where  $\beta: V'_d \rightarrow V_d$  is the blowing up at  $\alpha(C)$ .

Since  $V_d$  is of index 2 and  $-K_{V'_d} \sim \beta^*(-K_{V_d}) - 2D$  for the exceptional divisor  $D$  of  $\beta$ ,  $-K_{V'_d}$  is divisible by 2 in  $\text{Pic } V'_d$ . By I of § 2,  $V'_d$  is not a Fano 3-fold. Since  $Z' \cong Z$  is an elliptic curve and irrational  $X$  is not a Fano 3-fold by Proposition 4.5. q.e.d.

$X$  satisfying (1) in Proposition 7.1 are classified by Proposition 4.7, 6.8, 6.9 and 6.10.  $X$  satisfying (2) are classified by Corollary 4.6 and Proposition 4.7.

**Example 7.2.** Let  $Y$  be a Fano 3-fold in 2) of Theorem 1.7. Then  $Y$  is *terminal*, i.e., the blow up of  $Y$  along no smooth irreducible curve is a Fano 3-fold.

*Proof.*  $Y$  has two conic bundle structures  $\pi$  and  $\tau$  with ample discriminant locus. By Proposition 6.8, if the blow-up  $X$  of  $Y$  along  $C$  is a Fano 3-fold, then both  $\pi(C)$  and  $\tau(C)$  would be a point. But such a curve  $C$  does not exist. Hence  $Y$  is terminal. q.e.d.

**Example 7.3.** Let  $g$  be the  $P^1$ -bundle  $Y = P(\mathcal{O} \oplus \mathcal{O}(2)) \rightarrow P^2$  and  $D$  the negative section of  $f$ . If the blow-up  $X$  of  $Y$  along a smooth irreducible curve  $C$  is a Fano 3-fold, then we have

- (1)  $C$  is a subsection disjoint from  $D$ , and
- (2)  $\text{deg } g(C) \leq 4$

*Proof.* Assume that  $X$  is a Fano 3-fold. Since  $(-K_Y \cdot l) = 1$  for every line  $l$  in  $D$ ,  $C$  is disjoint from  $D$ . Hence we have (1) by Proposition 6.8. Put  $m = \text{deg } g(C)$ . Since  $-K_{Y/P^2} \sim 2D + g^*\mathcal{O}_P(2)$ ,  $(-K_{Y/P^2} \cdot C)$  is equal to  $2m$  by (1). Since  $(-K_{Y/P^2} \cdot C) = 2m \neq 2(m^2 + 1)$ , the elementary transform  $g': Y' \rightarrow P^2$  is a Fano  $P^1$ -bundle by Proposition 6.10 and  $(-K_{Y'})^3 = (-K_Y)^3 + 2m^2 - 8m = 62 + 2m^2 - 8m$  by (6.9). On the other hand by the classification of Fano 3-folds with  $B_2 = 2$ , for a Fano  $P^1$ -bundle  $Z$  over  $P^2$  we have  $(-K_Z)^3 = 62, 56, 54, 48, 46, 38$  or  $30$  (cf. the forthcoming paper). Therefore we have  $m \leq 4$ . q.e.d.

For every  $C$  satisfying (1) and (2) of Example 7.3, the blow-up of  $Y$  along  $C$  is a Fano 3-fold and we have

$\text{deg } g(C)$	1	2	3	4
$(-K_X)^3$	50	40	32	26
$Y'$	$V_7$	$P^1 \times P^2$	$V_7$	$P(\mathcal{O} \oplus \mathcal{O}(2))$

All Fano 3-folds with  $B_2 \geq 4$  are imprimitive. They can be classified by Propositions 4.7 and 6.8.

### § 8. Proof of Theorem 1.6.

Let  $X$  be a primitive Fano 3-fold with  $B_2 \geq 2$ . By definition,  $X$  has no extremal ray of  $E_{1,\alpha}$ -type.

