Classification of Fano 3-Folds with $B_2 \geq 2$, I

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§ 1. Introduction and statement of main results

The is the first part of the proof of the results announced in [8]. The proof will be completed in the forthcoming Part II.

In this paper we work over an arbitrary algebraically closed field k of characteristic 0. By Lefschetz's principle we may assume $k=\mathbf{C}$.

A nonsingular 3-dimensional projective variety X is called a *Fano 3-fold* if the anticanonical divisor $-K_X$ is ample. We refer the reader to [9] § 2 for the basic results.

Let X be a Fano 3-fold. We say that $-K_X$ has a *splitting* (resp. *free splitting*) if there are two non-zero effective divisors (resp. two non-zero base-point-free divisors) D_1 and D_2 such that $D_1+D_2 \sim -K_X$ (Definition 2.11).

The following three theorems are the main results to be proved by combining this paper and the forthcoming Part II.

Theorem 1. *There are exactly 87 classes of Fano 3-folds with $B_2 \geq 2$ up to deformations (Tables 1–5 in [8]).*

Theorem 2. *The Fano 3-folds in each class (mentioned in Theorem 1) are parametrized by an irreducible rational variety, that is, each "moduli space" is irreducible and unirational.*

Theorem 3. *If X is a Fano 3-fold with $B_2 \geq 2$, then $-K_X$ has a splitting. Furthermore, $-K_X$ has a free splitting if and only if $|-K_X|$ is base point free.*

In the forthcoming Part II, it will be shown that an arbitrary Fano 3-fold X with $B_2 \geq 2$ belongs to one of the 87 classes (Tables 2–5 in [8]). In the section 7 of this paper, the following assertions are proved:

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a) an arbitrary smooth 3-fold in each of the 87 classes is a Fano 3-fold with $B_2 \geq 2$,

b) the Fano 3-folds in each of the 87 classes are parametrized by a non-empty irreducible rational variety,

c) arbitrary two Fano 3-folds in different classes are not deformation equivalent to each other ((7.31)–(7.35)), and

d) $-K_X$ has a free splitting except for the cases (i) n°1 in Table 2 and (ii) $\mathbf{P}^1 \times S_1$ in Table 5. In these two cases, $-K_X$ has a splitting.

Theorems 1 and 2 follow from a), b) and c) modulo the forthcoming Part II. In the cases (i) and (ii) of d), it is easy to see that $|-K_X|$ has a base point. Thus Theorem 3 follow from d) and Theorem 1.

Notation. A linear system is *free from base points* if it is free from fixed components and base points. Hence the complete linear system $|L|$ associated to a line bundle L is free from base points if and only if L is generated by its global sections. The rational map associated to a linear system $|D|$ is denoted by $\Phi_{|D|}: X \to \mathbf{P}^N$ ($N = \dim |D|$). For a locally free sheaf E, $\mathbf{P}(E)$ (resp. $\mathbf{V}(E)$) is the projective bundle (resp. vector bundle) associated to E in the sense of EGA II. A divisor D (resp. a curve C) on the product

$$M = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$$

is of multi-degree $(a_1, \cdots, a_m)$ if the line bundle $\mathcal{O}_M(D)$ is isomorphic to $\otimes_{i=1}^m p_i^* \mathcal{O}_{\mathbf{P}}(a_i)$ (resp. if $(C \cdot p_i^* \mathcal{O}_{\mathbf{P}}(1)) = a_i$ for every $i = 1, \cdots, m$), where p_i is the projection of M onto the i-th factor.

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§2. Blowing up and down of Fano 3-folds

Let $f: X \to Y$ be the blowing up of a smooth projective 3-dimensional variety Y along a (non-empty) smooth irreducible curve C on Y. We will keep the meaning of these symbols in this section unless otherwise mentioned. We have an easy

Lemma 2.1. *The following assertions hold*:

(2.1.1) $-K_X \sim f^*(-K_Y) - D$ *for the exceptional divisor D of f,*

(2.1.2) $D \simeq \mathbf{P}(N^*_{C/Y})$ *and $\mathcal{O}_D(-D)$ is the tautological line bundle, where $N_{C/Y}$ is the normal bundle of C and $N^*_{C/Y}$ its dual, and*

(2.1.3) $(D^3) = -\deg(N_{C/Y}) = -(-K_Y \cdot C) + 2 - 2p_a(C)$,
$(D^2 \cdot -K_X) = 2p_a(C) - 2$, $(D \cdot (-K_X)^2) = (-K_Y \cdot C) + 2 - 2p_a(C)$,
$(-K_X)^3 = (-K_Y)^3 - 2\{(-K_Y \cdot C) - p_a(C) + 1\}$, *and*

(2.1.4) $B_2(X) = B_2(Y) + 1$, $B_3(X) = B_3(Y) + 2p_a(C)$.

Proof. (2.1.1) and (2.1.2) are well-known. Since $\mathcal{O}_D(-D)$ is the tautological line bundle of $N^*_{C/Y}$, one has

$$\begin{aligned}(D^3) &= -\deg(N_{C/Y})\\ &= (K_Y \cdot C) - \deg K_C\\ &= -(-K_Y \cdot C) + 2 - 2p_a(C)\,.\end{aligned}$$

By the same reason, one has $f_*(D^2) = -C$, whence $(D^2 \cdot f^*(-K_Y)) = -(C \cdot -K_Y)$. One also has $(D \cdot f^*(-K_Y)^2) = 0$ because $f_*D = 0$. Thus one immediately verifies equalities on $(D^2 \cdot -K_X)$, $(D \cdot (-K_X)^2)$, and $(-K_X)^3$ by (2.1.1). (2.1.4) is well-known. q.e.d.

Lemma 2.2. *Assume that X is a Fano 3-fold. Then we have*

(2.2.1) $(-K_Y \cdot C) > 2p_a(C) - 2$,

(2.2.2) *When C is rational, one has $(-K_Y \cdot C) \geq 0$, where the equality holds if and only if $N^*_{C/Y} \simeq \mathcal{O}(1)^{\oplus 2}$ ($N_{C/Y} \simeq \mathcal{O}(-1)^{\oplus 2}$) (or equivalently $D \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{O}_D(D) \simeq \mathcal{O}(-1, -1)$), and*

(2.2.3) $(-K_Y \cdot C) > p_a(C) - 1$ *and* $(-K_X)^3 > (-K_Y)^3$.

Proof. Since $-K_X$ is ample, one has $(D \cdot (-K_X)^2) > 0$. Thus (2.2.1) follows from (2.1.3).

Since $-K_X$ is ample, so is $\mathcal{O}_D(-K_X)$. Since $\mathcal{O}_D(-K_X)$ is a tautological line bundle of $f|_D \colon D \to C$, the direct image $F = (f|_D)_*\mathcal{O}_D(-K_X)$ is ample. If $C \simeq \mathbf{P}^1$, F is a sum of ample line bundles. Hence $((-K_X)^2 \cdot D) = \deg F \geq 2$ and the equality holds if and only if $F \simeq \mathcal{O}(1)^{\oplus 2}$. (2.2.2) follow from

$$\begin{aligned}F &\simeq N^*_{C/Y} \otimes \mathcal{O}_C(-K_Y)\\ &\simeq N_{C/Y} \otimes \omega_C^{-1}\end{aligned}$$

because $\deg N_{C/Y} = (-K_Y \cdot C) + 2p_a(C) - 2$. (2.2.3) follows from (2.1.3).

q.e.d.

The following tells exactly when the blow-down of a Fano 3-fold is again a Fano 3-fold.

Proposition 2.3. *If X is a Fano 3-fold, then one and only one of the following holds:*

(i) *Y is a Fano 3-fold,*

(ii) *$C \simeq \mathbf{P}^1$ and $N_{C/Y} \simeq \mathcal{O}(-1)^{\oplus 2}$ (hence $(C \cdot -K_Y) = 0$), or equivalently $D \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{O}_D(D) = \mathcal{O}_D(-1, -1)$.*

Proof. Since X is a Fano 3-fold, the cone of curves $NE(X)$ is generated by a finite number of extremal rational curves [7] (or, cf. §3). Since $f_*: NE(X) \to NE(Y)$ is surjective, $NE(Y)$ is generated by a finite number of half lines, and hence closed. Thus by Kleiman's criterion for ampleness, $-K_Y$ is ample if and only if $(-K_Y \cdot Z) > 0$ for every irreducible curves Z on Y. If $Z \neq C$, then the strict transform Z' of Z by f is not contained in D and

$$(-K_Y \cdot Z) = (f^*(-K_Y) \cdot Z') = (-K_X \cdot Z') + (D \cdot Z') > 0 .$$

Hence, if $(-K_Y \cdot C) > 0$, Y is a Fano 3-fold. Otherwise one has (ii) by (2.2.2).

q.e.d.

We give easy but useful necessary conditions for the blow-up X of a 3-fold Y to be a Fano 3-fold.

Proposition 2.4. *Assume that X is a Fano 3-fold and let E be an irreducible reduced curve on Y. Then one has*

(2.4.1) *If $(E \cdot -K_Y) = 1$, then $C = E$ or $C \cap E = \varnothing$, and*

(2.4.2) *if $(E \cdot -K_Y) = 2$, then $C = E$, $C \cap E = \varnothing$, or $C \cap E$ is one point p at which E is smooth and E and C intersect transversally.*

Proof. Assume that $(E \cdot -K_Y) \leq 2$, $C \neq E$, and $C \cap E \neq \varnothing$. Let E' be the proper transform of E by f. Then by (2.1.1), one has

$$\begin{aligned} 0 < (-K_X \cdot E') &= (f^*(-K_Y) \cdot E') - (D \cdot E') \\ &= (-K_Y \cdot E) - (D \cdot E') . \end{aligned}$$

Since $C \neq E$ and $C \cap E \neq \varnothing$, it follows that $(D \cdot E') > 0$ and hence $(-K_Y \cdot E) \geq 2$. Thus $(-K_Y \cdot E) = 2$ and $(D \cdot E') = 1$. Hence the proposition is proved.

By an *exceptional line* of $f: X \to Y$, we mean an irreducible reduced curve F on X such that $f(F)$ is a point.

Remark 2.5. *Let Z be an exceptional line of f with $p=f(Z)$.* Let $g: X'\to X$ (resp. $h: Y'\to Y$) be the blow-up of X along Z (resp. Y at p) and $C'\subset Y'$ the proper transform of C by h. Then it is well-known that X' dominates Y' and $X'\to Y'$ is the blow-up along C'.

Corollary 2.6. *Assume that X is a Fano 3-fold and that Y is the blow-up $g: Y\to Z$ of a Fano 3-fold Z along an irreducible smooth curve $F\subset Z$. Then $g(C)\cap F=\varnothing$ or C is an exceptional line of g. If $g': Y'\to Z$ is the blow-up of Z along $g(C)$, then X is the blow-up $f': X\to Y'$ along the proper transform F' of F by g'. Then one of the following holds:*

(2.6.1) *$g(C)\cap F=\varnothing$, and Y and Y' are Fano 3-folds, or*

(2.6.2) *C is an exceptional line of g, and Y is a Fano 3-fold; if Y' is not a Fano 3-fold, then $F\simeq \mathbf{P}^1$ and $N_{F/Z}\simeq \mathcal{O}\oplus\mathcal{O}$, (and hence $(F\cdot -K_Z)=2$).*

Proof. Applying (2.4.1) to an arbitrary exceptional line of g, one sees that $g(C)\cap F=\varnothing$ or $g(C)$ is a point on F. Then it is well-known that X is the blow-up of Y' along F' (Remark 2.5). If $g(C)\cap F=\varnothing$, one applies Proposition 2.3 to f and sees that Y is a Fano 3-fold because $(C\cdot -K_Y)=(g(C)\cdot -K_Z)>0$. Again by $(F'\cdot -K_{Y'})=(F\cdot -K_Z)>0$, Y' is a Fano 3-fold. This is (2.6.1). Assume that C is an exceptional line of g. Then Y is a Fano 3-fold by Proposition 2.3 because $(C\cdot -K_Y)=1$. If Y' is not a Fano 3-fold, then $F'\simeq\mathbf{P}^1$, $N_{F'/Y'}\simeq\mathcal{O}(-1)^{\oplus 2}$ by Proposition 2.3. Since Y' is the blow-up of Z at a smooth point of F, one sees

$$F\simeq F' \quad\text{and}\quad N_{F/Z}\simeq N_{F'/Y'}\otimes\mathcal{O}(1)\simeq\mathcal{O}^{\oplus 2}\,.$$

This is (2.6.2). q.e.d.

Corollary 2.7. *If Y contains $S\simeq\mathbf{P}^2$ such that $\mathcal{O}_S(S)\simeq\mathcal{O}_S(-1)$ and if X is a Fano 3-fold, then* (i) *$C\cap S=\varnothing$,* (ii) *$C\not\subset S$ and $(C\cdot S)=1$, or* (iii) *C is a line in $S\simeq\mathbf{P}^2$.*

This follows from (2.4.2) applied to an arbitrary line E in S because $(E\cdot -K_Y)=2$.

Proposition 2.8. *Assume that $C\simeq\mathbf{P}^1$ and $(C\cdot -K_Y)=1$. If X is a Fano 3-fold, then $N_{C/Y}\simeq\mathcal{O}\oplus\mathcal{O}(-1)$.*

Proof. Since $\deg N_{C/Y}=(-K_Y\cdot C)+\deg N_{C/Y}=-1$, one sees that $N_{C/Y}\simeq\mathcal{O}(n)\oplus\mathcal{O}(-1-n)$ for some $n\geq 0$.

Let S be the section of $\mathbf{P}^1$-bundle $D=\mathbf{P}(N^*_{C/Y})\to C$ corresponding to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(n+1) \longrightarrow N^*_{C/Y} \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \longrightarrow 0 .$$

Then one has

$$\begin{aligned} 0<(-K_X\cdot S) &= (f^*(-K_Y)\cdot S)-(D\cdot S) \\ &= (-K_Y\cdot C)+(\mathcal{O}_D(-D)\cdot S)_D = 1-n . \end{aligned}$$

Thus $n=0$. q.e.d.

For convenience, we state the point-blow-up version of Lemma 2.1 and Proposition 2.3.

Proposition 2.9. *Let $g: U \to V$ be the blow-up of a smooth projective 3-fold V at a point p, and let $D=g^{-1}(p)\simeq \mathbf{P}^2$. Then we have*

(2.9.1) $-K_U \sim g^*(-K_V)-2D$,

(2.9.2) $\mathcal{O}_D(D)\simeq \mathcal{O}_{\mathbf{P}}(-1)$,

(2.9.3) $(D^3)=1$, $(D^2\cdot -K_U)=-2$, $(D\cdot(-K_U)^2)=4$, $(-K_U)^3=(-K_V)^3-8$,

(2.9.4) $B_2(U)=B_2(V)+1$, $B_3(U)=B_3(V)$, *and*

(2.9.5) *If U is a Fano 3-fold, then so is V.*

Indeed (2.9.1) and (2.9.2) and well-known, (2.9.3) follows from them, and (2.9.4) is well-known. We omit the proof of (2.9.5), since it is very similar to that of Proposition 2.3.

Proposition 2.10. *Here we only assume that C is a smooth proper closed subscheme of Y. Let I_C be the sheaf of ideals of C in Y, and let L be an invertible sheaf of Y with the attached complete linear system $|L|$. Then the following are equivalent:*

(2.10.1) $H^0(L\otimes I_C)\otimes \mathcal{O}_Y \twoheadrightarrow L\otimes I_C$, *and*

(2.10.2) *$f^*L(-D)$ is generated by global sections.*

Proof. It is clear that (2.10.1) implies (2.10.2) because the natural map $f^*(L\otimes I_C)\to f^*L(-D)$ is surjective. Let us assume (2.10.2). Since $f_*(f^*L(-D))=L\otimes I_C$, there is a natural surjection

$$\alpha_p: H^0(L\otimes I_C)\otimes \mathcal{O}_{X_p} \longrightarrow\!\!\!\!\!\rightarrow f^*L(-D)\otimes \mathcal{O}_{X_p}$$

for all $p\in Y$, where $X_p=f^{-1}(p)\simeq \mathbf{P}^n$ ($n=0$, 1, or 2). Since $\dim X_p=n$, there are $n+1$ elements $s_0,\cdots,s_n\in H^0(L\otimes I_C)$ such that $\alpha_p(s_0),\cdots,\alpha_p(s_n)$ generate $f^*L(-D)\otimes \mathcal{O}_{X_p}$. Since $f^*L(-D)\otimes \mathcal{O}_{X_p}\simeq \mathcal{O}_{X_p}(1)$, $\alpha_p(s_0),\cdots,\alpha_p(s_n)$ have to generate $H^0(f^*L(-D)\otimes \mathcal{O}_{X_p})\simeq k^{(n+1)}$. Thus $H^0(\alpha_p)$ is surjective for all

$p \in Y$. Now

$$H^0(\alpha_p): H^0(L \otimes I_C) \otimes k_p \longrightarrow\!\!\!\!\rightarrow L \otimes I_C \otimes k_p,$$

whence follows (2.10.1). q.e.d.

Definition 2.11. (i) We say that C is an *intersection of members of* $|L|$ when the equivalent conditions in (2.10) are satisfied.

(ii) We say that a divisor D or its complete linear system $|D|$ has a *splitting* if there are two non-zero effective divisors D_1 and D_2 such that $D_1 + D_2 \in |D|$. The splitting is called *free* if $|D_1|$ and $|D_2|$ (given above) are free from base points.

Proposition 2.12. *Here C is only assumed to be a non-empty smooth subvariety of pure codimension r $(r=2, 3)$ of Y. If C is an intersection of members of a complete linear system $|L|$ such that $-K_Y-(r-1)L$ is ample, then X is a Fano 3-fold. If, furthermore, $|-K_Y-(r-1)L|$ is non-empty (resp. free from base points), then $-K_X$ has a splitting (resp. a free splitting).*

Proof. First of all, one has

$$-K_X \sim f^*\{-K_Y-(r-1)L\}+(r-1)(f^*L-D) \tag{2.12.1}$$

by (2.1.1) and (2.9.1). The last assertion is clear from this. Since $|f^*L-D|$ is free from base points and $-K_Y-(r-1)L$ is ample, some multiple of $-K_X$ is free from base points. It is now enough to show that $(-K_X \cdot Z)>0$ for all irreducible reduced curves Z of X. If $f(Z)$ is a point, then $(-K_X \cdot Z)=(r-1)\cdot(-D\cdot Z)>0$. If $f(Z)$ is not a point, then $(-K_X \cdot Z) \geq ((-K_Y-(r-1)L)\cdot f_*(Z))>0$. q.e.d.

Proposition 2.13. *Here C is assumed to be a non-empty disjoint union of irreducible smooth curves. Assume that Y has a structure of a $\mathbf{P}^1$-bundle $g: Y \to S$ over a smooth surface S, and that $g|_C: C \to S$ is an embedding. If there is a very ample divisor N on S such that C is an interesection of members of $|-K_X-g^*N|$, then X is a Fano 3-fold and $-K_X$ has a free splitting.*

Proof. First of all,

$$-K_X \sim \{f^*(-K_Y-g^*N)-D\}+f^*g^*N \tag{2.13.1}$$

shows that $-K_X$ has a free splitting. Let Z be an irreducible reduced curve in Y. If $f(Z)$ is a point, then $(-K_X \cdot Z)=1$. If $f(Z)$ is a fiber of $g: Y \to S$, then $(-K_X \cdot Z)=2-(D\cdot Z)\geq 1$ because $g|_C$ is an embedding. If $gf(Z)$ is not a point, then $(-K_X \cdot Z) \geq (N \cdot g_* f_* Z)>0$ by (2.13.1). q.e.d.