(8.1)  $X$  has an extremal ray of type  $C_1, C_2, D_1, D_2$  or  $D_3$ .

*Proof.* Let  $R_1, \dots, R_n$  be the extremal rays of  $X$  of type  $E_{1,\alpha}, E_2, E_3, E_4$  or  $E_5$  and  $D_1, \dots, D_n$  their exceptional divisors.

*Claim:*  $D_i \cap D_j = \emptyset$  if  $D_i \neq D_j$ .

Assume that  $s = D_i \cap D_j$  is not empty. Then  $(s \cdot D_i) = (s \cdot D_i|_{D_i})_{D_i}$  is negative because  $\mathcal{O}_{D_i}(D_i)$  is negative and  $(s \cdot D_i)$  is nonnegative because  $s$  moves in  $D_j$ , which is a contradiction. Hence  $D_i \cap D_j = \emptyset$ .

Let  $\Gamma$  be the subcone of  $NE(X)$  generated by  $R_1, \dots, R_n$ .

*Claim:*  $(Z \cdot D_i) \leq 0$  for every  $Z \in \Gamma$  and  $i$ .

We may assume that  $Z$  belongs to  $R_j$  for some  $j$ . If  $D_j = D_i$ , then  $(Z \cdot D_i) = (Z \cdot D_j) < 0$ . If  $D_j \neq D_i$ , then  $(Z \cdot D_i) = 0$  because  $D_i \cap D_j = \emptyset$ .

By the claim, we have  $\Gamma \neq NE(X)$ , e.g.  $(-K_X)^2 \notin \Gamma$ . Therefore, by (3.1),  $NE(X)$  has an extremal ray not of type  $E_{1,\alpha}, E_2, E_3, E_4$  or  $E_5$ . q.e.d.

(8.2) If  $X$  has an extremal ray  $R$  of type  $D_1, D_2$  or  $D_3$ , then  $X$  has another one of type  $C_1$  or  $C_2$ .

*Proof.* By definition of  $D$ -type,  $\rho(X) = 2$ . Let  $R'$  be another extremal ray of  $X$ .

*Claim:*  $R'$  is not of type  $E_2, E_3, E_4$  or  $E_5$ .

Assume to the contrary. By (3.4), the exceptional divisor  $D$  of  $R'$  is mapped to a point by  $\text{cont}_R: X \rightarrow P^1$ . Since every fibre of  $\text{cont}_R$  is irreducible,  $D$  is a fibre of  $\text{cont}_R$ , which contradicts (3.3).

*Claim:*  $R'$  is not of type  $D_1, D_2$  or  $D_3$ .

Assume to the contrary. Since  $R \neq R'$ , the morphism  $(\text{cont}_R, \text{cont}_{R'}): X \rightarrow P^1 \times P^1$  is surjective, which contradicts  $\rho(X) = 2$ .

By the above two claims,  $R'$  is of type  $C_1$  or  $C_2$ . q.e.d.

By (8.1) and (8.2),  $X$  has an extremal ray  $R$  of  $C_1$  or  $C_2$ -type. By definition of  $C$ -type, the conic bundle  $f = \text{cont}_R: X \rightarrow S$  satisfies

(8.3)  $\rho(X) = \rho(S) + 1$  and  $f^{-1}(C)$  is irreducible for every irreducible curve  $C$  on  $S$ .

(8.4)  $S \cong P^2$  or  $P^1 \times P^1$



*Proof.* By Proposition 3.5 and (3) of Proposition 6.6,  $S$  is a minimal rational surface. Hence  $S$  is isomorphic to  $P^2$  or  $P^1$ -bundle  $F_n = P(\mathcal{O} \oplus \mathcal{O}(n))$  over  $P^1$  ( $n \neq \pm 1$ ). By (1) of Proposition 6.6,  $S$  is isomorphic to  $P^2$  or  $F_0 \cong P^1 \times P^1$ . q.e.d.

(8.5) If  $B_2=3$ ,  $X$  has an extremal ray  $R' \neq R$  of type  $E_{1,b}$ ,  $C_1$ , or  $C_2$ . Moreover if  $R'$  is of type  $C_1$  or  $C_2$ , then  $\text{cont}_{R'}$  is a conic bundle over  $P^1 \times P^1$ .

*Proof.* Let  $R'$  be an extremal ray  $\neq R$ . It suffices to show that  $R'$  is not of type  $E_2, E_3, E_4$  or  $E_5$ . Assume to the contrary and let  $D$  be the exceptional divisor of  $R'$ . By (3.4),  $D$  is mapped to a point by  $f: X \rightarrow P^1 \times P^1$ , which is a contradiction. The latter part follows from (8.4). q.e.d.

Theorem 1.6 follows from (8.3), (8.4) and (8.5).

### § 9. Fano 3-folds with $B_2 \geq 6$

In this section, we prove Theorem 1.2. The following is the first step of the proof.

(9.1) An arbitrary Fano 3-fold  $Y$  with  $B_2=3$  satisfies one of the following:

- i)  $Y$  has a conic bundle structure.
- ii)  $Y$  is isomorphic to the blow-up of  $P^3$  along a disjoint union of a line and a conic.

*Proof (Outline).* By Propositions 7.1 and 6.8, we may assume that  $Y$  is isomorphic to the blow-up of  $V=P^3$  or  $Q$  along the disjoint union of two smooth curves  $C_1$  and  $C_2$ . Let  $Y_i$  be the blow-up of  $V$  along  $C_i$  for  $i=1, 2$ . Since  $\rho(Y_i)=2$ ,  $Y_i$  has a unique extremal ray whose contraction  $f_i$  is not the above map  $Y_i \rightarrow V$ . We may assume that  $f_i$  is not a conic bundle by Proposition 6.8. If both  $f_1$  and  $f_2$  are morphisms onto  $P^1$ , then  $Y$  has a morphism onto  $P^1 \times P^1$  and hence  $Y$  is a conic bundle over  $P^1 \times P^1$ . Hence we may assume that  $f_1$  is a birational morphism. Let  $D$  be the exceptional divisor of  $f_1$ . If  $D$  is not of  $E_2$ -type, then  $D$  is covered by curves  $Z$  with  $(-K_{Y_1} \cdot Z)=1$ . On the other hand, since the image  $\bar{D} \subset V$  is ample,  $(D \cdot C'_2) > 0$  for the proper transform  $C'_2$  of  $C_2$ . Hence  $X$  is not a Fano 3-fold by Proposition 4.7, which contradicts our assumption. Hence  $D$  is of  $E_2$ -type. Then we have  $(D \cdot C'_2)=1$ . For otherwise, there is a line in  $D$  which intersects  $C'_2$  in at least 2 points. For the proper transform  $Z$  of the line, we have  $(Z \cdot -K_X) \leq 0$  which contradicts our assumption. Hence  $V$  is isomorphic to  $P^3$ , the image  $\bar{D}$  is a plane in  $V$  and  $C'_2$  is the proper transform of a line  $C_2$  in  $V$ . Since  $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$  and  $\mathcal{O}_D(\bar{D}) \cong \mathcal{O}_D(1)$ ,  $C_1$  is a conic in  $\bar{D}$ . q.e.d.

By the following fact, we can apply the results in § 6 to prove Theorem 1.2.

(9.2) A Fano 3-fold  $X$  with  $B_2=4$  has a conic bundle structure.

*Proof.* By Theorem 1.6,  $X$  is imprimitive. Hence  $X$  is isomorphic to the blow-up of a Fano 3-fold  $Y$  with  $B_2=3$  along a smooth irreducible curve  $C$ . If  $Y$  satisfies i) of (9.1), then  $X$  has a conic bundle structure by (2) of Proposition 6.8. Assume that  $Y$  satisfies ii) of (9.1) and let  $\alpha: Y \rightarrow P^3$  the blowing up along a disjoint union of a line  $L$  and a conic  $Q$ .