Proposition 2.14. *Assume that Y has a structure of a $\mathbf{P}^2$-bundle $g: Y \to \mathbf{P}^1$, and that the curve $C \subset Y$ is an intersection of members of $|-K_Y - g^*\mathcal{O}(1)|$ and $g|_C: C \to \mathbf{P}^1$ is surjective and of degree ≤ 5. If there is irredudible divisor Q of Y containing C such that the fiber Q_t over every point $t \in \mathbf{P}^1$ is a smooth conic of $Y_t \simeq \mathbf{P}^2$, then X is a Fano 3-fold and $-K_X$ has a free splitting.*

Proof. By

(2.14.1) $$-K_X \sim \{f^*(-K_Y - g^*\mathcal{O}(1)) - D\} + f^*g^*\mathcal{O}(1),$$

$-K_X$ has a free splitting. Let Z be an irreducible reduced curve on X. We will show $(-K_X \cdot Z) > 0$ in the following. If $f(Z)$ is a point, then $(-K_X \cdot Z) = 1 > 0$ as in the proof of Proposition 2.13. If $f(Z)$ is a curve not in Q and if $gf(Z)$ is a point $t \in \mathbf{P}^1$, then

$$(Z \cdot D) \leq (Z \cdot f^*Q) = (f_*Z \cdot Q)$$

because $D \subset f^*Q$ and $Z \not\subset f^*Q$. Therefore $(-K_X \cdot Z) \geq (f_*Z \cdot -K_Y - Q)$ which is the degree of f_*Z in $Y_t \simeq \mathbf{P}^2$ and is positive. If $f(Z)$ is a curve in Q and if $gf(Z)$ is a point $t \in \mathbf{P}^1$, then $f_*Z = Q_t$ and

$$(D \cdot Z) = ((D \cdot Q') \cdot Z)_{Q'} = (C \cdot f_*Z)_Q = (C \cdot Q_t)_Q = \det(g|_C)$$

because $Q \simeq$ the proper transform Q' of Q by f and $f(D \cdot Q') = C$ where $(\cdot)_{Q'}$, for example, is the intersection considered on Q. Thus

$$(-K_X \cdot Z) = (-K_{Y_t} \cdot f_*Z) - (D \cdot Z) = 6 - \deg(g|_C) > 0$$

by (2.1.1) because $f_*Z = Q_t$. If $gf(Z)$ is not a point, then

$$(-K_X \cdot Z) \geq (g_*f_*Z \cdot \mathcal{O}(1)) > 0$$

by (2.14.1) q.e.d.

Definition 2.15. We say that a morphism $g: U \to V$ is a *basic blow-up* if U and V are Fano 3-folds and g is the blow-up along a non-empty irreducible smooth curve in V. A morphism $g: U \to V$ is called a *basic morphism* or a *successive basic blow-up* if g is the composition of a finite number (≥ 1) of basic blow-ups. In this case, U is called a *successive basic blow-up* of V.

§ 3. Extremal rays of Fano 3-folds

We recall [7] for Fano 3-folds. Let X be a Fano 3-fold with $B_2 \geq 2$. Let $N_{\mathbf{Z}}(X)$ (resp. $NE_{\mathbf{Z}}(X)$) be the set of numerical equivalence classes of 1-cycles (resp. effective 1-cycles) on X. Let $N(X) = N_{\mathbf{Z}}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ and $NE(X)$ the cone in $N(X)$ generated by $NE_{\mathbf{Z}}(X)$. Theorem 1.2 [7] says that $NE(X)$ is a closed polyhedral cone generated by a finite number of extremal rays, where an *extremal ray* R of X in our case is just a half line which is an edge of the polyhedral cone generated by a finite number of extremal rays, where an a uniquely determined morphism (up to an isomorphism) to a normal projective variety such that (i) $f_*\mathcal{O}_X=\mathcal{O}_Y$ and (ii) for any irreducible reduced curve C on X, $[C]\in R$ if and only if $f(C)$ is a point (Theorem 3.1 [7]). One has $\rho(X)=\rho(Y)+1$ (Theorem 3.2 [7]).

Set $\mu(R)=\min\{(-K_X\cdot Z)\mid Z \text{ is a rational curve such that } [Z]\in R\}$ and let $l=l_R$ be a rational curve such that $[l]\in R$ and $(-K_X\cdot l)=\mu(R)$.

Proposition 3.1. *There exists an exact sequence*

$$0\longrightarrow \operatorname{Pic} Y\xrightarrow{f^*}\operatorname{Pic} X\xrightarrow{(\cdot l)}\mathbf{Z}\longrightarrow 0\,,$$

where $(\cdot l)(D)=(D\cdot l)$ *for* $D\in\operatorname{Pic} X$.

The exactness is proved by Theorem 3.2 [7] except for the surjectivity of $(\cdot l)$. In the following, R and f will be classified into several types and the surjectivity of $(\cdot l)$ will be checked in the cases where $\mu(R)>1$.

Case dim $Y=3$ (*E-type*): There exists an irreducible reduced divisor D of X such that $f|_{X-D}$ is an isomorphism an $\dim f(D)\le 1$. Such D is uniquely determined by R and is called the *exceptional divisor* of R. Moreover f is the blow-up Y by the ideal defining $f(D)$ (given the reduced structure). f and D satisfy one and exactly one of the following (Theorem 3.3 and Corollary 3.4 [7])

type of R	f and D	$\mu(R)$	l
E_1	$f(D)$ is a smooth curve, Y is smooth and $f\vert_D\colon D\to f(D)$ is a $\mathbf{P}^1$-bundle.	1	exceptional line
E_2	$f(D)$ is a point, Y is smooth, $D\simeq\mathbf{P}^2$ and $\mathcal{O}_D(D)\simeq\mathcal{O}_{\mathbf{P}}(-1)$.	2	line in $D\simeq\mathbf{P}^2$
E_3	$f(D)$ is an ordinary double point. $D\simeq\mathbf{P}^1\times\mathbf{P}^1$, $\mathcal{O}_D(D)\simeq\mathcal{O}(-1,-1)$, and $s\times\mathbf{P}^1$ and $t\times\mathbf{P}^1$ are numerically equivalent for s, $t\in\mathbf{P}^1$.	1	$s\times\mathbf{P}^1$ in D ($s\in\mathbf{P}^1$)

E_4	$f(D)$ is a double point. D is an irreducible reduced singular quadric surface in $\mathbf{P}^3$, $\mathcal{O}_D(D)\simeq\mathcal{O}_D\otimes\mathcal{O}_{\mathbf{P}}(-1)$.	1	generator of the cone D
E_5	$f(D)$ is a quadruple point of Y, $D\simeq\mathbf{P}^2$ and $\mathcal{O}_D(D)\simeq\mathcal{O}_{\mathbf{P}}(-2)$.	1	line in $D\simeq\mathbf{P}^2$

If R is of E_1-type, then (i) Y is a Fano 3-fold, or (ii) $D\simeq\mathbf{P}^1\times\mathbf{P}^1$ and $\mathcal{O}_D(D)\simeq\mathcal{O}(-1,-1)$ (Proposition 2.3). We say that R is of $E_{1,a}$-type in the case (i) and of $E_{1,b}$-type in the case (ii). If $B_2(X)=2$, then Y is always a Fano 3-fold because Y is projective and $\rho(Y)=1$.

Proposition 3.2. *If R is of $E_{1,b}$-type for a Fano 3-fold X, then the horizontal section s generates a different extremal ray of $E_{1,b}$-type.*

Proof. Let l be a fiber of $f|_D$. One has $[s]\notin R$ by the property of f. It follows from Theorem 1.2 of [7] that the cone $NE(X)$ is spanned by s, l, and a finite number of irreducible curves Z such that $Z\not\subset D$. One need to show $S=\mathbf{R}_+[s]$ is an edge of $NE(X)$. If this is not the case, there are two positive numbers a and b and an effective 1-cycle Z on X such that $Z\not\subset D$ and $as\approx bl+Z$. Since $-K_X$ is ample and $(-K_X\cdot s)=(-K_X\cdot l)=1$, one has $a=b+(-K_X\cdot Z)>b$. From $(D\cdot s)=(D\cdot l)=-1$, follows

$$-a=-b+(D\cdot Z)\geqq -b\,.$$

This is a contradiction and S is an edge. q.e.d.

If R is of E_2-type, then Y is a Fano 3-fold by (2.9.5).
Proposition 3.1 for R of E_2-type is obvious because $(D\cdot l)=-1$.

Lemma 3.3. *One has*

(3.3.1) *$\mathcal{O}_D(-D)$ and ω_D^{-1} are both ample and every curve in D moves in D, if R is not of $E_{1,a}$-type,*
and

(3.3.2) *D is mapped to a point by every morphism g from X to a curve, if R is not of E_1-type.*

Proof. (3.3.1) is obvious from the description of D. If R is of type E_2, E_4, or E_5, then D has no surjective morphism to a curve. Let R be of type E_3 and g a morphism from X to a curve. Let $S\subset D$ be a curve sent to a point by g, and T an arbitrary curve in D. Then $\mathbf{R}_+[S]=\mathbf{R}_+[T]$ and $g_*S=0$, whence $g_*T=0$. Thus $g(D)$ is a point. q.e.d.

Case dim $Y=2$ (*C-type*): Y is a smooth projective surface and $f: X \to Y$ is a conic bundle (Theorem 3.5 [7]).

Lemma 3.4. *For every irreducible reduced curve C on Y, $f^{-1}(C)$ is irreducible reduced.*

Proof. Reducedness follows from general theory of conic bundles. Let D be an irreducible component of $f^{-1}(C)$, E an arbitrary irreducible component of $f^{-1}(x)$ for any $x \in C$, and F a fiber of f such that $f(F) \notin C$. By the property of f, $\mathbf{R}_+[E] = \mathbf{R}_+[F]$ and $(D \cdot F) = 0$. Thus $(D \cdot E) = 0$. Hence $E \subset D$ or $E \cap D = \varnothing$. Since E was an arbitrary irreducible curve in any fiber $\subset f^{-1}(C)$, one has $D = f^{-1}(C)$. q.e.d.

We have the following two cases:

type of R	f	$\mu(R)$	l
C_1	f has a singular fiber	1	an irreducible component of a reducible fiber or a reduced part of a multiple fiber
C_2	f is smooth	2	fiber

Proposition 3.5. (C_2-type). *Y is rational and f is a $\mathbf{P}^1$-bundle for Zariski topology. (Thus Proposition* 3.1 *holds when R is of C_2-type.)*

Proof. Since $q(X)=0$ and f is surjective, one has $q(Y)=0$. By (4.6), we have the formula $-4K_Y \approx f_*(-K_X)^2$. (we note that proofs of (4.5) and (4.6) do not use any other results of this paper.) Thus some positive multiple of $-K_Y$ is a non-zero effective divisor and hence all the plurigenera vanish. Thus Y is rational by Castelnuovo's criterion. Then the Brauer group of Y vanishes and hence f is a $\mathbf{P}^1$-bundle for Zariski topology. q.e.d.

Case dim $Y=1$ (*D-type*): Y is a smooth curve and $\rho(X) = \rho(Y)+1 = 2$. Every fiber of f is irreducible reduced and the generic fiber X_η is a del Pezzo surface (Theorem 3.5 [7]). We call $(K_{X_\eta})^2$ the *degree* of X_η. We have the following 3 cases:

type of R	f	$\mu(R)$
D_1	X_η is a del Pezzo furface of degree d $(1 \le d \le 6)$	1

D_2	f is a quadric bundle, i.e. every fiber is isomorphic to a normal quadric surface in $\mathbf{P}^3$	2
D_3	f is a $\mathbf{P}^2$-bundle	3

Since $q(X)=0$ and f is surjective, one has $q(Y)=0$. Thus $Y \simeq \mathbf{P}^1$ when $\dim Y=1$. Proposition 3.1 for R of D_2-type and D_3-type is reduced to the existence of a $k(\eta)$-rational point on X_η. If R is of D_3-type this is obvious by Tsen's Theorem. If R is of D_2-type, X_η is a smooth quadric of $\mathbf{P}^3_\eta$. Since $k(\eta)$ is a C_1-field [5], X_η has a $k(\eta)$-rational point. Thus Proposition 3.1 is proved now.

Definition 3.6. We say that a Fano 3-fold X is *imprimitive* if one of the following equivalent conditions is satisfied

(3.6.1) X is obtained as the blow-up of a Fano 3-fold Y along an irreducible smooth curve, and

(3.6.2) X has an extremal ray of $E_{1,a}$-type.

If X is not imprimitive, we say that X is *primitive*.

Indeed, (3.6.2) implies (3.6.1) by the definition of $E_{1,a}$-type. If $f: X \to Y$ is the morphism in (3.6.1), then f is a contraction of an extremal ray R by ([7], Corollary 3.6) because $\rho(X)=\rho(Y)+1$. This R must be of $E_{1,a}$-type by the property of f.

The classification of Fano 3-folds with $B_2 \geq 2$ starts with the following fundamental.

Theorem 3.7. *If X is a primitive Fano 3-fold with $\rho(X) \geq 2$, then X has an extremal ray of type C thus X has a structure of a conic bundle.*

Proof. Let us assume that X does not have an extremal ray of $E_{1,a}$, or C-type. Then we claim that X does not have an extremal ray R of D-type. Indeed, if R is such an extremal ray, then $\rho(X)=2$ ([7], Theorem 3.5) and $NE(X)$ is spanned by two extremal rays. Let R' be the other extremal ray. Let $l'=l_{R'}$ and $f=\mathrm{cont}_R: X \to \mathbf{P}^1$. Then $f(l')=\mathbf{P}^1$ because $[l'] \notin R$. If R' is of E-type, R' has to be of E_2, E_3, E_4, or E_5-type (cf. the remark preceding Proposition 3.2), and the associated exceptional divisor D' dominates $\mathbf{P}^1$ by f because $f(l')=\mathbf{P}^1$ and $l' \subset D'$. Thus contradicts (3.3.2), whence R' has to be of D-type. Let $f'=\mathrm{cont}_{R'}: X \to \mathbf{P}^1$. Then $g=(f, f'): X \to \mathbf{P}^1 \times \mathbf{P}^1$ has the property: if Z is an irreducible curve on X such that $g(Z)$ is a point, then $[Z] \in R \cap R' = \{0\}$. This implies that g is finite which is absurd. Thus X has no extremal rays of D-type and our claim is proved. Hence extremal rays of X

are $E_{1,b}$, E_2, $\cdots$, or E_5-type. Let $R_1, \cdots, R_n$ be all the extremal rays of X, and let $l_1, \cdots, l_n$ and $D_1, \cdots, D_n$ be the rational curves and the exceptional divisors associated with them. We claim that

$$(3.7.1) \qquad D_i \cap D_j = \emptyset \quad \text{if} \quad D_i \neq D_j .$$

Let $S = D_i \cdot D_j$ assuming $D_i \neq D_j$ and $D_i \cap D_j \neq \emptyset$. Then $(S \cdot D_i) \geq 0$ because S moves in D_j (3.3.1), and $(S \cdot D_i) = (S \cdot \mathcal{O}_{D_i}(D_i)) < 0$ because $\mathcal{O}_{D_i}(-D_i)$ is ample (3.3.1). This is a contradiction and one gets (3.7.1). Let $C = (-K_X)^2$. Then $C \in NE(X)$ and $C \approx \sum a_i l_i$ for $a_i \geq 0$, whence

$$(C \cdot D_j) = \sum_i a_i (l_i \cdot D_j) = (l_j \cdot D_j) \sum_{D_i = D_j} a_i \leq 0$$

for all j by (3.7.1). On the other hand, $(C \cdot D_j) > 0$ by ampleness of $-K_X$. This is a contradiction. Thus X has an extremal ray of C-type if X is primitive.

q.e.d.

We will need a description of the cone of curves for a cyclic branched covering.

Theorem 3.8. *Let $g: U \to V$ be an n-sheeted cyclic branched covering between non-singular projective varieties over k of characteristic 0 with branch locus $B \subset V$. Then*

(3.8.1) $\chi_{top}(U) = n\chi_{top}(V) - (n-1)\chi_{top}(B)$,

(3.8.2) $K_U \sim f^* K_V + (n-1)R$ *and* $nR = g^* B$, *and*

(3.8.3) *if* $\dim V \geq 3$ *and B is ample on V, then* $g^*: \mathrm{Pic}\, V \xrightarrow{\sim} \mathrm{Pic}\, U$ *and* $g^*: \overline{NE}(V) \xrightarrow{\sim} \overline{NE}(U)$.

Proof. (3.8.1) and (3.8.2) are well-known. Let us consider (3.8.3). If $\dim V \geq 3$ and R (resp. B) is ample on U (resp. V), one knows that the natural maps $\mathrm{Pic}\, U \to \mathrm{Pic}\, R$ and $\mathrm{Pic}\, V \to \mathrm{Pic}\, B \simeq \mathrm{Pic}\, R$ are injective and have torsion-free cokernels [2]. Thus $g^*: \mathrm{Pic}\, V \to \mathrm{Pic}\, U$ is injective and the cokernel is torsion-free, and the Galois group $\mathbf{Z}_n$ of g acts trivially on $\mathrm{Pic}\, U$ because $\mathrm{Pic}\, U \subset \mathrm{Pic}\, B$. Thus $nL \in g^* \mathrm{Pic}\, V$ for all $L \in \mathrm{Pic}\, U$, and one sees $g^*: \mathrm{Pic}\, V \xrightarrow{\sim} \mathrm{Pic}\, U$. Thus $g^*: N(V) \xrightarrow{\sim} N(U)$. The last assertion follows from [7, (1.9)].

§4. Fano conic bundle

First we recall some general properties of a conic bundle $f: X \to S$.

Definition 4.1. A morphism $f: X \to S$ from a smooth variety X onto a smooth surface S is a *conic bundle* if every fiber is isomorphic to a conic, i.e., a scheme of zeroes of a non-zero homogeneous form of degree 2 on $\mathbf{P}^2$. The set $\{s \in S \mid f^{-1}(s) \text{ is not smooth}\}$ is called the *discriminant locus* of f and denoted by Δ_f.

Proposition 4.2. *Let $f: X \to S$ be a conic bundle and $\omega_{X/S}$ the relative dualizing sheaf of f. Then we have*

(1) *f is flat, $f_*(\omega_{X/S}^{-1})$ is a vector bundle of rank 3 and the natural map $X \to \mathbf{P}(f_*(\omega_{X/S}^{-1}))$ is an embedding. In particular, X is projective if S is projective.*

(2) *If Δ_f is non-empty, then it is a curve with only ordinary double points and* Sing Δ_f *coincides with the set $\{s \in S \mid f^{-1}(s) \text{ is non-reduced}\}$.*

(3) *Let s be a singular point of Δ_f and u_1 and u_2 the local equations of the two branches of Δ_f at s respectively. Then u_1 and u_2 form a regular system of parameters of S at s, and the completion of $\mathcal{O}_{S,s}$ is $D = k[[u_1, u_2]]$ and the base change $f_D: X_D \to \operatorname{Spec} D$ of f by $\operatorname{Spec} D \to S$ is isomorphic to* $\operatorname{Proj}(D[U_1, U_2, W]/(u_1U_1^2 + u_2U_2^2 - W^2))$, $wt\, U_1 = wt\, U_2 = wt\, W = 1$.

For the proof see [1] Chapter I. The proof works in the general case though it was proven there only in the case $S = \mathbf{P}^2$.

If $f: X \to S$ is a conic bundle, then f is equidimensional and $\omega_{X/S}^{-1}$ is relatively ample. The converse is also true:

Proposition 4.3. *Let $f: X \to S$ be an equidimensional morphism from a smooth 3-fold X onto a smooth projective surface S. If ω_X^{-1} is f-ample then f is a conic bundle.*

Proof. We prove $h^0(\mathcal{O}_C) = 1$ for every fibre C of f. Then the proposition follows from the lemma below which is a characterization of conics. Since a generic fibre is $\mathbf{P}^1$ and f is flat, $\chi(\mathcal{O}_C) = 1$ for every fibre of f. Hence for the proof of $h^0(\mathcal{O}_C) = 1$ it suffices to show that $h^1(\mathcal{O}_C) = 0$ for every fiber C of f, that is, $R^1f_*\mathcal{O}_X = 0$. Let M be a line bundle on S. By Leray's spectral sequence,

$$H^0(S, M \otimes R^1f_*\mathcal{O}_X) \cong H^0(S, R^1f_*f^*M)$$

is a quotient of $H^1(X, f^*M)$ if $H^2(S, M) = 0$. Since ω_X^{-1} is f-ample, $\omega_X^{-1} \otimes f^*M$ is ample if M is sufficiently ample. Therefore by Kodaira's vanishing theorem, $H^1(X, f^*M)$ is zero and hence so is $H^0(S, M \otimes R^1f_*\mathcal{O}_X)$. This implies $R^1f_*\mathcal{O}_X = 0$.

Lemma 4.4. *Let C be a locally Gorenstein complete scheme of pure dimension 1 over k such that ω_C^{-1} is ample and $h^0(\mathcal{O}_C) = 1$. Then C is isomorphic to a conic of $\mathbf{P}^2$, as a scheme.*

Proof. By the Serre duality and the Riemann-Roch formula, we have

$$\chi(\mathcal{O}_C) = -\chi(\omega_C) = -\chi(\mathcal{O}_C) - \deg\omega_C .$$

Hence $-\deg\omega_C = 2\chi(\mathcal{O}_C) = 2\{1 - h^1(\mathcal{O}_C)\}$. Since ω_C^{-1} is ample, one sees that $h^1(\mathcal{O}_C) = 0$ and $\deg\omega_C^{-1} = 2$. We claim that an arbitrary irreducible reduced curve Z contained in C is isomorphic to $\mathbf{P}^1$. Indeed, if I is the sheaf of ideals of $\mathcal{O}_C$ defining Z, then the exact sequence $0 \to I \to \mathcal{O}_C \to \mathcal{O}_Z \to 0$ shows $H^1(\mathcal{O}_Z) = 0$ because $H^1(\mathcal{O}_C) = H^2(I) = 0$. Thus $Z \cong \mathbf{P}^1$ and the claim is proved. Hence from the ampleness of ω_C^{-1} and $\deg\omega_C^{-1} = 2$, it follows that C is (i) isomorphic to $\mathbf{P}^1$, (ii) $C_1 \cup C_2$ where C_1 and C_2 are distinct curves isomorphic to $\mathbf{P}^1$, or (iii) $2C_0$ as a cycle, where $C_0 \cong C_{\mathrm{red}} \cong \mathbf{P}^1$. In case (i), we are done. In case (ii), from the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_{C_1 \cap C_2} \longrightarrow 0 ,$$

we have $\chi(\mathcal{O}_{C_1 \cap C_2}) = \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2}) - \chi(\mathcal{O}_C) = 1$. Hence C_1 and C_2 intersect at one point and transversally. In case (iii), let I be the sheaf of defining ideals of C_0 in C. Then I is an invertible sheaf on C_0 because $C = 2C_0$ as a cycle and $\mathcal{O}_C$ has no embedded components. Since $\chi(I) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_0}) = 0$, we have $I \cong \mathcal{O}_{C_0}(-1)$. The obstruction to extend id: $C_0 \to C_0$ to a morphism $C \to C_0$ lies in $H^1(T_{C_0} \otimes \mathcal{O}_{C_0}(-1))$. But since

$$H^1(T_{C_0} \otimes \mathcal{O}_{C_0}(-1)) = H^1(\mathcal{O}_{C_0}(1)) = 0 ,$$

C has a structure of a C_0-scheme and $\mathcal{O}_C$ as $\mathcal{O}_{C_0}$-algebra is $\mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(-1)$ with $(\mathcal{O}_{C_0}(-1))^2 = 0$. This shows that C is exactly a double line in $\mathbf{P}^2$.

Now we investigate some properties of Δ_f as a divisor on S.

Proposition 4.5. *Let $f: X \to S$ be a conic bundle and $K_{X/S}$ the relative canonical class of f. Then we have*

$$\Delta_f \equiv -f_*(K_{X/S}^2) ,$$

where $K_{X/S} = K_X - f^*K_S$.

Proof. Let C be a smooth curve on S intersecting Δ_f transversally and such that the surface $Y = f^{-1}(C)$ is smooth. It is easy to see that linear equivalence classes of such curves generate $\operatorname{Pic} S$. Hence it suffices to show $(\Delta_f \cdot C) = (-f_*(K_{X/S})^2 \cdot C)$ for every such curves C. Since Y is isomorphic to the blow up of a $\mathbf{P}^1$-bundle, $-(K_{Y/C})^2$ is rationally equivalent to the sum of the singular points of fibers with all coefficients 1. It follows that $-f_*(K_{Y/C})^2 \sim \Delta_f \cdot C$. Since $\omega_{Y/C}$ is canonically isomorphic to $\omega_{X/S}|_Y$, we have

$$\Delta_f \cdot C \sim -f_*(K_{Y/C})^2 \sim -f_*(K_{X/S}^2 \cdot Y) \sim -f_*(K_{X/S}^2 \cdot f^*C) \sim -(f_*K_{X/S}^2) \cdot C .$$

In particular, $(\Delta_f \cdot C) = -(f_* K^2_{X/S} \cdot C)$, that is, $\Delta_f \equiv -f_*(K_{X/S})^2$. q.e.d.

Since $K_{X/S} = K_X - f^* K_S$ and $f_* K_X \sim -2S$, we have

$$f_*(K_{X/S})^2 \sim f_*(K_X - f^* K_S)^2 \sim f_* K_X^2 + 4K_S .$$

Hence we have by the proposition

Corollary 4.6. $-4K_S \equiv f_*(K_X)^2 + \Delta_f$.

Proposition 4.7. *Let $f: X \to S$ be a conic bundle.*

(1) *If a smooth rational curve C is a connected component of Δ_f, then $f^{-1}(C)$ is reducible.*

(2) *Let C be an irreducible component of Δ_f and $\bar{C}$ the closure in X of the set of singular points of $f^{-1}(s)$, $s \in C_{reg}$. Then we have*

$$(K_{X/S} \cdot \bar{C}) = n + \frac{1}{2}(C \cdot \Delta_f - C),$$

where n is the number of ordinary double points of C.

Remark. We use the notation of (3) of Proposition 4.2 and make the identification $X_D = \mathrm{Proj}(D[U_1, U_2, W]/(u_1 U_1^2 + u_2 U_2^2 - W^2))$, where $D = k[[u_1, u_2]]$, and denote the origin of Spec D again by s. Then Δ_f is the union of Δ_1 and Δ_2 and $f_D^{-1}(\Delta_f)$ is the union of $F_1 = f^{-1}(\Delta_1)$ and $F_2 = f^{-1}(\Delta_2)$, where Δ_i is defined by $u_i = 0$ in Spec D $(i = 1, 2)$. Let $\bar{\Delta}_i$ be the closure of $\{\mathrm{Sing} f_D^{-1}(t) \mid t \in \Delta_i - \{s\}\}$ in X_D. Then we have

a) $\bar{\Delta}_i$ is defined by $u_i = U_{3-i} = W = 0$. In particular, $\bar{\Delta}_i$ intersects with F_{3-i} only at t_i (defined by $u_1 = u_2 = U_{3-i} = W = 0$) and the intersection is transversal, and

b) let $\pi_i: \tilde{X}_D \to X_D$ be the blow up along $\bar{\Delta}_i$, T_i the strict transform of F_i, and E_i the exceptional divisior of π_i. Then $E_i \cap T_i$ is a double cover $\bar{\Delta}_i$ and ramifies exactly at t_i.

Proof. Let C be a smooth rational curve which is a connected component of Δ_f. By (2) of Proposition 4.2, $f^{-1}(s)$ is a union of two smooth rational curves for every $s \in C$. Let $\tilde{C} \subset \mathrm{Hilb}\, X$ be the parametrizing space of those rational curves. Since $C \cong \mathbf{P}^1$ and $\tilde{C}$ is an etale double cover of C, $\tilde{C}$ is disconnected. Hence $f^{-1}(C)$ is reducible, which shows (1).

Let C be an irreducible component of Δ_f and $p_1, \cdots, p_n$ the ordinary double points of C. By a) of the previous remark, $\bar{C}$ is smooth and meets $f^{-1}(p_i)$ transversally at two distinct points, say q_i and r_i, for $i = 1, \cdots, n$ and $f|_{\bar{C}}$ is just the normalization of C. Let $\pi: \tilde{X} \to X$ be the blow-up of X along $\bar{C}$,

E the exceptional divisor $\pi^{-1}(\bar{C})$ and T the strict transform of $f^{-1}(C)$. We show (2) by calculating the arithmetic genus of $D=T\cap E$ in two different ways:

i) It is easy to see that $\pi|_D\colon D\to\bar{C}$ is an etale double cover over $(f|_{\bar{C}})^{-1}(C_{\mathrm{reg}})$. By the previous remark, D contains the exceptional lines $\pi^{-1}(q_i)$ and $\pi^{-1}(r_i)$ for every $i=1,\cdots,n$,

$$\bar{D}=D-\sum_{i=1}^{n}(\pi^{-1}(q_i)+\pi^{-1}(r_i))$$

is smooth and the morphism $\pi|_{\bar{D}}\colon\bar{D}\to\bar{C}$ branches at q_i, r_i $(i=1,\cdots,n)$ and $\bar{C}\cap f^{-1}$ (closure of (Δ_f-C)). Hence by the Hurwitz formula we have

(4.7.1) $$p_a(D)=p_a(\bar{D})+2n$$
$$=2p_a(\bar{C})-1+n+\frac{1}{2}(\bar{C}\cdot f^*(\Delta_f-C))+2n$$
$$=2p_a(C)-1+n+\frac{1}{2}(C\cdot\Delta_f-C).$$

ii) By the adjunction formula, $\omega_D\cong\mathcal{O}_D(K_{\bar{X}}+T+E)$. Since $K_{\bar{X}}\sim\pi^*K_X+E$ and $T\sim\pi^*f^*C-2E$, we have

$$\omega_D\cong\mathcal{O}_D(\pi^*K_X+\pi^*f^*C)\cong(\pi|_D)^*\mathcal{O}_{\bar{C}}(K_X+f^*C).$$

Since $\pi|_D$ is of degree 2, we have

$$2p_a(D)-2=\deg\omega_D$$
$$=2(K_X+f^*C\cdot\bar{C})$$
$$=2(K_{X/S}+f^*(K_S+C)\cdot\bar{C})$$
$$=2\{(K_{X/S}\cdot\bar{C})+(K_S+C\cdot f_*\bar{C})\}$$
$$=2\{(K_{X/S}\cdot\bar{C})+2p_a(C)-2\}.$$

It follows that

(4.7.2) $$p_a(D)=2p_a(C)-1+(K_{X/S}\cdot\bar{C}).$$

(2) of the proposition follows inmediately from (4.7.1) and (4.7.2).

q.e.d.

Next we investigate some properties of a conic bundle $f\colon X\to S$ such that $f^{-1}(C)$ is reducible for an irreducible curve C on S.

Proposition 4.8. *Let $f: X \to S$ be a conic bundle over a projective surface S and Z a curve in a fibre. Then the followings are equivalent:*

(1) *$f^{-1}(C)$ is irreducible for every irreducible curve C on S,*

(2) *$\rho(X) = \rho(S) + 1$, and*

(3) *$A = \mathbf{R}_+[Z]$ is an extremal ray and $f = \mathrm{cont}_A$.*

Proof. (1) $\Rightarrow$ (2) Let l be the generic fibre of f. Assuming (1) we show that every divisor D with $(D \cdot l) = 0$ is linearly equivalent to f^*C for a divisor C on S. It is clear that (2) follows from this. In the case D is effective and irreducible, $(D \cdot l) = 0$ means that $D \cap l = \varnothing$, that is, $C = f(D)$ is a curve. Hence by our assumption $D = f^*C$. Therefore if D is effective and $(D \cdot l) = 0$, then $D = f^*C$ for an effective divisor C on S. In the case D is general, we consider the direct image $f_*\mathcal{O}_X(D)$. If $(D \cdot l) = 0$, then the sheaf $f_*\mathcal{O}_X(D)$ is non-zero. Hence for a sufficiently ample divisor A on S, we have $H^0(X, \mathcal{O}_X(D + f^*A)) \cong H^0(S, (f_*\mathcal{O}_X(D)) \otimes A) \neq 0$, that is, $|D + f^*A|$ is not empty. Hence by what we have shown, $D + f^*A$ is linearly equivalent to f^*C for a divisor C on S. This completes our proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) See [7] Corollary 3.6.

(3) $\Rightarrow$ (1) We have proved it in Lemma 3.4. q.e.d.

By the proposition, there is an irreducible curve C such that $f^{-1}(C)$ is reducible if $\rho(X) > \rho(S) + 1$.

Proposition 4.9. *Let $f: X \to S$ be a conic bundle over a projective surface S and C an irreducible curve on S such that $f^{-1}(C)$ is reducible. Then we have*

(1) *C is a smooth connected component of Δ_f,*

(2) *$f^{-1}(C)$ is a union of effective divisors E_1 and E_2 such that $f|_{E_i}: E_i \to C$ is a $\mathbf{P}^1$-bundle for $i = 1, 2$,*

(3) *there are a conic bundle $g_i: Y_i \to S$ and a morphism $\alpha_i: X \to Y_i$ which is the contraction of all fibres of $f|_{E_i}$, such that $g_i \circ \alpha_i = f$ for both $i = 1, 2$,*

(4) *$\Delta_f = \Delta_{g_i} \coprod C$ and $\rho(X) = \rho(Y_i) + 1$ for $i = 1, 2$,*

(5) *$\Delta_{g_1} = \Delta_{g_2}$ and $\rho(Y_1) = \rho(Y_2)$, and*

(6) *$(-K_{Y_1/S} \cdot C_1) + (-K_{Y_2/S} \cdot C_2) = (C^2)_S$, where $C_i = \alpha_i(E_i)$ for $i = 1, 2$.*

Proof. Since every fibre of f is a conic, $f^{-1}(C) = E_1 + E_2$ for irreducible divisors E_1 and E_2. Let l_i be a generic fibre of $f|_{E_i}: E_i \to C$, $i = 1, 2$. Then $l_i \cong \mathbf{P}^1$ by (2) of Proposition 4.2 and $(l_1 \cdot E_2) = (l_2 \cdot E_1) = 1$. Since $(l_1 + l_2 \cdot E_i) = 0$, $(l_i \cdot E_i) = -1$ for $i = 1, 2$. If m is a double line of $f|_{f^{-1}(C)}$, then $m \equiv r_i l_i$ for some $r_i \in \mathbf{Q}_+$ and $i = 1, 2$. Hence $f|_{f^{-1}(C)}$ has no double line (otherwise l_1 and l_2 are numerically equivalent). Therefore C is a smooth connected component of Δ_f by (2) of Proposition 4.2 and $f|_{E_i}$ is a $\mathbf{P}^1$-bundle for $i = 1, 2$ because every fibre of $f|_{E_i}$ is isomorphic to $\mathbf{P}^1$, which shows (1) and (2). Since $f|_{E_i}$ is a $\mathbf{P}^1$-bundle and $(E_i \cdot l_i) = -1$, there is a contraction $\alpha_i: X \to Y_i$ (in the category of algebraic spaces) of all fibres of $f|_{E_i}$. Then Y_i is projective by (1) of Proposition 4.2. It will be clear that there is a morphism $g_i: Y_i \to S$ such that $g_i \circ \alpha_i = f$ and that g_i is a conic bundle which satisfies (4) and (5). It remains to prove (6). Put $\bar{C} = E_1 \cap E_2$. By (2) of Proposition 4.7 and (1), we have $(K_{X/S} \cdot \bar{C}) = 0$. On the other hand we have

$$(4.9.1) \qquad 2K_{X/S} \sim \alpha_1^* K_{Y_1/S} + \alpha_2^* K_{Y_2/S} + f^* C$$

because $K_{X/S} \sim \alpha_i^* K_{Y_i/S} + E_i$ for $i = 1, 2$. Since $\alpha_{i*}\bar{C} = C_i$ for $i = 1, 2$, (6) follows immediately from (4.9.1). q.e.d.

By an induction on $\rho(X) - \rho(S)$, we obtain the following from Propositions 4.8 and 4.9:

Proposition 4.10. *Let $f: X \to S$ be a conic bundle over a projective surface S and $n = \rho(X) - \rho(S) - 1$. Then there exist n distinct smooth irreducible curves $C_1, \cdots, C_n$ on S such that $f^{-1}(C_i)$ are reducible. $C_1, \cdots, C_n$ are connected components of Δ_f and $f^{-1}(C)$ is irreducible for every irreducible curve C other than $C_1, \cdots, C_n$.*

The elementary transformation of $\mathbf{P}^1$-bundles are generalized to conic bundles.

Definition 4.11. Let $g: Y \to S$ be a conic bundle. A curve C on Y is a *subsection* if $g|_C$ is an embedding. A subsection is *regular* if C is smooth and does not meet any singular fiber of g.

Let C be a regular subsection of a conic bundle $g: Y \to S$ and $\alpha: X \to Y$ be the blowing up along C. Then $f = \alpha \circ g: X \to S$ is a conic bundle. Since $g^{-1}(g(C))$ is reducible, we have by Proposition 4.9 that there are a conic bundle $g': Y' \to S$ and a morphism $\alpha': X \to Y'$ satisfying the two conditions (a) $g \circ \alpha = g' \circ \alpha'$ and (b) α' is birational and an irreducible reduced curve Z on X is contracted to a point by α' if and only if Z is a strict transform of a smooth fibre of g meeting C. This conic bundle $g': Y' \to S$ is called *the elementary transform of g with center C.*

Proposition 4.12. *Let the situation be as above. Then we have*

(1) $\Delta_{g'} = \Delta_g$ *and* $\rho(Y') = \rho(Y)$, *and*

(2) $(-K_{Y'})^3 = (-K_Y)^3 + 2(g(C)^2)_S - 4(-K_{Y/S} \cdot C)$.

Proof. (1) will be obvious. Applying (2.1.3) for α and α', we have

$$(-K_X)^3=(-K_Y)^3-2\{(-K_Y\cdot C)-p_a(C)+1\}$$
$$=(-K_{Y'})^3-2\{(-K_{Y'}\cdot C')-p_a(C')+1\}\ ,$$

where C' is the center of the blowing up α' : $X\to Y'$. Since $C\cong C'\cong g(C)$ and $(g^*K_S\cdot C)=(g'^*K_S\cdot C')$, we have

$$(-K_{Y'})^3=(-K_Y)^3-2(-K_{Y/S}\cdot C)+2(-K_{Y'/S}\cdot C')\ .$$

Hence (2) follows from (6) of Proposition 4.9. q.e.d.

We compute the Euler-Poincaré characteristic $\chi_{\text{top}}(X)$ and the Betti numbers $B_i(X)$ of a conic bundle f: $X\to S$.

(4.13) a) $\chi_{\text{top}}(X)=2\{\chi_{\text{top}}(S)-p_a(\Delta_f)+1\}$,

b) $B_1(X)=B_1(S)$, *and*

c) $B_3(X)=2\{B_1(S)+(B_2(X)-B_2(S))+p_a(\Delta_f)-2\}$.

(*We understand* $p_a(\Delta_f)=1$ *if* Δ_f *is empty*.)

Proof. a) S is the disjoint union of $S-\Delta_f$, $(\Delta_f)_{\text{reg}}$ and $(\Delta_f)_{\text{sing}}$. Hence we have

$$\chi_{\text{top}}(X)=\chi_{\text{top}}(f^{-1}(S-\Delta_f))+\chi_{\text{top}}(f^{-1}((\Delta_f)_{\text{reg}}))+\chi_{\text{top}}(f^{-1}((\Delta_f)_{\text{sing}}))$$
$$=2\chi_{\text{top}}(S-\Delta_f)+3\chi_{\text{top}}((\Delta_f)_{\text{reg}})+2\chi_{\text{top}}((\Delta_f)_{\text{sing}})$$
$$=2\chi_{\text{top}}(S)+\chi_{\text{top}}((\Delta_f)_{\text{reg}})$$
$$=2\{\chi_{\text{top}}(S)-p_a(\Delta_f)+1\}\ .$$

b) Since f is a conic bundle, $R^1f_*\mathcal{O}_X=0$. Hence we have $B_1(X)=2q(X)=2q(S)=B_1(S)$.

c) c) follows immediately from a) and b). q.e.d.

Next we study some properties of Fano conic bundles.

Definition 4.14. A conic bundle (resp. $\mathbf{P}^1$-bundle) $f: X\to S$ is a *Fano conic bundle* (resp. *Fano* $\mathbf{P}^1$*-bundle*) if X is a Fano 3-fold, i.e., $-K_X$ is ample.

The following is an immediate consequence of Proposition 4.3.

Proposition 4.15. *Let* f : $X\to S$ *be a morphism from a Fano* 3*-fold onto a smooth surface* S. *If* f *is equidimensional, then* f *is a Fano conic bundle.*

Proposition 4.16. *If* f: $X\to S$ *is a Fano conic bundle, then* S *is a del Pezzo surface.*

Proof. We claim that some multiple of $-K_S$ is numerically equivalent to an effective divisor and that $(-K_S \cdot C) > 0$ for every irreducible curve C on S. The proposition will follow from this and Nakai's criterion. By Corollary 4.6, we have

$$4(-K_S) \equiv f_*(-K_X)^2 + \Delta_f . \tag{4.16.1}$$

Since $-K_X$ is ample, $|m(-K_X)|$ is very ample for a positive integer m. Let D_1 and D_2 be two distinct general members of $|m(-K_X)|$. Then by (4.16.1), $4m^2(-K_S)$ is numerically equivalent to the effective divisor $f_*(D_1 \cdot D_2) + m^2 \Delta_f$, which shows the first half of our claim. Let C be an irreducible curve on S. If C is not contained in Δ_f, then by (4.16.1), we have

$$\begin{aligned} 4(-K_S \cdot C) &= (f_*(-K_X)^2 \cdot C) + (\Delta_f \cdot C) \\ &\geqq ((-K_X)^2 \cdot f^* C) . \end{aligned}$$

If C is an irreducible component of Δ_f, then by Proposition 4.7, we have

$$\begin{aligned} (-K_S \cdot C) &= (f^*(-K_S) \cdot \bar{C}) \\ &= (-K_X \cdot \bar{C}) + (K_{X/S} \cdot \bar{C}) \\ &\geqq (-K_X \cdot \bar{C}) . \end{aligned}$$

Since $-K_X$ is ample, $(-K_S \cdot C)$ is positive in both cases. This shows the second half of our claim. q.e.d.