*Claim:*  $C$  is an exceptional line of  $\alpha$ .

Assume to the contrary. Then by Corollary 4.8,  $C$  is disjoint from the exceptional divisor of  $\alpha$ . Let  $Z$  be a line which meets  $L$  and  $\alpha(C)$  and is contained in the plane containing  $Q$ . Since  $Z$  meets  $Q$  at two points counted with multiplicity, we have  $(-K_X \cdot Z') = (\beta^* \mathcal{O}_P(4) \cdot Z') - (\beta^{-1}(Q) \cdot Z') - (\beta^{-1}(L) \cdot Z') - (\beta^{-1}(\alpha(C)) \cdot Z') \leq 4 - 2 - 1 - 1 = 0$  for the proper transform  $Z'$  of  $Z$  by  $\beta$ , where  $\beta: X \rightarrow P^3$  is the blowing-up along  $Q \amalg L \amalg \alpha(C)$ . This contradicts our assumption that  $X$  is a Fano 3-fold.

By the claim,  $X$  is isomorphic to the blow-up of  $V_7$ , the blow-up of  $P^3$  at a point, along the proper transform of  $L \amalg Q$ . Since  $V_7$  is a  $P^1$ -bundle over  $P^2$ ,  $X$  has a conic bundle structure by Proposition 6.8. q.e.d.

We prepare the following two lemmas for the proof of (9.5), a stronger version of (9.2).

**Lemma 9.3.** *Let  $f: X \rightarrow S$  be a  $P^1$ -bundle over  $S$  such that  $S$  has a  $P^1$ -bundle structure  $\pi: S \rightarrow P^1$ . If  $f|_{f^{-1}(l)}$  is trivial for every fibre  $l$  of  $\pi$ , then  $X$  is isomorphic to  $Z \times_{P^1} S$  for a  $P^1$ -bundle  $Z$  over  $P^1$ .*

*Proof.* Let  $E$  be a vector bundle such that  $P(E) \cong X$ . By assumption, twisting  $E$  by some line bundle, we may assume that  $E|_l \cong \mathcal{O}_l^{\oplus 2}$  for every fibre of  $\pi$ . By the base change theorem, we have  $E \cong \pi^* \pi_* E$ . It follows that  $X \cong P(\pi_* E) \times_{P^1} S$ . q.e.d.

**Lemma 9.4.** *Let  $f: X \rightarrow S$  be a Fano  $P^1$ -bundle over  $S$  which has a  $P^1$ -bundle structure  $\pi: S \rightarrow P^1$ . Then one of the following holds:*

- (a)  $f^{-1}(l) \cong P^1 \times P^1$  for every fibre  $l$  of  $\pi$ , and
- (b)  $f^{-1}(l) \cong F_1$  for every fibre  $l$  of  $\pi$ .

*Proof.* The  $P^1$ -bundle  $f^{-1}(l)$  over  $l$  is a fibre of  $\pi \circ f: X \rightarrow P^1$  and hence a del Pezzo surface. Therefore  $f^{-1}(l) \cong P^1 \times P^1$  or  $F_1$ . Since  $P^1 \times P^1$  and  $F_1$  cannot be deformed to each other, we have either (a) or (b). q.e.d.

(9.5) An arbitrary Fano 3-fold with  $B_2=4$  has a conic bundle structure  $f: X \rightarrow S$  satisfying one of the following:

- (1)  $S \not\cong P^1 \times P^1$ , and
- (2)  $S \cong P^1 \times P^1$  and  $\Delta_f$  is ample.

*Proof.* By (9.2), we may assume that  $X$  has a conic bundle structure  $f: X \rightarrow P^1 \times P^1$  with  $\Delta_f$  not ample.  $\Delta_f$  is a disjoint union  $\coprod_{i=1}^n s_i \times P^1$  (or  $\coprod_{i=1}^n P^1 \times s_i$ ) for  $s_1, \dots, s_n \in P^1$ . By (3) of Proposition 6.2 and Proposition 6.3,  $X$  is isomorphic to the blow-up of a Fano  $P^1$ -bundle  $g: Y \rightarrow P^1 \times P^1$  along a disjoint union of subsections  $C_i$  over  $s_i \times P^1$ ,  $i=1, \dots, n$ . Since  $B_2(X)=4$ , we have  $n=1$ . By Lemma 9.4, we have either (a)  $g^{-1}(P^1 \times t) \cong P^1 \times P^1$  for every  $t \in P^1$  or (b)  $g^{-1}(P^1 \times t) \cong F_1$  for every  $t \in P^1$ .  $C_1$  meets  $g^{-1}(P^1 \times t)$  transversally at one point. Since  $X$  is a Fano 3-fold, the point does not lie on the exceptional curve of the first kind on  $F_1$  in the case (b). Let  $g': Y' \rightarrow P^1 \times P^1$  be the elementary transform of  $g$  with center  $C_1$ .  $g'^{-1}(P^1 \times t)$  is isomorphic to the elementary transform of  $g^{-1}(P^1 \times t)$  with center  $C_1 \cap g^{-1}(P^1 \times t)$ . Therefore in the case (b),  $g'^{-1}(P^1 \times t) \cong P^1 \times P^1$  for every  $t \in P^1$ . Therefore by Lemma 9.3,  $Y \cong P^1 \times Z$  in the case (a) and  $Y' \cong P^1 \times Z$  in the case (b) for a  $P^1$ -bundle  $Z$  over  $P^1$ . In both cases,  $X$  is isomorphic to the blow-up of  $P^1 \times Z$  along a smooth irreducible curve  $C$ .

*Case  $Z \not\cong P^1 \times P^1$ .*  $P^1 \times Z$  is a  $P^1$ -bundle over  $Z$ . Hence  $X$  is isomorphic to a conic bundle over  $Z$  or the blow-up of  $Z$  at a point. Hence  $X$  satisfies (1).

*Case  $Z \cong P^1 \times P^1$ .*  $X$  is isomorphic to the blow-up of  $P^1 \times P^1 \times P^1$  along  $C$ .  $P^1 \times P^1 \times P^1$  has three  $P^1$ -bundle structure  $\pi_1, \pi_2, \pi_3: P^1 \times P^1 \times P^1 \rightarrow P^1 \times P^1$ . It is easily seen that  $\pi_i(C)$  is a point or an ample divisor for some  $i=1, 2$  or  $3$ . By Proposition 6.8, if  $\pi_i(C)$  is a point, then  $X$  satisfies (1) and if  $\pi_i(C)$  is ample, then  $X$  satisfies (2). q.e.d.

Since a Fano 3-fold with  $B_2 \geq 5$  is imprimitive, we have by Proposition 6.8 and (9.5):

(9.6) A Fano 3-fold  $X$  with  $B_2 \geq 5$  has a conic bundle structure  $f: X \rightarrow S$  such that  $S \not\cong P^1 \times P^1$ .

Let  $f: X \rightarrow S$  be an arbitrary Fano conic bundle. By (1) of Corollary 6.7 and the classification of relatively minimal rational surface, we have

(9.7)  $S \cong P^2, F_1$  or  $P^1 \times P^1$  if  $\rho(S) \leq 2$  and

(9.8) there is a morphism  $\alpha: S \rightarrow P^1 \times P^1$  which is a blowing-up at  $\rho(S) - 2$  points on  $P^1 \times P^1$  if  $\rho(S) \geq 3$ .