Proposition 4.17. *Let the situation be the same as in Proposition* 4.9 *and assume, in addition, that* X is a *Fano 3-fold. Then we have*

(1) *If* Y_1 *is not a Fano 3-fold, then* $C \cong \mathbf{P}^1$, $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ *and* $\mathcal{O}_{E_1}(E_1) \cong \mathcal{O}(-1, -1)$.

(2) *Either* Y_1 *or* Y_2 *is a Fano 3-fold.*

Proof. (1) is an immediate consequence of (2.3).

Assume that neither Y_1 nor Y_2 is a Fano 3-fold. Since every fibre of f is connected, $s = E_1 \cap E_2$ is not empty. Since $\mathcal{O}_{E_1}(E_1)$ is negative by (1), we have that $(s \cdot E_1) = (s \cdot E_1 |_{E_1})_{E_1}$ is negative. On the other hand $(s \cdot E_1)$ is non-negative because s can move in $E_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$. This contradiction shows (2). q.e.d.

The following is another important property of Fano conic bundles:

Proposition 4.18. *Let* $f: X \to S$ *be a Fano conic bundle and* E *an exceptional curve of the* 1*st kind on* S *such that* $f^{-1}(E)$ *is irreducible. Then we have*

(1) $f|_{f^{-1}(E)}$ *is a* $\mathbf{P}^1$*-bundle over E, and*

(2) *there are Fano conic bundle* $f': X' \to S'$ *and a morphism* $A: X \to X'$ *which is the contraction of the horizontal sections of* $f|_{f^{-1}(E)}$ *(cf. (1)) such that* $f = f' \times_{S'} S$, *where* $\alpha: S \to S'$ *is the blow down of E.*

Proof. Let Z be a curve on X *such that* $f(Z) = E$. *By* [7, (1.2)], there are extremal rational curves C_i such that $Z \approx \sum_{i=1}^{n} a_i C_i$ for some positive rational numbers a_i, $i = 1, \cdots, n$. Since $(f_* Z \cdot E)$ is negative, $(f_* C_i \cdot E)$ is negative for some i. It follows that there is an extremal rational curve C belonging to an extremal ray R such that $f(C) = E$. Since $(C \cdot f^{-1}(E))$ is negative, R is of type E_1, E_2, E_3, E_4 or E_5 and $f^1(E)$ is the exceptional divisor of R by the classification of extremal rays. Since $f^{-1}(E)$ has a morphism φ onto E, R is not of type E_2, E_4 or E_5. Moreover, since fibres of the morphism φ are not numerically equivalent to C, R is not of type E_3, either. It follows that R is of type E_1. Hence $f^{-1}(E)$ has a $\mathbf{P}^1$-bundle structure $\psi: f^{-1}(E) \to T$ over a smooth curve T which contracts C to a point. It is easy to see that the morphism $(\varphi, \psi): f^{-1}(E) \to E \times T$ is an isomorphism and both E and T are rational, which shows (1). Since $N_{f^{-1}(E)/X}$ is isomorphic to $\varphi^* N_{E/S}$ and is not negative, the contracted variety X' in $A = \mathrm{cont}_R: X \to X'$ is a Fano 3-fold by (2.3). It is clear that there is a morphism $f': X' \to S'$ with $f' \circ A = \alpha \circ f$, f' is a conic bundle and $f = f' \times_{S'} S$. q.e.d.

Corollary 4.19. *Let* $f: X \to S$ *be a Fano conic bundle. If E is an exceptional curve of the first kind on S, then E is disjoint from* Δ_f *or a connected component of* Δ_f.

Proof. If $f^{-1}(E)$ is irreducible, E is disjoint from Δ_f by (1) of Proposition 4.18. If $f^{-1}(E)$ is reducible, E is a connected component of Δ_f by (1) of Proposition 4.9. q.e.d.

The goal for the moment is to prove the following:

Theorem 4.20. *If* $f: X \to S$ *is a Fano conic bundle and* $S \not\cong \mathbf{P}^2$, $\mathbf{F}_1$ *or* $\mathbf{P}^1 \times \mathbf{P}^1$, *then f is trivial, i.e.,* $X \cong \mathbf{P}^1 \times S$ *and f is the projection to the second factor.*

The following is the first step for the proof of the theorem:

Lemma 4.20.1. *If* $f: X \to S$ *is a Fano conic bundle and* $S \not\cong \mathbf{P}^2$, $\mathbf{F}_1$ *or* $\mathbf{P}^1 \times \mathbf{P}^1$, *then f is a* $\mathbf{P}^1$*-bundle.*

Proof. By our assumption and Proposition 4.16, there is a morphism $\alpha: S \to \mathbf{P}^1 \times \mathbf{P}^1$ which is a blowing up at n points $x_1, \cdots, x_n$, where $n =$

$\rho(S)-2\geqq 1$. Put $E_i=\alpha^{-1}(x_i)$, $x_i=(y_i, z_i)$ and let L_i and M_i be the strict transforms of $y_i\times\mathbf{P}^1$ and $\mathbf{P}^1\times z_i$ by α, respectively, for $i=1,\cdots,n$. Then E_i, L_i and M_i are exceptional curves of the first kind.

Claim: Δ_f is disjoint from E_i, L_i and M_i for every i.

Assume that Δ_f meet E_i. Then by Corollary 4.19, Δ_f contains E_i as a connected component. Hence Δ_f meets L_i. But L_i is a connected component of neither E_i nor Δ_f-E_i, which contradicts Corollary 4.19. It follows that Δ_f is disjoint from E_i. In the cases of L_i and M_i the proof is same.

By the claim, Δ_f is contained in

$$S-\bigcup E_i-\bigcup L_i-\bigcup M_i\cong\mathbf{P}^1\times\mathbf{P}^1-\bigcup y_i\times\mathbf{P}^1-\bigcup\mathbf{P}^1\times z_i\,.$$

Since this surface is affine and Δ_f is complete, Δ_f is empty, which shows the lemma. q.e.d.

So the proof of the theorem is reduced to the case f is a $\mathbf{P}^1$-bundle.

Lemma 4.20.2. *Let $\pi: T\to\mathbf{P}^1$ be a $\mathbf{P}^1$-bundle and $g: Y\to T$ a $\mathbf{P}^1$-bundle over T. Then we have*

(1) *If $g|_{g^{-1}(C)}$ is trivial for every fibre C of π, then $g=g_0\times_{\mathbf{P}^1}T$ for a $\mathbf{P}^1$-bundle $g_0: Y_0\to\mathbf{P}^1$ over $\mathbf{P}^1$.*

(2) *If Y is a Fano 3-fold, then one of the following holds*:

a) *$g^{-1}(C)\cong\mathbf{P}^1\times\mathbf{P}^1$ for every fibre C of π, or*

b) *$g^{-1}(C)\cong\mathbf{F}_1$ for every fibre C of π.*

Proof. (1) Since T is rational, the Brauer group of T is zero. Hence Y is isomorphic to $\mathbf{P}(E)$ for a vector bundle E of rank 2 on T. We consider the natural homomorphism $\varphi: \pi^*\pi_*E\to E$. By the base change theorem and by our assumption, π_*E is a vector bundle of rank 2 and φ is an isomorphism on every fibre of π. Hence φ is an isomorphism and we have $\mathbf{P}(E)\cong\mathbf{P}(\pi_*E)\times_{\mathbf{P}^1}T$.

(2) The $\mathbf{P}^1$-bundle $g^{-1}(C)$ is a fibre of $\pi\circ g: Y\to\mathbf{P}^1$ and hence a del Pezzo surface. Hence $g^{-1}(C)$ is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$ or $\mathbf{F}_1$. Since $\mathbf{P}^1\times\mathbf{P}^1$ and $\mathbf{F}_1$ cannot be deformed to each other, we have either a) or b). q.e.d.

(4.20.3) *Proof of the theorem*: We may assume that f is a $\mathbf{P}^1$-bundle by Lemma 4.20.1 Let α, y_i, z_i, L_i and M_i be as in the proof of the lemma. By using (2) of Proposition 4.18 repeatedly, there is a Fano $\mathbf{P}^1$-bundle $g: Y\to\mathbf{P}^1\times\mathbf{P}^1$ such that $f=g\times_{(\mathbf{P}^1\times\mathbf{P}^1)}\alpha$. Since $f|_{f^{-1}(L_1)}$ and $f|_{f^{-1}(M_1)}$ are trivial $\mathbf{P}^1$-bundles by (1) of Proposition 4.18, so are $g|_{g^{-1}(y_1\times\mathbf{P}^1)}$ and $g|_{g^{-1}(\mathbf{P}^1\times z_1)}$. Hence $g|_{g^{-1}(y\times\mathbf{P}^1)}$ is trivial for every $y\in\mathbf{P}^1$ by (2) of Lemma

4.20.2 and there is a $\mathbf{P}^1$-bundle $g_0: Y_0 \to \mathbf{P}^1$ such that $g = g_0 \times_{\mathbf{P}^1} (\mathbf{P}^1 \times \mathbf{P}^1)$ by (1) of Lemma 4.20.2 Since

$$Y_0 \cong g^{-1}(\mathbf{P}^1 \times z_1) \cong \mathbf{P}^1 \times \mathbf{P}^1 ,$$

g_0 is trivial. It follows that g and f are trivial $\mathbf{P}^1$-bundles. q.e.d.

For the classification of imprimitive Fano 3-folds (cf. §3) with $B_2 \geq 3$, it is necessary to classify the curves C on a Fano conic bundle Y such that the blow-up of Y along C is a Fano 3-fold. Propositions 4.22 and 4.23 give strong necessary conditions on $C \subset Y$.

Proposition 4.21. *Let $g: Y \to S$ be a conic bundle and C a smooth irreducible curve on Y. Assume that the blow-up X of Y along C is a Fano 3-fold. Then we have*

(1) *C does not meet any singular fibre of g:*

and

(2) *C is either* (i) *a smooth fibre of g or* (ii) *a regular subsection of g* (*Definition* 4.11). *In the case* (i) *X is a conic bundle over S', the blow-up of S at $g(C)$. In the case* (ii) *$f = g \circ \alpha$ is a conic bundle such that $\Delta_f = \Delta_g \coprod g(C)$, where $\alpha: X \to Y$ is the blowing up along C.*

Proof. (1) Assume that C meets a singular fibre. Then C meets Z, an irreducible component of a reducible fibre or a reduced part of a multiple fibre. In both cases, $(-K_X \cdot Z) = 1$. Hence if $C \neq Z$, then X is not a Fano 3-fold by (2.4.1). If $C = Z$ and Z is an irreducible component of a reducible fibre, C meets another component of the reducible fibre and hence X is not a Fano 3-fold. If $C = Z$ and Z is a reduced part of a multiple fibre, then $N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$ [7, (3.25)] and hence X is not a Fano 3-fold by (2.8).

(2) If $g(G)$ is a point, then C is a smooth fibre by (1). Assume that $g(C)$ is not a point and that X is a Fano 3-fold. Since $(g^{-1}(s) \cdot -K_X) = 2$, C is disjoint from $g^{-1}(s)$ or meet C transversally at one point (2.4.2). Therefore $g|_C$ is an embedding and by (1), C is a regular subsection. The latter half of (2) is almost clear. q.e.d.

Corollary 4.22. *Let $g: Y \to S$ be a Fano conic bundle and $A: X \to Y$ be a successive basic blow-up. Then X has a conic bundle structure $f: X \to S'$ such that $g \circ A = \alpha \circ f$, where $\alpha: S' \to S$ is a blowing up of S at a finite set of points.*

Corollary 4.23. *Let $g: Y \to S$ be a conic bundle and C a smooth irreducible curve on Y. If every curve on S meets the discriminant locus Δ_g of g and if the blow-up X of Y along C is a Fano 3-fold, then C is a smooth fibre of g.*

Theorem 4.20 and Corollary 4.23 give us the following criteria.

Proposition 4.24. *Let $g: Y \to S$ be a non-trivial Fano conic bundle over $S = \mathbf{F}_1$ or $\mathbf{P}^1 \times \mathbf{P}^1$. If every curve meets Δ_g, then no blow-up of Y along irreducible smooth curve is a Fano 3-fold.*

Proof. Let C be a smooth irreducible curve on Y and X the blow-up of Y along C. If C is not a smooth fibre of g, then X is not a Fano 3-fold by Corollary 4.23. If C is a smooth fibre, then X has the conic bundle structure $f = g \times_S S': X \to S'$, where S' is the blow-up of S at the point $g(C)$. Since g is not trivial, f is not trivial either. Hence by Theorem 4.20, X is not a Fano 3-fold. Therefore the blow-up of Y is not a Fano 3-fold for any C. q.e.d.

Proposition 4.25. *Let $g: Y \to S$ be a Fano conic bundle and C an irreducible regular subsection of g. If the blow-up X of Y along C is a Fano 3-fold, then the elementary transform $g': Y' \to S$ satisfies one of the following:*

(1) *Y' is also a Fano 3-fold, or*

(2) $C \cong \mathbf{P}^1$, *$g|_{g^{-1}(g(C))}$ is a trivial $\mathbf{P}^1$-bundle and* $(-K_{Y/S} \cdot C) = 2\{(g(C)^2)_S + 1\}$.

Proof. Let $\alpha': X \to Y'$ be the contraction of the strict transforms of the fibres of g meeting C, E' the exceptional divisor of α' and $C' = \alpha'(E')$ (Proposition 4.12). Assume that Y' is not a Fano 3-fold, then $C' \cong \mathbf{P}^1$, the $\mathbf{P}^1$-bundle $\alpha'|_{E'}: E' \to C'$ is trivial and $(-K_{Y'} \cdot C') = 0$ by (4.17). Since $C \cong C'$ and $g|_{g^{-1}(g(C))} \cong \alpha'|_{E'}$, we have that $C \cong \mathbf{P}^1$ and $g|_{g^{-1}(g(C))}$ is a trivial $\mathbf{P}^1$-bundle. The last equality of (2) follows from (6) of Proposition 4.9:

$$\begin{aligned}(-K_{Y/S} \cdot C) &= (g(C)^2)_S - (-K_{Y'/S} \cdot C') \\ &= (g(C)^2)_S - (g'^* K_S \cdot C') - (-K_{Y'} \cdot C') \\ &= (g(C)^2)_S - (K_S \cdot g(C)) \\ &= 2\{(g(C)^2)_S + 1\}\end{aligned}$$

because $(g(C)^2) + (g(C) \cdot K_S) = -2$ by $g(C) \cong \mathbf{P}^1$. q.e.d.

Proposition 4.26. *Let E be a rank 2 vector bundle on a surface S and L the tautological line bundle of the $\mathbf{P}^1$-bundle $\pi: X = \mathbf{P}(E) \to S$. Then we have*

(1) $\omega_X = L^{-2} \otimes \pi^*(\omega_S \otimes \det E)$, *and*

(2) *assume that* i) *E is generated by its global sections and* ii) $\omega_S^{-1} \otimes (\det E)^{-1}$ *is ample and generated by its global sections. Then X is a Fano 3-fold and $-K_X$ has a free splitting.*

Proof. (1) *Let* $\Omega_{X/S}$ be the sheaf of differentials of X over S. Then assertion follows from the two natural exact sequences

$$0 \longrightarrow \pi^*\Omega_S \longrightarrow \Omega_X \longrightarrow \Omega_{X/S} \longrightarrow 0$$

and

$$0 \longrightarrow \Omega_{X/S} \longrightarrow L^{-1} \otimes \pi^* E \xrightarrow{L^{-1}\otimes\alpha} \mathcal{O}_X \longrightarrow 0,$$

where $\alpha: \pi^*E \to L$ is the homomorphism induced from the natural isomorphism $\pi_* L \cong E$.

(2) By (1), we have

$$(*) \quad -K_X \sim 2L + \pi^*(-K_S - \det E).$$

By the assumption, both $|L|$ and $|\pi^*(-K_S - \det E)|$ are free from base points. Hence (*) is a free splitting of $-K_X$ and $|-K_X|$ is free from base points. Let Z be a curve on X. If Z is a fibre of π, then $(-K_X \cdot Z) = 2$. Otherwise, we have

$$(-K_X \cdot Z) \geqq (\pi^*(-K_S - \det E) \cdot Z) = (-K_S - \det E \cdot \pi_* Z) > 0,$$

because $|L|$ is base point free and $-K_S - \det E$ is ample. Therefore, $-K_X$ is ample and X is a Fano 3-fold. q.e.d.

§ 5. Some comments on Fano 3-folds with $B_2 = 1$ and index $= 1$.

In this section, we shall show that a Fano 3-fold with $B_2 = 1$ and index $= 1$ is terminal in the following sense.

Definition 5.1. A Fano 3-fold X is *terminal* if X satisfies the following equivalent conditions:

1) the blow-up $B_C(X)$ of X is not a Fano 3-fold for any smooth irreducible curve C,

2) the blow-up $B_C(X)$ of X is not a Fano 3-fold for any smooth curve, C, and

3) if a morphism $f: Y \to X$ from a Fano 3-fold Y is a composition of blow-ups along a smooth curve, then f is an isomorphism.

The equivalence of these three conditions follows from Corollary 2.6.

Theorem 5.2. *An arbitrary Fano 3-fold X with $B_2 = 1$ and index $= 1$ is terminal.*

We recall the following results (A) and (B) in [4] and [11] on Fano 3-folds X with $B_2 = 1$ and index $= 1$, which are essential for our proof of the theorem:

(A) if $(-K_X)^3 \geqq 8$, then the anticanonical linear system $|-K_X|$ is very ample and the image of $\Phi_{|-K_X|}$ is a (scheme-theoretic) intersection of quadrics [4], and

(B) under the situation of (A), the image of $\Phi_{|-K_X|}$ contains a line [11].

(B) is equivalent to saying that X contains a smooth rational curve l such that $(-K_X \cdot l)=1$.

Let C be a smooth irreducible curve on X. In the sequal we shall show that the blow-up $B_C(X)$ is not a Fano 3-fold. We have the following 4 cases:

a) $(-K_X)^3 \leq 6$,

b) $(-K_X)^3 \geq 8$ and $(-K_X \cdot C) > 1$,

c) $(-K_X)^3 \geq 8$, $(-K_X \cdot C)=1$, $C \simeq \mathbf{P}^1$ and $N_{C/X} \not\simeq \mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-1)$, and

d) $(-K_X)^3 \geq 8$, $(-K_X \cdot C)=1$, $C \simeq \mathbf{P}^1$ and $N_{C/X} \simeq \mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-1)$.

In the case c), the blow-up Y of X along C is not a Fano 3-fold by Proposition 2.8. In the case a), we have $(-K_Y)^3 < (-K_X)^3 = 6$ by Lemma 2.2 and hence $(-K_Y)^3 \leq 4$. By [4], if Y is a Fano 3-fold with $(-K_Y)^3 = 2$ (resp. 4), then Y is isomorphic to a double cover of $\mathbf{P}^3$ (resp. a smooth quartic hypersurface in $\mathbf{P}^4$ or a double cover of a smooth quadric hypersurface in $\mathbf{P}^4$) and, in particular, $B_2(Y)=1$ by Theorem 3.8. In our case $B_2(Y)=2$, hence Y is not a Fano 3-fold. In the case b), by virtue of (B), there exists a curve $l \simeq \mathbf{P}^1$ with $(-K_X \cdot l)=1$. Since $h^0(N_{l/X}) - h^1(N_{l/X}) > 0$, l can move in X. Let S be the union of all deformations of l. Then $\dim S \geq 2$ and C intersects with S because $B_2(X)=1$. Hence C intersects with a deformation l' of l. Since $(-K_X \cdot l') = (-K_X \cdot l) = 1$ and since $-K_X$ is ample, l' is irreducible. Therefore, the blow-up Y of X along C is not a Fano 3-fold by Proposition 2.4. In the remaining case d), Y is not a Fano 3-fold by Proposition 2.4 and the proposition below, because the ampleness of $-K_X$ is an open condition under the deformation of X.

Proposition 5.3. *Let X be a Fano 3-fold with very ample $-K_X$ and C a smooth rational curve with $N_{C/X} \simeq \mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-1)$. Let T_0 be an irreducible component of* $\mathrm{Hilb}_{X/k}$ *which contains* $[C]$. *Let S be the union of all deformations C_t, $t \in T_0$ and assume that $(S \cdot C) \geq 0$. Then, if $t \in T_0$ is general, C_t intersects with another $C_{t'}$ $(\neq C_t)$, $t' \in T_0$.*

Proof. Since $h^0(N_{C/X})=1$ and $h^1(N_{C/X})=0$, T_0 is smooth and 1-dimensional at the point $[C]$. Let $T \to T_0$ be the normalization of T_0 and put $Z = U \times_{T_0} T \subset X \times T$, where $U \subset X \times \mathrm{Hilb}_{X/k}$ is the universal closed subscheme. Denote by $\pi: Z \to X$ (resp. $\psi: Z \to T$) the restriction of the projection of $X \times T$ onto the 1st (resp. 2nd) factor. For every $t \in T$, we denot $Z \cap X \times t$ (or $\psi^{-1}(t)$) by Z_t and $\pi(Z_t)$ by C_t.

Claim: For every $t \in T$, C_t is isomorphic to $\mathbf{P}^1$.

Let C_t° be the 1-cycle associated to C_t. Then we have $(C_t^\circ \cdot -K_X) = (C \cdot -K_X) = 1$. Since $-K_X$ is very ample, C_t° is isomorphic to $\mathbf{P}^1$. Since ψ is flat, we have $\chi(\mathcal{O}_{C_t}) = \chi(\mathcal{O}_C) = 1 = \chi(\mathcal{O}_{C_t^\circ})$. Hence the natural surjection $\alpha: \mathcal{O}_{C_t} \to \mathcal{O}_{C_t^\circ}$ is an isomorphism because the support of Ker α is 0-dimensional. Thus the claim is proved.

By the claim, ψ is a $\mathbf{P}^1$-bundle. We have the following 3 cases for the morphism $\pi: Z \to X$:

case 1) $\deg \pi > 1$,

case 2) $\deg \pi = 1$ and π is not finite, and

case 3) π is finite and birational onto the image $S = \pi(Z)$.

In the case 1), n distinct C_t's pass through the generic point of $\pi(Z)$, where $n = \deg \pi > 1$. In the case 2), every C_t passes through the fundamental points of $\pi^{-1}: \pi(Z) \cdots \to Z$. In both cases, our proposition is clear. Hence we consider the case 3). Let $\Delta \subset Z$ be the closed subscheme of Z defined by the conductor ideal of π. Let f be the generic fiber of ψ. By the adjunction and the residue formulae, we have

$$\pi^*(\omega_X(S)) \simeq \pi^* \omega_S \simeq \omega_Z(\Delta)$$

and hence

$$(C \cdot S) - 1 = (\pi(f) \cdot S + K_X) = (f \cdot K_Z + \Delta) = -2 + (f \cdot \Delta)\,.$$

By our assumption, we have $(f \cdot \Delta) = (C \cdot S) + 1 > 0$. Hence $f \cap \Delta$ is not empty. Let $z \in f \cap \Delta$. Then S is not normal at the point $\pi(z)$. On the other hand π is unramified near f, because $N_{\pi(f)/X} \simeq \mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-1)$ by our assumption and the natural map

$$H^0(\pi(f), N_{\pi(f)/X}) \longrightarrow N_{\pi(f)/X} \otimes k(p)$$

is an isomorphism for every $p \in \pi(f)$. Therefore, there exists a point z' such that $z \neq z'$ and $\pi(z) = \pi(z')$. Since the restriction $\pi|_f : f \to X$ is an embedding, z' does not lie on f. Hence $C_{\psi(z')}$ is different from $\pi(f)$ and intersects with $\pi(f)$ which is a generalization of C. q.e.d.

Now the proof of Theorem 5.2 is finished, but it heavily depends on the existence of lines. Let us comment that it is used only in case b) and that the proof of the existence of lines in [11] needs an extract argument using a very precise projective geometry in the case $(-K_X)^3 \leq 12$. So we give an alternative proof of case b) under the assumption $(-K_X)^3 \leq 12$.

Proposition 5.4. *Let X be a Fano 3-fold with $B_2=1$, index $=1$ and $(-K_X)^3=8$, 10, 12 and C a smooth curve on X with $(-K_X\cdot C)>1$. Then the blow-up Y of X along C is not a Fano 3-fold.*

Proof. Assume that Y is a Fano 3-fold.

Claim: $H^1(C, \mathcal{O}_C(-K_X))=0$.

By (2.1.1), $H^2(X, \mathcal{O}_X(-K_X)\otimes I_C)$ is isomorphic to $H^2(Y, \mathcal{O}_Y(-K_Y))$, where I_C is the defining ideal of C. Since X and Y are Fano 3-folds, we have

$$H^2(X, \mathcal{O}_X(-K_X)\otimes I_C)=H^1(X, \mathcal{O}_X(-K_X))=0$$

by Kodaira's vanishing and Serre's duality. Hence the claim follows from the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X)\otimes I_C \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_C(-K_X) \longrightarrow 0 .$$

By (2.1.3) and the Riemann-Roch theorem, we have the formula

(5.4.1) $$(-K_Y)^3=(-K_X)^3-2\chi(\mathcal{O}_C(-K_Y)) .$$

Since $(-K_X\cdot C)>1$ and $-K_X$ is very ample, $h^0(\mathcal{O}_C(-K_X))\geq 3$. Hence by the claim and by our assumption, we have $(-K_Y)^3\leq 12-6=6$.

Claim: $|-K_Y|$ has no base points.

By [4], if $|-K_Y|$ has base points, then Y is isomorphic to $\mathbf{P}^1\times S_1$ or the blow-up $\tilde{V}_1$ of V_1 along a complete intersection Z of two members of $|-\frac{1}{2}K_{V_1}|$, where S_1 is a del Pezzo surface with $(-K_{S_1})^2=1$ and V_1 is a Fano 3-fold of index 2 with $(-\frac{1}{2}K_{V_1})^3=1$. Since $B_2(Y)=2$, Y is not isomorphic to $\mathbf{P}^1\times S_1$. The Picard number of $\tilde{V}_1$ is equal to 2 and hence $\tilde{V}_1$ has exactly two extremal rays (§3). One corresponds to the blowing up $\tilde{V}_1\to V_1$ and the other to the del Pezzo fibration $\tilde{V}_1\to\mathbf{P}^1$ induced by $H^0(\mathcal{O}_{V_1}(-\frac{1}{2}K_{V_1})\otimes I_Z)$, which follows from [7] Corollary 3.6. Hence $\tilde{V}_1$ is not isomorphic to the blow-up of a Fano 3-fold $\not\simeq V_1$. Therefore, $|-K_Y|$ has no base points.

By the claim, $\Phi_{|-K_Y|}$ is a morphism. Since $(-K_Y)^3\leq 6$, the image $\Phi_{|-K_Y|}$ is a complete intersection if Y is not hyperelliptic. If Y is hyperelliptic, then the image is either $\mathbf{P}^3$, a smooth quadric $Q\subset\mathbf{P}^4$ or the Segre embedding $\mathbf{P}^1\times\mathbf{P}^2\subset\mathbf{P}^5$ by [4]. Since $B_2(Y)=2$, the last case is possible. Let us denote the double cover $Y\to\mathbf{P}^1\times\mathbf{P}^2$ by ϕ. Then $-K_Y\sim\phi^*\mathcal{O}(1,0)+\phi^*\mathcal{O}(0,1)$ is a free splitting of $-K_Y$, which induces a splitting of $-K_X$. This contradicts the assumption that $B_2(X)=1$ and index $(X)=1$. q.e.d.

§ 6. Curves on Fano 3-folds

In this section, we consider the irreducibility and unirationality of the several families of pairs of a Fano 3-fold of index ≥ 2 and a curve on it. These will be used in section 7.

Lemma 6.1. *Let T be an affine algebraic scheme over k of characteristic 0 and $Z \subset T \times \mathbf{P}^3$ a closed subscheme flat over T such that $Z_t \subset \mathbf{P}_t^3$ is an irreducible smooth curve for an arbitrary geometric point $t \in T$. Then the set V of geometric points $t \in T$ such that Z_t is the scheme-theoretic intersection of cubics of $\mathbf{P}_t^3$ containing Z_t is open.*

Proof. One may assume that T is irreducible and reduced. Let $f: X \to T \times \mathbf{P}^3$ be the blow-up along Z, $D \subset X$ the exceptional divisor, $g = p_1 \circ f: X \to T$, and $L = f^* p_2^* \mathcal{O}_{\mathbf{P}}(3) \otimes \mathcal{O}_X(-D)$, where $p_i: T \times \mathbf{P}^3 \to \mathbf{P}^3$ is the i-th projection. Then for a geometric point $t \in T$, $t \in V$ if and only if L_t, the induced invertible sheaf on X_t (the blow-up of $\mathbf{P}_t^3$ along Z_t) is generated by global sections. Now let a be a geometric point of T in V. Then X_a is a Fano 3-fold by Proposition 2.12, and hence $L_a \otimes \mathcal{O}(-K_{X_a})$ is ample. By Kodaira vanishing theorem, one has $H^i(X_a, L_a) = 0$ for $i > 0$. Now by Grothendieck's base change theorem, the natural map $g_* L \otimes k_a \to H^0(X_a, L_a)$ is surjective. Thus there is an open neighbourhood U of a in T such that L_t is generated by global sections for $t \in U$. Whence $U \subset V$, and V is open. q.e.d.

We consider 2 families of curves on $\mathbf{P}^3$.

Corollary 6.2. *Let L be a fixed projective 3-space $\mathbf{P}^3$. Let U_1 (resp. U_2) be the set of irreducible smooth curves C_1 (resp. C_2) in L of degree 7 (resp. 6) and genus 5 (resp. 3) which are scheme-theoretically intersections of cubics. Then U_1 and U_2 are irreducible unirational algebraic sets.*

Proof. By the genus formula for plane curves, C_1 and C_2 span L. Let $M_i = \mathcal{O}_{C_i} \otimes \mathcal{O}_L(1)$ for $i = 1, 2$. Let us consider U_1. By the Riemann-Roch formula: $\chi(M_1) = 7 - 4 = 3$, whence $h^1(M_1) \neq 0$ and $K_{C_1} - M_1 \sim P$ for some $P \in C_1$. Since M_1 is very ample,

$$\dim |K_{C_1} - P - Q - R| = \dim |K_{C_1} - P| - 2 \qquad \text{for all} \quad Q, R \in C_1,$$

i.e. $h^1(\mathcal{O}(P+Q+R)) = h^1(\mathcal{O}(P)) + 2$ by Serre duality. This implies that $\dim |P+Q+R| = 0$ for all $Q, R \in C_1$, i.e. C_1 is not hyperelliptic or trigonal. By reversing the argument, one sees that $K_C - P$, with $p \in C$, is very ample if C is a non-singular curve of genus 5 which is not hyperelliptic or trigonal. Such pairs (C, P) form an irreducible set by the irreducibility of $\mathfrak{m}_5$, and hence U_1

is an irreducible algebraic set by the preceding lemma. For unirationality, it is enough to see that there is a rational variety parametrizing general (C, P)'s. So let W_1 be the set of triples (Q_1, Q_2, Q_3) of quadrics Q_1, Q_2, Q_3 of $\mathbf{P}^4$ passing through $(1, 0, 0, 0, 0)$ such that $Q_1 \cap Q_2 \cap Q_3$ is a smooth curve. Then

$$(Q_1, Q_2, Q_3) \longmapsto (Q_1 \cap Q_2 \cap Q_3, (1, 0, 0, 0, 0))$$

gives a parametrization of general (C, P)'s by a rational variety W_1 because general canonical curve of genus 5 is a complete intersection in $\mathbf{P}^4$ of type $(2, 2, 2)$. Thus U_1 is unirational. Let us now consider U_2. By the Riemann-Roch formula, one has $h^0(M_2)=4$ and $h^1(M_2)=0$. Since M_2 is very ample,

$$\dim|M_2-P-Q|=\dim|M_2|-2=1 \qquad \text{for all} \quad P, Q \in C_2 .$$

Since $\deg(M_2-P-Q)=4=\deg K_{C_2}$, this implies that there are no $P, Q \in C_2$ such that $M_2-P-Q \sim K_{C_2}$. Again by reversing the argument, one sees that an arbitrary line bundle M of degree 6 on C of genus 3 is very ample if there are no $P, Q \in C$ such that $M \sim K_C+P+Q$. The rest is similar to the argument for U_1. One only has to see that general (C, M)'s are parametrized by an open dense subset of

$$W_2=\{(p_1, \cdots, p_6, F) \mid F \text{ is a quartic curves of } \mathbf{P}^2, p_1, \cdots, p_6 \in F\} ,$$

because general curves of genus 3 are plane quartic curves. W_2 is rational because $\dim|\mathcal{O}_{\mathbf{P}^2}(4)|=14>6$. q.e.d.

Remarks 6.3. (i) It is known [3] that the image $\bar{C}$ in $\mathbf{P}^3$ of a curve C of genus 3 by a very ample line bundle M of degree 6 is scheme-theoretically an intersection of cubics if C is not hyperelliptic. This shows $U_2 \neq \emptyset$.

(ii) Proposition 7.4 shows that $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$.

We recall

Theorem 6.4 ([10]). *Let $C \subset \mathbf{P}^N$ be an embedding of a complete non-singular curve of genus g by a complete linear system of degree d. If $d \geq 2g+2$, then C is projectively normal and a scheme-theoretic intersection of quadrics.*

Proposition 6.5. *Let S be an irreducible reduced quadric surface of $\mathbf{P}^3$ which is singular at a point p. Let $C \subset \mathbf{P}^3$ be an irreducible smooth (closed) curve in S of degree m. Then*

$$p_a(C)=\left[\frac{1}{4}(m-2)^2\right],$$

where $[r]$ *denotes the integer* n *such that* $n \leq r < n+1$.

Proof. The blow-up $p: F \to S$ of S at p is $\mathbf{F}_2$, and let $f \subset F$ *be a fiber and* $s \subset F$ the negative section so that $(f^2)=0$, $(f \cdot s)=1$, and $(s^2)=-2$. Let C' be the proper transform of C by p. Then $(C' \cdot s)=0$ or 1 since C is smooth. Whence

$$C' \sim n(s+2f)+\varepsilon f \qquad \text{for some} \quad n \in \mathbf{Z} \quad \text{and} \quad \varepsilon=0, 1 .$$

It is well-known that $K_F \sim -2s-4f$ and $p^*\mathcal{O}_S(1) \sim s+2f$. Thus

$$m=(C' \cdot s+2f)=2n+\varepsilon$$

and

$$p_a(C)=1+(C')^2/2+(C' \cdot K_F)/2=(n-1)^2+\varepsilon(n-1)=[(n-1+\varepsilon/2)^2] ,$$

whence $p_a(C)=[(m-2)^2/4]$. q.e.d.

We consider next 3 families on Q.

Corollary 6.6. *Let* L *be a fixed projective* 4*-space* $\mathbf{P}^4$. *Let* C_0 *(resp.* C_1, C_2*) be an irreducible smooth curve of degree* 4 *(resp.* 5, 6*) and genus* 0 *(resp.* 1, 2*) lying in a smooth quadric* 3*-fold* $Q \subset L$. *Then*

(6.6.1) C_1 *and* C_2 *span* L *and are intersections of quadrics,*

(6.6.2) *the algebraic set* U_1 *(resp.* U_2*) of such pairs* (C_1, Q) *(resp.* (C_2, Q)*) is non-empty, irreducible, and unirational,*
and

(6.6.3) C_0 *is scheme-theoretically an intersection of quadrics if* C_0 *spans* L, *and the algebraic set* U_0 *of pairs* (C_0, Q) *of such* C_0 *and* $Q \supset C_0$ *is non-empty, irreducible and unirational.*

Proof. Let us prove (6.6.3). Modulo $\operatorname{Aut} L$, there is only one C_0 spanning L: the image of $\mathbf{P}^1 \to \mathbf{P}^4$ by $(s, t) \longmapsto (s^4, s^3t, s^2t^2, st^3, t^4)$. C_0 is an intersection of quadrics, and quadrics containing C_0 form a projective space, whence U_0 is non-empty, irreducible, and unirational. Let us consider (6.6.1). If C_1 (resp. C_2) does not span L, then it lies in a quadric surface S in L. This is impossible, because the degree and the genus do not match by the genus formula for curves in $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, or a quadric cone (Proposition 6.5). Then C_1 and C_2 are embedded in L by complete linear systems and (6.6.1) follows from Theorem 6.4. Let us consider the irreducibility. If M is a line bundle of degree 5 (resp. 6) on a curve C of genus 1 (resp. 2) with a basis $s=(s_0, s_1, \cdots, s_4)$ of $H^0(C, M)$, then M is very ample, s defines an embedding $C \hookrightarrow L \simeq \mathbf{P}^4$, and the image C is projectively normal and an intersection of quadrics (6.4). Let I_C be the sheaf of ideals for C in L. Then a quadric $Q \supset C$ is defined by a non-zero element of $H^0(L, I_C \otimes \mathcal{O}(2)) \simeq k^5$ (resp. k^4). Thus $U_1 \neq \emptyset$ (resp. $U_2 \neq \emptyset$) and the irreducibility of U_1 (resp. U_2) follow from the irreducibility of $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$). For unirationality, it is enough to see that general pairs (C, M) as above are parametrized by a rational variety. Let

$$W_1 = \{(p_1, \cdots, p_5, C) \mid C \text{ is a cubic curve in } \mathbf{P}^2, p_1, \cdots, p_5 \in C\},$$

$$W_2 = \left\{(p_1, \cdots, p_6, C) \,\middle|\, \begin{array}{l} C \text{ is a curve of degree 6 in } \mathbf{Q}(1, 1, 3) \\ p_1, \cdots, p_6 \in C \end{array}\right\},$$

(cf. Remark 6.7 for $\mathbf{Q}(1, 1, 3)$). These are rational varieties because

$$\dim |\mathcal{O}_{\mathbf{P}^2}(3)| = 9 > 5 \quad \text{and} \quad \dim |\mathcal{O}_{\mathbf{Q}(1,1,3)}(6)| = 11 > 6$$

(cf. Remark 6.7). An open dense subset of W_1 (resp. W_2) parametrizes general (C, M)'s (cf. Remark 6.7 for W_2) in the obvious ways. q.e.d.

Remark 6.7. Let $k[x, y, z]$ be the graded polynomial ring with $\deg x = \deg y = 1$, $\deg z = 3$, and let $\mathbf{Q}(1, 1, 3) = \operatorname{Proj} k[x, y, z]$. The singular locus is one point $\{x = y = 0\}$, and $|\mathcal{O}_{\mathbf{Q}(1,1,3)}(6)|$ is the base-point-free linear system associated to $k[x, y, z]_6$, the homogeneous part of degree 6. On affine set $D_+(X)$, the generic member C of the linear system is written as $\bar{z}^2 = g_3(\bar{y})\bar{z} + g_6(\bar{y})$, where $\bar{y} = y/x$, $\bar{z} = z/x^3$, and $g_i(\bar{y})$ is a polynomial of degree i ($i = 3, 6$). Thus C is a general curve of genus 2. (cf. [6] for general results.)

Let X be an irreducible smooth algebraic variety over k and $\Sigma = (L, V)$ a linear system over X, where L is an invertible sheaf on X and V a vector subspace of $H^0(X, L)$. The *base locus* $\mathrm{Bs}(\Sigma)$ of Σ is, by definition, the scheme-theoretic intersection of all members of Σ.

Proposition 6.8 (Characteristic 0). *Assume that* $\mathrm{Bs}(\Sigma)$ *is of dimension* ≤ 1 *and has only isolated singularities, and*

$$\text{emb-dim}_x \mathrm{Bs}(\Sigma) \leq \dim X - 1 \qquad \textit{for all} \quad x \in \mathrm{Bs}(\Sigma),$$

where $\text{emb-dim}_x \mathrm{Bs}(\Sigma)$ *denotes the embedding dimension of* $\mathrm{Bs}(\Sigma)$ *at* x. *Then the generic member of* Σ *is smooth.*

Proof. Let $n = \dim X$. We first assume that $C = \mathrm{Bs}(\Sigma)$ is smooth and of pure dimension 1. Let $f: Y \to X$ be the blow-up along C and $D \subset Y$ the exceptional set. Since $C = \mathrm{Bs}(\Sigma)$, one sees that $V \subset H^0(f^*L(-D))$ and the induced map $V \otimes \mathcal{O}_Y \to f^*L(-D)$ is surjective. Thus the linear system $(f^*L(-D), V)$ on Y is free from base point. By Bertini's theorem, the generic member E of $(f^*L(-D), V)$ is smooth and intersects transversally with D.