**Lemma 9.9.** *If  $\rho(S) \geq 3$ , then  $f$  is a  $P^1$ -bundle.*

*Proof.* Let  $z_i = (x_i, y_i), \dots, z_n = (x_n, y_n) \in P^1 \times P^1$  be the center of  $\alpha$ . Put  $E_i = \alpha^{-1}(z_i)$  and let  $L_i$  and  $M_i$  be the proper transforms of  $x_i \times P^1$  and  $P^1 \times y_i$  by  $\alpha$ , respectively, for  $i = 1, \dots, n$ . Both  $L_i$  and  $M_i$  are exceptional curves of the first kind by (1) of Corollary 6.7.

*Claim:*  $\Delta_f$  is disjoint from  $E_i, L_i$  and  $M_i$  for every  $i$ .

Assume that  $\Delta_f$  meets  $E_i$ . Then by (2) of Corollary 6.7,  $\Delta_f$  contains  $E_i$  as a connected component. Hence  $\Delta_f$  meets  $L_i$ . But  $L_i$  is a connected component of neither  $E_i$  nor  $\Delta_f - E_i$ , which contradicts Corollary 6.7. It follows that  $\Delta_f \cap E_i = \emptyset$ . In the cases of  $L_i$  and  $M_i$ , the proof is the same.

By the claim,  $\Delta_f$  is contained in  $S - \bigcup E_i - \bigcup L_i - \bigcup M_i = P^1 \times P^1 - \bigcup x_i \times P^1 - \bigcup P^1 \times y_i$ . Since this surface is affine and  $\Delta_f$  is complete,  $\Delta_f$  is empty, which shows the lemma. q.e.d.

The following is a key to the proof of Theorem 1.2.

**Proposition 9.10.** *If  $f: X \rightarrow S$  is a Fano conic bundle and  $\rho(S) \geq 3$ , then  $f$  is a trivial  $P^1$ -bundle.*

*Proof.* By Lemma 9.9 and (3) of Proposition 6.6,  $X$  is isomorphic to  $Y \times_a S$  for a Fano  $P^1$ -bundle  $g: Y \rightarrow P^1 \times P^1$ . Since both  $f|_{f^{-1}(L_i)}$  and  $f|_{f^{-1}(M_i)}$  are trivial  $P^1$ -bundles, so are both  $g|_{g^{-1}(x_i \times P^1)}$  and  $g|_{g^{-1}(P^1 \times y_i)}$ . By Lemma 9.4,  $g|_{g^{-1}(x \times P^1)}$  is trivial for every  $x \in P^1$ . By Lemma 9.3,  $Y \cong Z \times P^1$  for a  $P^1$ -bundle  $Z$  over  $P^1$ . Since  $Z \cong g^{-1}(P^1 \times y_i) \cong P^1 \times P^1$ ,  $g$  is trivial. It follows that  $f$  is a trivial  $P^1$ -bundle. q.e.d.

*Proof of Theorem 1.2.* Let  $X$  be a Fano 3-fold with  $B_2 \geq 6$ . By (9.6),  $X$  has a conic bundle structure  $f: X \rightarrow S$  with  $S \not\cong P^1 \times P^1$ .

*Claim:*  $\rho(S) \geq 3$ .

Assume to the contrary. Then, by Corollary 6.4, the number of connected components of  $\Delta_f$  is not less than  $\rho(X) - \rho(S) - 1 \geq 3$ . Since every curve on  $P^2$  is connected,  $S \not\cong P^2$ . Since every curve on  $F_1$  disjoint from the exceptional curve of the first kind is connected,  $S \not\cong F_1$  by Corollary 6.7. This contradicts (9.7).

By the claim and Proposition 9.10,  $f$  is a trivial  $P^1$ -bundle. Hence  $X$  is isomorphic to  $P^1 \times S$ . Since  $X$  is a Fano 3-fold,  $S$  is a del Pezzo surface, which completes the proof of Theorem 1.2.

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