We claim that $E \cdot D$ does not contain any fiber F of the $\mathbf{P}^{n-2}$-bundle morphism $D \to C$. Indeed if $E \cdot D \supset F$, then F is an irreducible component of $F \cdot D$ and $E \cdot D$ can not be smooth because $\mathcal{O}(E \cdot D)$ is a tautological line bundle of $D \to C$. This is a contradiction and hence E does not contain any fiber of $D \to C$. Now $\bar{E} = f(E) \subset X$ is the generic member of Σ and $f^*(\bar{E}) = E + D$. $\bar{E}$ is smooth in $X - C$. $\bar{E}$ is smooth at any point of C because E does not contain any fiber of $D \to C$.

Let us now consider the general case. Let S be the union of singular points and the isolated points of $\mathrm{Bs}(\Sigma)$. Let $X^* = X - S$ and Σ^* the induced linear system by Σ on X^*. The pervious case applied to Σ^* shows that the generic member E of Σ is smooth outside S. Let $v_1, \cdots, v_m$ be a basis of V. Let x be an arbitrary point in S, then one can write v_i as $v_i = e_i s$ $(1 \le i \le m)$, where $e_1, \cdots, e_m \in \mathcal{O}_{X,x}$ and s a generator of $L \otimes \mathcal{O}_{X,x}$. Then E is defined near x by the generic linear combination of $e_1, \cdots, e_m$, and $\mathrm{Bs}(\Sigma)$ is defined by $e_1, \cdots, e_m$. Since $\text{emb-dim}_x \mathrm{Bs}(\Sigma) \le n-1$, there is an i such that $e_i \notin I_x^2$, where I_x is the maximal ideal of $\mathcal{O}_{X,x}$. Thus the equation of E does not belong to I_x^2 and E is smooth at x.

q.e.d.

Proposition 6.9. *Let Z be a smooth conic on a Fano 3-fold $Q \subset \mathbf{P}^4$, $V_4 \subset \mathbf{P}^5$, or $V_5 \subset \mathbf{P}^6$, or a twisted cubic on $V_5 \subset \mathbf{P}^6$. Then Z is scheme-theoretically the intersection of hyperplane sections containing Z.*

Remark 6.10. V_d $(d=4, 5)$ actually contains a line, a smooth conic, and a twisted cubic. Let S be a smooth hyperplane section of V_d. Then S is a del Pezzo surface of degree d and is the blow-up of $\mathbf{P}^2$ at $9-d$ points in general position. Let C_m be a line in $\mathbf{P}^2$ passing through exactly m points $(m=0, 1, 2)$ of the center of the blowing up. Then the proper transform Z_m on S is a smooth rational curve of degree $3-m$.

Proof. Let $\langle Z \rangle$ be the linear span of Z. Since $X = V_5$ (resp. V_4, Q) is defined by quadrics, so is $\langle Z \rangle \cap X$ in $\langle Z \rangle$. Since $\mathrm{Pic}\, X \simeq \mathbf{Z}$ [4], X does not contain $\langle Z \rangle$. Thus

$$\langle Z \rangle \cap X = Z \quad \text{for a conic} \quad Z \quad \text{because} \quad \langle Z \rangle \cap X \supset Z .$$

Let Z be a twisted cubic and assume that $\langle Z \rangle \cap V_5 \supsetneqq Z$. $\langle Z \rangle \cap V_5$ is defined by quadrics and there are at most ∞^2 quadrics of $\langle Z \rangle$ passing through Z. Hence one sees that $\langle Z \rangle \cap V_5$ is either a quadric surface, the union of Z and a line l, or Z. The first case does not occur because $\mathrm{Pic}\, V_5$ is generated by hyperplane section and $\deg V_5 = 5$. Let us assume that second case. Let Σ be the linear system of hyperplane sections of V_5 passing through Z. Then $\mathrm{Bs}(\Sigma) = \langle Z \rangle \cap V_5 = Z \cup l$ has only planer singularities because Z and l are smooth. By Proposition 6.8 applied to Σ, Σ has a smooth member S. Now $\mathrm{Tr}_S\Sigma$ has fixed component $Z+l$ and $\mathrm{Tr}_S\Sigma \subset |-K_S|$ by the adjunction formula, whence $|-K_S-Z-l|$ is free from base points. Let $C \in |-K_S-Z-l|$. Since $-K_S$ is very ample of degree 5 and $(-K_S\cdot Z+l)=4$, one sees that C is a line by

$$(-K_S\cdot C)=(-K_S)^2-(-K_S\cdot Z+l)=1\,.$$

Then $(C^2)=-1$ by $(C^2)+(K_S\cdot C)=2p_a(C)-2=-2$. This is a contradiction because $|C|$ is base point free. Thus the second case does not occur, and one has $Z=\langle Z\rangle \cap V_5$. q.e.d.

We finally consider families of V_5 together with curves on it.

Proposition 6.11. *Let d be 3, 4, or 5, and L_d a fixed projective space $\mathbf{P}^{d+1}$. Let $A(d, 1)$ (resp. $A(d, 2)$, $A(d, 3)$) be the algebraic set of pairs of a line (resp. smooth conic, twisted cubic) Z in L_d and a Fano 3-fold V_d in L_d such that $Z\subset V_d$. Then $A(3, 1)$, $A(4, i)$ $(i=1, 2)$, $A(5, j)$ $(j=1, 2, 3)$ are, non-empty, irreducible, and unirational.*

Proof. The proof for $A(3, 1)$, $A(4, 1)$, $A(4, 2)$ are similar, and we consider $A(4, 2)$ only. By Proposition 6.9, $A(4, 2)$ is parametrized by an open dense subset U of $\{(Q_1, Q_2, W)\,|\,Q_1, Q_2$ are quadric hypersurfaces, W a 2-dimensional linear subspace of L_4, and $Q_2\supset W\}$. Indeed general $(Q_1, Q_2, W)\in U$ gives $V_4=Q_1\cap Q_2$ and $Z=Q_1\cap W$. It is clear that U is, non-empty, irreducible, and rational. Let $d=5$ now. It is known that $V_5\subset L_5$ is $L_5\cap \mathrm{Gr}(4, 1)$ for some linear embedding $L_5\to \mathbf{P}^9$, where $\mathrm{Gr}(4, 1)\hookrightarrow \mathbf{P}^9$ is the Grassman variety embedded by Plücker coordinates. Hence $Z\subset V_5$ in L_5 is induced by $Z\subset \mathrm{Gr}(4, 1)$ and a linear subspace $H\simeq \mathbf{P}^6$ of $\mathbf{P}^9$ containing Z, and an isomorphism $H\simeq L_5$ such that $H\cap \mathrm{Gr}(4, 1)$ is a transverse intersection. Hence it is enough to show the non-emptiness, irreducibility and unirationality of the set of lines Z (resp. conics, twisted cubics) in $\mathrm{Gr}(4, 1)\subset \mathbf{P}^9$, which lie on some V_5. By Propositions 6.8 and 6.9, one can see that Z lies on some V_5 iff $\langle Z\rangle \cap \mathrm{Gr}(4, 1)=Z$. Thus by (i)$\Leftrightarrow$(iii) of Proposition 6.12, one can see that Z is unique modulo $\mathrm{Aut}\,\mathrm{Gr}(4, 1)$. q.e.d.

Proposition 6.12. *Let Q (resp. S) be the universal quotient bundle (resp. universal subbundle) of $\mathrm{Gr}(4,1)$, and $\mathrm{Gr}(4, 1)\subset \mathbf{P}^9$ the Plücker embedding. Let $f\colon \mathbf{P}^1\to \mathrm{Gr}(4, 1)\subset \mathbf{P}^9$ be a morphism, $\bar{Q}=f^*Q$, and $\bar{S}=f^*S$. Assume that $m=\deg f^*\mathcal{O}_{\mathbf{P}}(1)$ is 1, 2, or 3. Then the following are equivalent:*

(i) *f is an isomorphism of $\mathbf{P}^1$ to a curve $C\subset \mathbf{P}^9$ such that the linear span $\langle C\rangle$ of C is $\mathbf{P}^m$ and $C=\langle C\rangle \cap \mathrm{Gr}(4, 1)$ as a scheme,*

(ii) $\bar{Q} \simeq \mathcal{O}(1) \oplus \mathcal{O}(m-1)$ *and* $\bar{S} \simeq \mathcal{O}^{3-m} \oplus \mathcal{O}(-1)^m$, *and*

(iii) $\bar{Q} \simeq \mathcal{O}(1) \oplus \mathcal{O}(m-1)$ *and the natural map* $\mathcal{O}^5 \to \bar{Q}$ *induces a surjection* $k^5 \to H^0(\bar{Q})$.

Proof. We consider the case $m = 3$ only. Other cases are similar and easier.

(i) $\Rightarrow$ (ii): Since C is defined by hyperplanes by (i), $I_C \otimes \mathcal{O}_{P^9}(1)$ and hence $N_C^* \otimes \mathcal{O}_{P^1}(m)$ are generated by global sections, where I_C (resp. N_C) is the defining ideal (resp. normal bundle) of C in Gr (4, 1). Hence N_C is a direct sum of $\mathcal{O}_{P^1}(i)$'s with $i \le m = 3$. Since the tangent bundle of Gr(4, 1) is $S^* \otimes Q$, $\bar{S}^* \otimes \bar{Q}$ fits in the exact sequence

$$0 \longrightarrow T_{P^1} \longrightarrow \bar{S}^* \otimes \bar{Q} \longrightarrow N_C \longrightarrow 0 .$$

Thus $\bar{S}^* \otimes \bar{Q}$ is a direct sum of $\mathcal{O}_{P^1}(i)$'s with $i \le 3$. On the other hand $\bar{Q} \simeq \mathcal{O} \oplus \mathcal{O}(3)$ or $\mathcal{O}(1) \oplus \mathcal{O}(2)$, $\bar{S}^* \simeq \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(3)$, $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$, or $\mathcal{O}(1)^{\oplus 3}$ because $\deg \bar{Q} = \deg \bar{S}^* = m = 3$ and $\bar{Q}$ and $\bar{S}^*$ are generated by global sections. Hence $\bar{Q} \simeq \mathcal{O}(1) \oplus \mathcal{O}(2)$ and $\bar{S}^* \simeq \mathcal{O}(1)^{\oplus 3}$.

(ii) $\Rightarrow$ (iii): Since $h^0(\bar{S}) = 0$, the map $k^5 \simeq H^0(\mathcal{O}^5) \to H^0(\bar{Q}) \simeq k^5$ is injective, whence follows (iii).

(iii) $\Rightarrow$ (i): Modulo the action of Aut $\mathbf{P}^1$ and Aut Gr(4, 1), there exists only one morphism f with property (iii). It is, therefore, enough to show that there is one f with property (i) because we settled (i) $\Rightarrow$ (iii). This is done in *Remark* 6.10.

§ 7. 3-folds in Tables 2–5 are Fano 3-folds.

(7.0) We will show (7.1)–(7.29) that an arbitrary smooth X described in each class in Tables 2, 3, 4, 5 is a Fano 3-fold with described $(-K_X)^3$, B_2, and B_3, and that X's in each class (N^{os} 1 through 36 in Table 2, N^{os} 1 through 31 in Table 3, N^{os} 1 through 12 in Table 4, and 8 classes in Table 5) are parametrized by a non-empty (irreducible) rational variety, (we will say that each class is *non-empty, irreducible*, and *unirational*). At the same time, $-K_X$ is shown to have a splitting, and, furthermore, it has a free splitting if (i) X is not in N^o 1 in Table 2 and (ii) $X \not\simeq \mathbf{P}^1 \times S_1$ (Remark 7.30). In (7.31)–(7.35), we will show that different classes are not deformation equivalent (cf. 7.31)).

Let us first consider Table 2 in several cases.

(7.1) N^{os} 24, 32, 34, 35, 36 in Table 2 ($\mathbf{P}^1$-bundles over $\mathbf{P}^2$).

One sees that $B_1(X) = 0$ and $B_2(X) = 2$ are obvious, and $B_3(X) = 0$ follows from $2 + 2B_2(X) - B_3(X) = \chi_{top}(X) = \chi_{top}(\mathbf{P}^1) \cdot \chi_{top}(\mathbf{P}^2) = 6$ because X

is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$. 3-folds in N^{os} 35, 36 are Fano 3-folds by (4.26), and the rest is easy.

(7.2) N^{os} 2, 6, 8, 18 in Table 2 (double covers).

Let us consider N^o 6 first. Let L be the line bundle over $\mathbf{P}^2\times\mathbf{P}^2$ of bidegree (1,1). Now general $p\in H^0(\mathbf{P}^2\times\mathbf{P}^2, L^{\otimes 2})$, $q\in H^0(\mathbf{P}^2\times\mathbf{P}^2, L)$, and $r\in k$ define a smooth complete intersection $X(p, q, r)$ in L defined by $z^2=p$ and $rz=q$, where z is the fibre coordinate of L. They parametrize X's in N^o 6 ($X(p, q, r)$ is in (6, a) if $r\neq 0$, and $X(p, q, 0)\in$ (6, b)). Thus N^o 6 is irreducible and unirational. Obviously an arbitrary $X\in$ (6, a) is a Fano 3-fold by the adjunction formula. Since $(-K_X)^3$, B_2, and B_3 are deformation invariants, the rest of (7.2) follows from (3.8.2) in a similar way, and easy examples show that both (8.a) and (8.b) in N^o 8 really occur. Let us consider N^o 2 for illustration. Let $f: X\to\mathbf{P}^1\times\mathbf{P}^2$ be a double cover with branch locus $B\subset\mathbf{P}^1\times\mathbf{P}^2$ a divisor of bidegree (2, 4). By (3.8.2),

$$-K_X\sim f^*(2H_1+3H_2)-f^*(2H_1+4H_2)/2\sim f^*(H_1+H_2)\,,$$

where $H_1=p_1^*\mathcal{O}(1)$ and $H_2=p_2^*\mathcal{O}(1)$. Thus $-K_X$ is ample, has a free splitting, and $(-K_X)^3=6$. $B_2(X)=2$ by (3.8.3). Since $B\sim 2H_1+4H_2$, one has

$$\chi(\mathcal{O}_B)=1-\chi(-2H_1-4H_2)=1-\chi(\mathcal{O}_{\mathbf{P}^1}(-2))\cdot\chi(\mathcal{O}_{\mathbf{P}^2}(-4))=4\,.$$

From $K_B\sim H_2\cdot B$, follows $(K_B^2)=2$. Thus by Noether's formula $\chi_{\rm top}(B)=12\chi(\mathcal{O}_B)-(K_B^2)=46$, whence $\chi_{\rm top}(X)=-34$ by (3.8.1). Hence $B_3(X)=40$, because $\chi_{\rm top}(X)=2+2B_2(X)-B_3(X)$. X's in N^o 2 are parametrized by an open dense subset of $|B|$. This proves the assertion for N^o 2.

(7.3) For the rest of Table 2, X's are given as blow-ups of $\mathbf{P}_3$, Q, or V_d ($d=1,\cdots,5$) along irreducible smooth curves whose genera are explicitly given or can be calculated immediately from the description. Thus $B_2(X)$, $B_3(X)$, and $(-K_X^3)$ for these are calculated from formulae (2.1.3) and (2.1.4), and the values of $B_3(V_d)$ [12].

(7.4) N^{os} 9, 12, 13, 17, 21 in Table 2 (blow-ups).

By Proposition 2.12 (plus Corollary 6.6 for N^{os} 13, 17, 21), X is a Fano 3-fold and $-K_X$ has a free splitting in these cases. N^{os} 9, 12, 13, 17, 21 are irreducible and unirational, and N^{os} 13, 17, 21 are non-empty by Corollaries 6.2 and 6.6.

N^{os} 9 and 12 are non-empty by:

Proposition 7.5. *Let X be a smooth complete intersection on* $\mathbf{P}^3\times\mathbf{P}^2$ *(resp.* $\mathbf{P}^3\times\mathbf{P}^3$, $Q\times\mathbf{P}^2$*) of the form* $(1,1)\cdot(2,1)$ (resp. $(1,1)^3$, $(1,1)^2$), *where* (a, b) *denotes some member in* $|aH_1+bH_2|$ *and* H_i *is the ample generator of* *the Picard group of the i-th factor* ($i=1, 2$). *Then X is a Fano 3-fold in* N° 9 (*resp.* 12, 13).

Proof. It is clear that $-K_X \sim H_1 + H_2$, X is a Fano 3-fold, and $\rho(X)=2$ [2]. Since other cases are treated in the same way, we consider N° 9 only. Let $f: X \to Y = \mathbf{P}^3$ be the first projection. From

$$(f^*\mathcal{O}_Y(1)^3) = (H_1^3 \cdot H_1 + H_2 \cdot 2H_1 + H_2) = 1 ,$$

f is a birational morphism. By $\rho(X)=2$, f is the blow-up at a point or along an irreducible smooth curve on Y [7, Corollary 3.6]. Since

$$(-K_X)^3 = ((H_1 + H_2)^4 \cdot (2H_1 + H_2)) = 16 ,$$

f is the blow-up along an irreducible smooth curve C on Y by (2.9.3). Let $D \subset X$ be the exceptional divisor of f. Then

$$D \sim f^*(-K_Y) + K_X \sim (3H_1 - H_2)_X ,$$

where $(\ \ )_X$ denotes the restriction to X. It is easy to see $(D^2 \cdot -K_X)=8$ like $(-K_X)^3=16$. By (2.1.3), it follows that C is of degree 7 and genus 5. One also sees that $f^*\mathcal{O}_Y(3) - D \sim (H_2)_X$ is free from base points, whence C is an intersection of cubics by Proposition 2.9. q.e.d.

(7.6) N^os 27 and 31 in Table 2 (blow-ups).

Lines (resp. twisted cubics) in Q (resp. $\mathbf{P}^3$) are defined by hyperplanes (resp. quadrics), and unique modulo Aut Q (resp. Aut $\mathbf{P}^3$). Indeed it is easy to see that any two maximal isotropic subspaces ($\simeq k^2$) of k^5 with respect to Q are conjugate to each other. Thus by Proposition 2.12, each of N^os 27 and 31 contains exactly one Fano 3-fold whose anti-canonical divisor has a free splitting.

(7.7) N^os 11, 16, 19, 20, 22, 26 in Table 2 (blow-ups of V_d).

These are blow-ups of V_d along lines, conics, or twisted cubics. The assertion in these cases follows from Propositions 2.12 and 6.11 because $\mathcal{O}(-K_{V_d}) \simeq \mathcal{O}_{V_d}(2)$, where $\mathcal{O}_{V_d}(1)$ is the pull-back of $\mathcal{O}(1)$ by the natural map $V_d \subset \mathbf{P}^{d+1}$.

(7.8) The rest of Table 2 (blow-ups along complete intersections).

In these cases, X is obtained in the following way. Let Y be $\mathbf{P}^3$, Q, or V_d, and H the ample generator of Pic $Y \simeq \mathbf{Z}$. Let $C \subset Y$ be a smooth complete intersection of two divisors $A \in |aH|$ and $B \in |bH|$, where a and b are positive integers with $a, b <$ index Y. X is the blow-up of Y along C. Thus X is a Fano 3-fold, $-K_X$ has a splitting for N° 1 and a free splitting for the others

(Proposition 2.12), and it is clear that each family is non-empty, irreducible, and unirational [4]. It is clear that each of the cases (15.a) and (15.b) in N° 15, and (23.a) and (23.b) in N° 23 really occurs.

We now consider Table 3.

(7.9) N^{os} 1, 27, 28, 31 in Table 3 (obvious cases).

X is a double covering of $\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$ for N° 1, and the argument is almost the same as that of (7.2). N^{os} 27, 28, 31 are well-known Fano 3-folds.

(7.10) N^{os} 6, 10, 12, 15, 18, 20, 25 in Table 3 (blow-ups of $\mathbf{P}^3$ and Q along non-connected curves).

In N^{os} 6, 12, 18, 25, X is the blow-up of $\mathbf{P}^3$ along a curve which is the disjoint union of a line l and a curve C, where C is a smooth complete intersection of (2, 2) type, twisted cubic, a conic or a line. Since C is an intersection of quadrics, $l \amalg C$ is an intersection of cubics. Thus X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12. $B_2(X)$, $B_3(X)$ and $(-K_X)^3$ are calculated by (2.1.3) and (2.1.4). It is clear that such curves are parametrized by a non-empty irreducible rational variety in each case. If C is a line (N° 25), X has a morphism to $\mathbf{P}^1\times\mathbf{P}^1$ because each blow-up along a line gives a morphism to $\mathbf{P}^1$, which is induced by the linear projection of the line. Since lines are complete intersections of hyperplanes, two exceptional divisors E_1 and E_2 are $\mathbf{P}^1\times\mathbf{P}^1$ and $N_{E_1/X}$ and $N_{E_2/X}$ are of bidegree $(1, -1)$. It is easy to see that $X\to\mathbf{P}^1\times\mathbf{P}^1$ is a $\mathbf{P}^1$-bundle with 2 disjoint sections E_1 and E_2. Thus $X\simeq\mathbf{P}(\mathcal{O}(1,0)\oplus\mathcal{O}(0,1))$ as stated in N° 25. In N^{os} 10, 15, 20, X is the blow-up of Q along the disjoint union of a conic and a conic, a line and a conic, or a line and a line. Lines and conics on Q are intersection of hyperplanesections (Corollary 6.9). Thus the argument is the same as that for $\mathbf{P}^3$ above, once we show that N° 20 contains only one member up to isomorphisms. And this follows from the fact that any non-degenerate quadratic form q on $L\simeq k^5$ with two isotropic subspaces L_1 and L_2 of dimension 2 such that $L_1\cap L_2=\{0\}$ can be put in a standard form

$$q(X_0, X_1, X_2, X_3, X_4)=X_0X_2+X_1X_3+X_4^2,\ L_1=\{X_2=X_3=X_4=0\},$$

and $L_2=\{X_0=X_1=X_4=0\}$ for some coordinate system $(X_0,\cdots,X_4)$ of L.

(7.11) N° 19 in Table 3 (2-points-blow-up of Q).

Let $f: X\to Q$ be the blow-up at 2 points p_1 and p_2 on Q which are not colinear, and let $P_i=f^{-1}(p_i)$ $(i=1, 2)$ and $H=f^*\mathcal{O}_Q(1)$. We claim that

(7.11.1) $|H-P_1-P_2|$ is free from base points.

Let $L\simeq\mathbf{P}^1$ be the linear span of p_1 and p_2 in $\mathbf{P}^4$. Since p_1 and p_2 are not colinear on Q, $L\cap Q \subsetneqq L$. Since $\deg Q=2$, this means that $L\cap Q=\{p_1, p_2\}$ as schemes, whence we get (7.11.1) by Proposition 2.10.

Since $\{p_1, p_2\}$ is an intersection of members of $\mathcal{O}_Q(1)$ and $-K_Q-2\mathcal{O}_Q(1)\sim\mathcal{O}_Q(1)$ is very ample, X is a Fano 3-fold and $-K_X$ has a free splitting by (2.12). The rest for N° 19 follows by (2.9.3) and (2.9.4).

(7.12) N^{os} 3, 8, 17 in Table 3 (divisors).

Since X, in our case, is given as a smooth member of a divisor class of $\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2$ or $\mathbf{F}_1\times\mathbf{P}^2$, it is easy to see that X is a Fano 3-fold with described $(-K_X)^3$ and $-K_X$ has a free splitting and that X is parametrized by a non-empty irreducible rational variety. To see that $B_2(X)=3$ and $B_3(X)=0$ for N° 8, it is enough to consider a generic X in N° 8 because $B_2(X)$ and $B_3(X)$ are topological invariants. If $\mathbf{F}_1\times\mathbf{P}^2\to\mathbf{P}^2\times\mathbf{P}^2$ is the blow-up along $t\times\mathbf{P}^2$ for some $t\times\mathbf{P}^2$, then the generic X is the blow-up of a smooth divisor Y of bidegree $(1, 2)$ in $\mathbf{P}^2\times\mathbf{P}^2$ (N° 24 in Table 2) along a smooth intersection $C=Y\cdot(t\times\mathbf{P}^2)$. Since C is a smooth conic, one has $B_2(X)=3$ and $B_3(X)=0$ by (2.1.4). B_2 and B_3 for N^{os} 3 and 17 are calculated immediately by (2.1.4) and:

Proposition 7.13. *Let Y be a smooth projective 3-fold, H an ample divisor on Y, and X a smooth divisor on $Y\times\mathbf{P}^1$ such that $X\sim p_1^*H+p_2^*\mathcal{O}(1)$, where p_i is the i-th projection from $Y\times\mathbf{P}^1$ to Y or $\mathbf{P}^1$.*

Then the induced morphism $f: X\to Y$ is the blow-up of Y along an irreducible smooth curve C and C is a smooth complete intersection of two members $\in|H|$. In particular, one has

(7.13.1) $$2p_a(C)-2=(H^2\cdot 2H+K_Y).$$

Proof. X is irreducible because it is smooth and ample on $Y\times\mathbf{P}^1$. By the Künneth formula, there are global sections h_0 and $h_1\in H^0(Y, \mathcal{O}(H))$ such that X is the zero-set of the section $h_0x_0-h_1x_1$ of $p_1^*\mathcal{O}(H)\otimes p_2^*\mathcal{O}(1)$, where $\{x_0, x_1\}$ is a basis for $H^0(\mathbf{P}^1, \mathcal{O}(1))$. Since X is an irreducible smooth divisor on $Y\times\mathbf{P}^1$, one sees that $h_0=h_1=0$ in $\mathcal{O}(H)$ defines a smooth curve C by Jacobian criterion. Since C is a complete intersection of two members $h_0=0$ and $h_2=0$ of $|H|$, C is irreducible. It is clear that X is the blow-up of Y along C by the expression $h_0x_0=h_1x_1$ of X in $Y\times\mathbf{P}^1$. q.e.d.

(7.14) N° 2 in Table 3 (conic bundle).

Let L, X, Y be as in N° 2 in Table 3, and let

$$f: V=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-1, -1)^{\oplus 2})\longrightarrow\mathbf{P}^1\times\mathbf{P}^1$$

be the $\mathbf{P}^2$-bundle given there. Then $N=L+f^*\mathcal{O}(1, 1)$ is free from base points because it is the tautological line bundle of $\mathbf{P}(\mathcal{O}(1, 1)\oplus\mathcal{O}^{\oplus 2})$. One sees that

(7.14.1) $$-K_V \sim 3N+f^*\mathcal{O}(1, 1),$$

and the relation

(7.14.2) $$N^3-f^*\mathcal{O}(1, 1)\cdot N^2=0.$$

From this, it is easy to calculate

(7.14.3) $$((-K_V-X)^3\cdot X)=14, \quad (X^2\cdot Y\cdot -K_V)=4, \quad (X^4)=64.$$

By $X\sim 2N+f^*\mathcal{O}(0, 1)$ and (7.14.1), $|X|$ and $|-K_V|$ are free from base points. Hence for a general member X of $|X|$, X and $X\cdot Y$ are irreducible smooth by Bertini's theorem because $|X|$ and $\mathrm{Tr}_Y|X|$ are not composed of pencils by $(\mathcal{O}_Y(X)^2\cdot\mathcal{O}_Y(-K_V))>0$ and $(X^4)>0$ (7.14.3). This means that smooth $X\in|X|$ with irreducible $X\cdot Y$ is parametrized by a non-empty rational variety. One has $(-K_X)^3=14$ by (7.14.3). We note

$$Y\simeq\mathbf{P}(\mathcal{O}(-1, -1)^{\oplus 2})\simeq\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1,$$

where the induced $\mathbf{P}^1$-subbundle morphism is the projection to the first and the second factors. Then

(7.14.4) $$N\otimes\mathcal{O}_Y\simeq\mathcal{O}(0, 0, 1) \quad\text{and}\quad \mathcal{O}_V(X)\otimes\mathcal{O}_Y\simeq\mathcal{O}(0, 1,2).$$

This means that $X\cap Y\simeq\mathbf{P}^1\times\mathbf{P}^1$, and let us identify $D=X\cap Y$ with $\mathbf{P}^1\times\mathbf{P}^1$ by the projection from $Y\simeq\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$ to the first and the third factors. Then it is easy to see

(7.14.5) $$f^*\mathcal{O}(1, 0)|_D\simeq\mathcal{O}(1, 0), \quad f^*\mathcal{O}(0, 1)|_D\simeq\mathcal{O}(0, 2), \quad N\otimes\mathcal{O}_D\simeq\mathcal{O}(0, 1)$$

from (7.14.4). It is clear that $N\otimes\mathcal{O}_D\simeq\mathcal{O}(0, 1)$ by the identification $D\simeq\mathbf{P}^1\times\mathbf{P}^1$. Thus

(7.14.6) $$N_{D/X}\simeq\mathcal{O}_V(Y)\otimes\mathcal{O}_D\simeq\mathcal{O}(-1, -1)$$

by (7.14.5) and $Y\sim N-f^*\mathcal{O}(1, 1)$. Let $g=f|_X\colon X\to\mathbf{P}^1\times\mathbf{P}^1$. Then

(7.14.7) $$-K_X\sim D+g^*\mathcal{O}(2, 1)\sim N|_X+g^*\mathcal{O}(1, 0)$$

by (7.14.1), (7.14.6), and $N|_X\sim D+g^*\mathcal{O}(1, 1)$. Thus $-K_X$ has a free splitting.

We show that $-K_X$ is ample. Since $-K_X$ is free from base points (7.14.7), it is enough to show $(-K_X\cdot Z)>0$ for any irreducible curve Z on X. If $Z\subset D$, then $(-K_X\cdot Z)>0$ because $\mathcal{O}_D(-K_X)\simeq\mathcal{O}(1, 1)$ is ample by (7.14.5), (7.14.6), and (7.14.7). If $g(Z)$ is a point, then $(-K_X\cdot Z)=(D\cdot Z)>0$ by (7.14.7) because D is induced by a tautological line bundle L. If $Z\not\subset D$ and if

$g(Z)$ is not a point, then

$$(-K_X \cdot Z) \geqq (g_* Z \cdot \mathcal{O}(2, 1)) > 0$$

by (7.14.7). Thus X is a Fano 3-fold. To see that $B_2(X) = 3$ and $B_3(X) = 6$, it is enough to find one X with these values because B_2 and B_3 are topological invariants. Let E be a vector subbundle of

$$f_* \mathcal{O}(X) = S^2(\mathcal{O}(1, 1) \oplus \mathcal{O}^{\oplus 2}) \otimes \mathcal{O}(0, 1)$$

(We note that $X \sim 2N + f^* \mathcal{O}(0, 1)$) defined by

$$E = [S^2(\mathcal{O}(1, 1)) \oplus S^2(\mathcal{O})^{\oplus 2}] \otimes \mathcal{O}(0, 1).$$

Then $H^0(E)$ gives a linear subsystem Σ of $|X|$. Since E is generated by global sections and members of Σ induce "Fermat" conics like $\alpha x^2 + \beta y^2 + \gamma z^2$ on fibers of f, Σ is free from base points. Thus for the generic X' of Σ, X' and $X' \cap Y$ are smooth irreducible, and $g' = f|_{X'}: X' \to \mathbf{P}^1 \times \mathbf{P}^1$ is a conic bundle. Since $E \simeq \mathcal{O}(2, 3) \oplus \mathcal{O}(0, 1)^{\oplus 2}$, the discriminant locus $\Delta' \subset \mathbf{P}^1 \times \mathbf{P}^1$ of g' for generic X' is a sum of 3 distinct smooth curves C_1, C_2, C_3 intersecting transversally pairwise, and $C_1 \sim \mathcal{O}(2, 3)$, $C_2 \sim C_3 \sim \mathcal{O}(0, 1)$. Thus Δ' is connected and not smooth, whence $g'^{-1}(C')$ is irreducible for any irreducible curve C' on $\mathbf{P}^1 \times \mathbf{P}^1$ (4.9). This means that $B_2(X) = \rho(X) = 3$ (4.8). The above description of Δ' shows that $P_a(\Delta') = 4$. Hence $B_3(X') = 2p_a(\Delta') - 2 = 6$ (4.13).

(7.15) N° 4 in Table 3 (blow-up along a fiber of a conic bundle).

Let $f: Y \to \mathbf{P}^1 \times \mathbf{P}^2$ be a double cover with branch locus (2, 2) and let $H_1 = f^* \mathcal{O}(1, 0)$ and $H_2 = f^* \mathcal{O}(0, 1)$. Then $-K_Y \sim H_1 + 2H_2$ (3.8.2). If $C \subset Y$ is a smooth fibre of $Y \to \mathbf{P}^2$, then C is an intersection of members of H_2. Thus the blow-up X of Y along C is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12. By Lemma 2.1, it follows that $(-K_X)^3 = (-K_Y)^3 - 6 = 18$, $B_2(X) = B_2(Y) + 1$, and $B_3(X) = B_3(Y)$. Since fibres are parametrized by $\mathbf{P}^2$, X in N° 4 is parametrized by a non-empty rational variety by (7.2).

(7.16) N° 9 in Table 3 (blow-up along a subsection of a $\mathbf{P}^1$-bundle).

The blow-up Y of W_4 at the vertex is in N° 36 of Table 2 and is a $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbf{P}^2$. Let $f: Y \to \mathbf{P}^2$ be the structure morphism and L the tautological line bundle. The inverse image $S \simeq \mathbf{P}^2$ of R_4 by $Y \to W_4$ is linearly equivalent to L. Thus $L \otimes \mathcal{O}_S \simeq \mathcal{O}_S(2)$.

From the exact sequence

$$0 \longrightarrow L \longrightarrow L^{\otimes 2} \longrightarrow \mathcal{O}_S(4) \longrightarrow 0,$$

one has a surjection $H^0(L^{\otimes 2}) \to H^0(\mathcal{O}_S(4))$. Indeed one has $H^1(L) = 0$ by

Kodaira vanishing because $L+(-K_Y)$ is ample. Thus any quartic C in $S \simeq R_4$ is an intersection of members of $|2S|$. Hence the blow-up X along smooth C is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.13 because $-K_Y \sim 2S+f^*\mathcal{O}(1)$. B_2, B_3, and $(-K_X)^3$ are calculated by Lemma 2.1. It is clear that X is parametrized by a non-empty rational variety.

(7.17) N^{os} 5, 21, 22 in Table 3 (blow-ups of $\mathbf{P}^1 \times \mathbf{P}^2$).

Let $H_1 = \mathcal{O}(1,0)$ and $H_2 = \mathcal{O}(0,1)$ on $Y = \mathbf{P}^1 \times \mathbf{P}^2$. One knows that $-K_Y \sim 2H_1 + 3H_2$.

Let us consider N° 22. Because there are divisors $A \sim H_1$ and $B \sim 2H_2$ such that $A \cdot B$ is the center C, and hence C is an intersection of members of $H_1 + 2H_2$, X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12. The rest for N° 22 in easy. Let us consider N° 21. Let C be a curve of bidegree (2, 1). The second projection $C \to \mathbf{P}^2$ is an isomorphism to a line l and C is a divisor of bidegree (1, 2) in $\mathbf{P}^1 \times l$ (cf. Introduction). Now it is easy to see that C's are parametrized by a non-empty rational variety and C is an intersection of members of $|H_1 + 2H_2|$ and one applies Proposition 2.12 to see that X is a Fano 3-fold and $-K_X$ has a free splitting. Invariants are calculated by Lemma 2.1. Let us consider N° 5. Let C be as in N° 5. Then the image of $C \to \mathbf{P}^2$ is a smooth conic q of $\mathbf{P}^2$ and C in $Q = \mathbf{P}^1 \times q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is a divisor of bidegree (1, 5). Then $\mathcal{O}_Q(\mathrm{H}_1 + 3H_2) \simeq \mathcal{O}(1,6)$, and from the exact sequence:

$$0 \longrightarrow \mathcal{O}(H_1 + H_2) \longrightarrow \mathcal{O}(H_1 + 3H_2) \longrightarrow \mathcal{O}_Q(\mathrm{H}_1 + 3H_2) \longrightarrow 0,$$

one sees the surjection $H^0(\mathcal{O}(H_1 + 3H_2)) \twoheadrightarrow H^0(\mathcal{O}_Q(H_1 + 3H_2))$. Indeed $H^1(\mathcal{O}(H_1 + H_2)) = 0$ by Kodaira vanishing because $H_1 + H_2 - K_Y \sim 3H_1 + 4H_2$ is ample. Thus $\mathrm{Tr}_Q|H_1 + 3H_2|$ is a complete linear system and C is an intersection of members of $|H_1 + 3H_2| = |-K_Y - H_1|$. Hence the blow-up X along C is a Fano 3-fold and $-K_x$ has a free splitting by Proposition 2.14. Other assertions about N° 5 are easy to check.

(7.18) N^{os} 7, 13, 24 in Table 3 (blow-ups of W).

W is a divisor of bidegree (1, 1) in $\mathbf{P}^2 \times \mathbf{P}^2$, and $-K_W \sim 2H_1 + 2H_2$, where $H_1 = \mathcal{O}_W \otimes \mathcal{O}(1,0)$ and $H_2 = \mathcal{O}_W \otimes \mathcal{O}(0,1)$. The center C of the blow-up is a complete intersection of two members of $|H_1 + H_2|$ in the case of N° 7. In the case of N° 24, C can be considered to be a fiber of the first projection, by symmetry. Then C is a complete intersection of two members of $|H_1|$. In these two cases, X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12 and other assertions are easy to check. Let us consider N° 13. Let C, p_1, p_2 be as in N° 13. By the condition on C, $q_2 = p_2(C)$ is a conic, and let $Q=p_2^{-1}(q_2)$. We claim that $\mathcal{O}_Q(C)\simeq\mathcal{O}_Q(H_1)$. Since $p_2|_Q$: $Q\to q_2\simeq P^1$ is a $\mathbf{P}^1$-bundle, $\mathcal{O}_Q(H_2)=(p_2|_Q)^*\mathcal{O}(1)$ and the tautological line bundle $\mathcal{O}_Q(H_1)$ form a basis for Pic Q. It is easy to check $(\mathcal{O}_Q(H_1)^2)=(\mathcal{O}_Q(H_1)\cdot\mathcal{O}_Q(H_2))=2$. Thus $\mathcal{O}_Q(C)\simeq\mathcal{O}_Q(H_1)$ as claimed because $(C\cdot H_1)=(C\cdot H_2)=2$. Since $K_Q\sim-2H_1$ by the adjunction formula, one has $H^i(\mathcal{O}_Q(H_1))=H^{2-i}(\mathcal{O}_Q(-3H_1))=0$ for $i=1$, 2 by Ramanujam vanishing theorem. Thus $h^0(\mathcal{O}_Q(H_1))=4$ by the Riemann-Roch formula. Hence the map

$$H^0(\mathcal{O}_W(H_1))\simeq k^3\to H^0(\mathcal{O}_Q(H_1))$$

is not surjective, and the generic $C'\in|\mathcal{O}_Q(H_1)|$ are not sent onto a line by p_1. Thus C's in N° 13 form an open dense subset of a $\mathbf{P}^3$-bundle $\bigcup_Q|\mathcal{O}_Q(H_1)|$ over $|Q|\simeq\mathbf{P}^5$. Since the invariants of the blow-up X is calculated by Lemma 2.1, it remains to show that X is a Fano 3-fold and $-K_X$ has a free splitting. By the exact sequence

$$0\longrightarrow\mathcal{O}_W(H_1)\longrightarrow\mathcal{O}_W(H_1+2H_2)\longrightarrow\mathcal{O}_Q(H_1+2H_2)\longrightarrow 0\,,$$

the map $H^0(\mathcal{O}_W(H_1+2H_2))\to H^0(\mathcal{O}_Q(H_1+2H_2))$ is a surjection, because $H^1(\mathcal{O}_W(H_1))=0$ by Kodaira vanishing theorem and ampleness of $H_1-K_W\sim 3H_1+2H_2$. Since $Q\sim 2H_2$, C is an intersection of members of $|H_1+2H_2|$ because $|H_1+2H_2-2H_2|$ and $|H_1+2H_2-H_1|$ are free from base points. Hence X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.13 applied to p_1: $W\to\mathbf{P}^2$.

(7.19) The rest of Table 3 (blow-ups of V_7).

In these cases, X is the blow-up of $\mathbf{P}^1$-bundle f: $Y=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbf{P}^2$ along subsections C. Since it is easy to calculate the invariants of X and check that X is parametrized by a non-empty rational variety in each case, we only show that X is a Fano 3-fold and $-K_X$ has a free splitting in each case. Let L be the tautological line bundle of Y, $H=f^*\mathcal{O}(1)$, and $D\simeq\mathbf{P}^2$ the negative section. Then one has $L\sim H+D$ and $-K_Y\sim 2L+2H$. It is clear that $|L|$ and $|H|$ are free from base points and $L+H$ is ample. In N° 11 (resp. N° 26), C is the complete intersection of two members of $|L+H|$ (resp. $|L|$), whence X is a Fano 3-fold and $-K_X$ has a free splitting for N^{os} 11 and 26 by Proposition 2.12. Let us consider N^{os} 16, 23, 30. Let g: $Y\to\mathbf{P}^3$ be the blow-up at $p\in\mathbf{P}^3$. The exceptional divisor is D, and $L\simeq g^*\mathcal{O}(1)$. Let $\bar{C}=g(C)$, and $I_{\bar{C}}$, I_C the sheaf of ideals of $\bar{C}$ and C in $\mathbf{P}^3$ and Y respectively. Since $\bar{C}$ is an intersection of quadrics and $(g^*I_{\bar{C}})\mathcal{O}_Y=I_C(-D)$, C is an intersection of members of $|2L-D|=|L+H|$. Thus X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12. Let us consider N° 29. C, in this case, is the complete intersection of D and the proper transform M of some hyperplane of $\mathbf{P}^3$ passing through $p \in \mathbf{P}^3$. Since $M \sim L-D \sim H$, C is an intersection of members of $|L+H|$ because $|L+H-D|=|2H|$ and $|L+H-M|=|L|$ are free from base points. Hence X is a Fano 3-fold and $-K_X$ has a free splitting again by Proposition 2.12. Now N° 14 is left, and let S be an in N° 14. Then $S \sim L$. Let $E \subset \mathbf{P}^3$ be the cone over $C \subset S$ with vertex p, and $E' \subset Y$ the proper transform by g. Then $E' \sim 3L-3D$. C is the complete intersection of $S \sim L$ and $E'+2D \sim 2L+H$. Thus C is an intersection of members of $|2L+H|$. Since C is a subsection of f, X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.13.

Let us consider Table 4.

(7.20) N^{os} 1 and 10 in Table 4 (a divisor and a product).

The only thing which is not clear is that $B_2(X)=4$ and $B_3(X)=2$ for N° 1, which follows from Proposition 7.13.

(7.21) N° 2 in Table 4 (a blow-up along a subsection).

Let $f\colon Y=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(1,1))\to\mathbf{P}^1\times\mathbf{P}^1$ (N° 31 in Table 3). Let L be the tautological line bundle and $S\in|L|$ a smooth member. Then X is the blow-up of Y along a smooth divisor C of bidegree (2, 2) of $S\simeq\mathbf{P}^1\times\mathbf{P}^1$. From the exact sequence

$$0 \longrightarrow \mathcal{O}(L) \longrightarrow \mathcal{O}(2L) \longrightarrow \mathcal{O}_S(2L) \longrightarrow 0\,,$$

one gets the surjectivity of the natural map $H^0(2L)\to H^0(\mathcal{O}_S(2L))\simeq H^0(\mathcal{O}_S(2,2))$ because $H^1(\mathcal{O}(L))=0$ by Kodaira vanishing theorem and ampleness of $L-K_Y$. Thus C is an intersection of members of $|2L|$. Since $-K_Y\sim 2L+f^*\mathcal{O}(1,1)$, X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.13. The rest follows from Lemma 2.1.

(7.22) N^{os} 3 and 8 in Table 4 (blow-ups of $\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$).

Let $Y=\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$, $H_1=\mathcal{O}_Y(1,0,0)$, $H_2=\mathcal{O}_Y(0,1,0)$, $H_3=\mathcal{O}_Y(0,0,1)$, and let $f\colon Y\to\mathbf{P}^1\times\mathbf{P}^1$ be the projection to the first two factors. Let $C\subset Y$ be the center of the blow-up. In N° 8, C is the complete intersection of two divisors $D_1\in|H_1|$ and $D_2\in|H_2+H_3|$, and hence an intersection of members of $|H_1+H_2+H_3|=|-K_Y/2|$. Thus X is a Fano 3-fold and $-K_X$ has a free splitting by Proposition 2.12 and the rest is easy for N° 8. In N° 3, $q=f(C)$ is a divisor of bidegree (1, 1) on $\mathbf{P}^1\times\mathbf{P}^1$ and C is a divisor of bidegree (2, 1) on $Q=q\times\mathbf{P}^1\neq\mathbf{P}^1\times\mathbf{P}^1$ (cf. Introduction). From the exact sequence

$$0 \longrightarrow \mathcal{O}(H_3) \longrightarrow \mathcal{O}(H_1+H_2+H_3) \longrightarrow \mathcal{O}_Q(2,1) \longrightarrow 0\,,$$

one obtains the surjectivity of $H^0(\mathcal{O}(H_1+H_2+H_3))\to H^0(\mathcal{O}_Q(2,1))$ because

$H^1(\mathcal{O}(H_3)) \simeq H^1(\mathcal{O}_{\mathbf{P}^1}(1)) = 0$. Thus C is the complete intersection of two divisors $Q \in |H_1 + H_2|$ and $D_3 \in |H_1 + H_2 + H_3|$. The rest of the argument is the same as that for N° 8.

(7.23) N° 4 in Table 4 (a successive blow-up of Q).

Invariants are calculated by Proposition 2.9. Since p and q are not colinear on quadric $Q \subset \mathbf{P}^4$, smooth conics C passing through p and q exist and are intersections of Q with linear 2-spaces ($\simeq \mathbf{P}^2$) of $\mathbf{P}^4$ passing through p and q. So C's are parametrized by a rational variety. We will see in (7.28) that $-K_X$ is ample and has a free splitting.

(7.24) N° 5 in Table 4 (a blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$).

Let $Y = \mathbf{P}^1 \times \mathbf{P}^2$, $H_1 = \mathcal{O}(1, 0)$, $H_2 = \mathcal{O}(0, 1)$, and C_1 and C_2 disjoint irreducible curves of bidegree $(2, 1)$ and $(1, 0)$ on Y respectively. Let $f: Y \to \mathbf{P}^2$ be the projection. Then $l = f(C_1)$ is a line disjoint from the point $p = f(C_2)$ in $\mathbf{P}^2$. In $L = f^{-1}(l) \simeq \mathbf{P}^1 \times \mathbf{P}^1$, C_1 is a divisor of bidegree $(1, 2)$ on it. We claim that the natural map

$$\text{(7.24.1)} \qquad \alpha: H^0(\mathcal{O}_Y(H_1 + 2H_2)) \longrightarrow H^0(\mathcal{O}_{L \amalg C_2}(H_1 + 2H_2))$$

is surjective. Because the natural map $\beta: H^0(\mathcal{O}_Y(H_1 + 2H_2)) \to H^0(\mathcal{O}_L(H_1 + 2H_2))$ is surjective by $H^1(\mathcal{O}_Y(H_1 + H_2)) = 0$, and the induced map

$$H^0(\mathcal{O}_Y(H_1 + H_2)) \simeq \operatorname{Ker} \beta \to H^0(\mathcal{O}_{C_2}(H_1 + 2H_2)) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(1)) \otimes \mathcal{O}_{\mathbf{P}^2}(2H_2)$$

is surjective by $Q \cap C_2 = \varnothing$. By surjectivity of α, there is a divisor $D \in |H_1 + 2H_2|$ such that $C_1 = D \cdot Q$ and $D \supset C_2$. Thus $C_1 \amalg C_2$ in D is an intersection of members of $\operatorname{Tr}_D |2H_2|$, and $C_1 \amalg C_2$ in Y is an intersection of members of $|H_1 + 2H_2|$. Thus $-K_X$ is ample and has a free splitting by Proposition 2.12. The rest for N° 5 is easy.

(7.25) N$^{\text{os}}$ 6 and 7 in Table 4.

The sum of 3 disjoint lines in $\mathbf{P}^3$ is an intersection of members of $|\mathcal{O}(3)|$ and the sum of two disjoint curves in $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(0, 1)$ and $(1, 0)$ is an intersection of members of $|\mathcal{O}_W(1, 1)|$. Thus, in these cases, $-K_X$ is ample and has a free splitting by Proposition 2.12. The only thing to be checked is that X in N° 6 is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ along a curve of tridegree $(1, 1, 1)$. In N° 6, X contains 3 exceptional divisors E_1, E_2, E_3 coming from lines, and the divisor H the pull back of $\mathcal{O}_{\mathbf{P}^3}(1)$. For each i, $|H - E_i|$ is free from base points and induces a morphism $f_i: X \to \mathbf{P}^1$ which is induced by the projection of $\mathbf{P}^3$ from the line. Now

$$f = (f_1, f_2, f_3): X \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

is a birational morphism because $(H-E_1)\cdot(H-E_2)\cdot(H-E_3)=(H^3)=1$. The assertion that f is the blow-up along a curve of tridegree (1, 1, 1) is shown by the argument in the proof of Proposition 7.5.

(7.26) N^{os} 9 and 12 in Table 4 (successive blow-ups of $\mathbf{P}^3$).

By Lemma 2.1, the only thing to be checked here is that $-K_X$ is ample and has a free splitting. Let m and n be two disjoint lines on $\mathbf{P}^3$, and p and q two distinct points on m. Let us blow up $\mathbf{P}^3$ first at p and then along the proper transform of m. Let $f\colon U\to\mathbf{P}^3$ (resp. $g\colon V\to U$) be the first (resp. second) blow-up and $h=f\circ g$. Let $H=h^*\mathcal{O}(1)$ and let P (resp. M) be the irreducible exceptional divisor on V lying over p (resp. dominating m). Let n' and q' be smooth rational curves $h^{-1}(n)$ and $h^{-1}(q)$ on V, respectively. Now X in N° 9 (resp. N° 12) is the blow-up of V along n' (resp. q'). It is easy to see that $-K_V\sim 4H-2P-M$ and both of n' and q' are intersections of members of $|H|$. Then $-K_X$ is ample and has a free splitting for N° 9 and 12 by Proposition 2.12 if $-K_V-H\sim 3H-2P-M$ is ample on V. Now $|H-P|$ (resp. $|H-M-P|$) is free from base points and induced by the linear projection of $\mathbf{P}^3$ from p (resp. m). Since

$$(7.26.1)\qquad -K_V-H\sim H+(H-P)+(H-M-P),$$

$|-K_V-H|$ is free from base points. Let $Z\subset V$ be an arbitrary irreducible curve. If $g(Z)$ is a point, then Z is an exceptional line of g and $(-K_V-H\cdot Z)=(-K_V\cdot Z)=1$. If $g(Z)$ is a curve and $h(Z)$ is a point, then $g(Z)$ lies in the exceptional divisor $\bar{P}=g(P)$ of f and

$$(-K_V-H\cdot Z)\geqq(f^*\mathcal{O}(1)-\bar{P}\cdot g_*Z)=(-\bar{P}\cdot g_*Z)>0$$

by (7.29.1) and (2.9.2). If $h(Z)$ is a curve, then $(-K_V-H\cdot Z)\geqq(H\cdot h_*Z)>0$ by (7.26.1). Thus $-K_V-H$ is ample.

(7.27) N° 11 in Table 4.

Let t and e be as in N° 11 in Table 4, and $f\subset\mathbf{F}_1$ a fiber of $\mathbf{F}_1\to\mathbf{P}^1$. Since $t\times e=(t\times\mathbf{F}^1)$. $(\mathbf{P}^1\times e)$, $t\times e$ is an intersection of members of $|t\times\mathbf{F}_1+\mathbf{P}^1\times(e+f)|$ because $|f|$ and $|e+f|$ are free from base points. It is easy to see that $-K_{F_1}-(e+f)\sim e+2f$ is ample. Thus the blow-up X along $t\times e$ is a Fano 3-fold by Proposition 2.12. The rest is easy.

We consider Table 5. N° 3 and the cases $6\leqq B_2\leqq 10$ are obvious, and we consider N^{os} 1 and 2.

(7.28) N° 1 in Table 5 (with N° 4 in Table 4).

Let C be a smooth conic on a quadric $Q\subset\mathbf{P}^4$, and let $p_1,\cdots,p_n$ be distinct n points on C ($n=2, 3$). Let $f_n: Z_n \to Q$ be the blow-up of Q at these n points, and let $H_n = f_n^* \mathcal{O}_Q(1)$, $P_i = f_n^*(p_i)$ $(1 \leqq i \leqq n)$, and $C_n' \subset Z_n$ the proper transform of C by f_n. X in N° 1 in Table 5 (resp. N° 4 in Table 4) is obtained as the blow-up $g_n: X_n \to Z_n$ along C_n for $n=3$ (resp. 2) by Remark 2.5. We need only to show that X_2 and X_3 are Fano 3-folds and their anti-canonical bundles have free splittings. Let $n=2$ or 3. Since C is an intersection of members of $|\mathcal{O}_Q(1)|$, C_n is an intersection of members of $|H_n - P_1 - \cdots - P_n|$. Since $-K_{Z_n} \sim 3H_n - 2P_1 - \cdots - 2P_n$, X_n is a Fano 3-fold and $-K_{X_n}$ has a free splitting by Proposition 2.12 if $M_n \sim 2H_n - P_1 - \cdots - P_n$ is ample on Z_n. Since $p_1, \cdots, p_n \in C$, no two of $p_1, \cdots, p_n$ are colinear on Q by Corollary 6.9. Hence as in (7.11.1), $|H_n - P_i - P_j|$ is free from base points if $i \neq j$. Then by

$$(7.28.1) \qquad \begin{cases} M_2 \sim H_2 + (H_2 - P_1 - P_2), \\ 2M_3 \sim H_3 + (H_3 - P_1 - P_2) + (H_3 - P_2 - P_3) + (H_3 - P_3 - P_1), \end{cases}$$

$|2M_n|$ is free from base points. For ampleness, it is enough to show $(M_n \cdot l) > 0$ for an arbitrary irreducible curve l on Z_n. If $g_n(l)$ is a point, then $(M_n \cdot l) > 0$ by g_n-ampleness of M_n (2.9.2). If $g_n(l)$ is not a point, then $(2M_n \cdot l) \geqq (\mathcal{O}_Q(1) \cdot g_{n*} l) > 0$ by (7.28.1).

(7.29) N° 2 in Table 5.

Let $\phi: Y \to \mathbf{P}^3$ be the blow-up along two disjoint lines m_1 and m_2, and let $E_i = \phi^{-1}(m_i)$ $(i=1, 2)$ and $H = \phi^* \mathcal{O}(1)$. Assume that the two exceptional liens l and l' are in E_1 and let $p = \phi(l)$ and $p' = \phi(l')$. Since $\{p, p'\}$ is an intersection of members of $|\mathcal{O}(2)|$, $l \amalg l'$ is an intersection of members of $|2H|$. Since $-K_Y \sim 4H - E_1 - E_2$, $-K_Y - 2H \sim (H - E_1) + (H - E_2)$. Y has a structure of a $\mathbf{P}^1$-bundle over $\mathbf{P}^1 \times \mathbf{P}^1$, and the pull backs of $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ are $H - E_1$ and $H - E_2$ (7.10). Thus $-K_Y - 2H$ is the pull back of the ample divisor $\mathcal{O}(1, 1)$ and E_1 is a section of the $\mathbf{P}^1$-bundle (7.10). Hence $-K_X$ is ample and has a free splitting by Proposition 2.13. The rest for N° 2 is clear.

Remark 7.30. Now we obtained, for $X \not\simeq \mathbf{P}^1 \times S_d$ $(1 \leqq d \leqq 7)$, the splitting or the free splitting of $-K_X$ as stated in (7.0) during the proof of ampleness of $-K_X$ using Propositions 2.12, 2.13, and 2.14. One has a splitting or a free splitting of $-K_X$ for $X \simeq \mathbf{P}^1 \times S_d$ because X is a product.

(7.31) We now show that different classes in Tables 2, 3, 4, 5 are not deformation equivalent. To be exact, let $k = \mathbf{C}$ and let $f: V \to S$ be a proper smooth holomorphic map between connected complex varieties. If V_s and $V_{s'}$ are Fano 3-folds in Tables 2, 3, 4, 5 for some s and s' of S, then we show that V_s and $V_{s'}$ belong to the same class. Since $B_2(X)$, $B_3(X)$, $(-K_X)^3$ are deformation invariants, we need to check the inequivalence for each of the following sets {N^{os} 22 and 24}, {N^{os} 30 and 31}, {N^{os} 33 and 34} in Table 2; {N^{os} 17 and 18}, {N^{os} 19, 20, and 21}, {N^{os} 23 and 24}, {N^{os} 27 and 28}, {N^{os} 29 and 30} in Table 3; {N^{os} 4 and 5} in Table 4; {N^{os} 2 and 3} in Table 5.

For these, we use finer deformation invariants: free $\mathbf{Z}$-module $H^2(X, \mathbf{Z}(1))$ with an element, the class of K_X, and the cubic intersection form. For a Fano 3-fold X, one has $H^2(X, \mathbf{Z}(1)) \simeq \operatorname{Pic} X$.

(7.32) N^{os} 33 and 34 in Table 2 and N^{os} 27 and 28 in Table 3.

X in N° 33 has a divisor H (pull back of $\mathcal{O}_{\mathbf{P}^3}(1)$) such that $(H^3)=1$, while one has $(\mathcal{O}(a, b))^3 = 3ab^2 \neq 1$ for $X = \mathbf{P}^1 \times \mathbf{P}^2$ in N° 34. One has $(\mathcal{O}(a, b, c))^3 = 6abc \neq 3$ for $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ in N° 27, while X in N° 28 has a divisor H (pull back of $\mathcal{O}(1, 1)$ by $X \to \mathbf{P}^1 \times \mathbf{P}^2$) such that $(H^3)=3$. Thus these are inequivalent.

(7.33) To distinguish the rest, we define the following invariant $d(X)$. Let $r = B_2(X)$ and $D_1, \cdots, D_r$ a $\mathbf{Z}$-basis for $\operatorname{Pic} X$. Let $M(D_1, \cdots, D_r)$ be the $r \times r$ matrix whose (i,j)-entry $M(D_1, \cdots, D_r)_{i,j}$ is $(-K_X \cdot D_i \cdot D_j)$. Then $d(X) = \det M(D_1, \cdots, D_r) \in \mathbf{Z}$ does not depend on the choice of $D_1, \cdots, D_r$ and is a deformation invariant. For the actual calculation, we need the following lemma.

Lemma 7.34. *Let $f: X \to Y$ be the blow-up of Y along an irreducible smooth curve C (resp. at a point p) and D the exceptional divisor. Let $E_1, \cdots, E_r$ be a $\mathbf{Z}$-basis for* Pic Y, *and let $D_i = f^*(E_i)$ $(i=1, \cdots, r)$ and $D_{r+1} = D$. Then $D_1, \cdots, D_{r+1}$ is a basis for* Pic X *and*

$$M(D_1, \cdots, D_{r+1})_{i,j} = \begin{cases} M(E_1, \cdots, E_r)_{i,j} & \text{if } i,j \leqq r \\ (E_j \cdot C) \ (\text{resp. } 0) & \text{if } i=r+1, \ j \leqq r \\ (E_i \cdot C) \ (\text{resp. } 0) & \text{if } i \leqq r, \ j=r+1 \\ 2p_a(C)-2 \ (\text{resp. } -2) & \text{if } i=j=r+1 . \end{cases}$$

The proof follows easily by projection formula and Lemma 2.1 and Proposition 2.9.

(7.35) By Lemma 7.33 and Theorem 3.8, one can easily calculate $d(X)$ and the result is as follows.

Table 2	$d(X) = -24$	for N° 22,	-21	for N° 24
	-12	N° 30,	-13	N° 31

Table 3	$d(X)=$ 28	N° 17,	26	N° 18		
	24	N° 19,	28	N° 20,	22 for N° 21	
	20	N° 23,	22	N° 24		
	12	N° 29,	14	N° 30		
Table 4	−40	N° 4,	−39	N° 5		
Table 5	44	N° 2,	48	N° 3.		

Thus we have shown that different classes are inequivalent.

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Representations of Weyl Groups on Zero Weight Spaces of $\mathfrak{g}$-Modules

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Introduction

The subject of this article is to analyze the representation of the Weyl group on the zero weight space of a finite-dimensional representation of a complex simple Lie group.

B. Kostant [5] and E. A. Gutkin [3] has dealt with this subject. B. Kostant established a relationship between the eigenvalues of the Coxeter elements as acting on the zero weight space and his "generalized exponents". E. A. Gutkin treated the type A and converted the problem into a decomposition problem of certain induced representations of symmetric groups. They both gave a set of irreducible $SL(n, \mathbf{C})$-modules whose zero weight spaces form a complete set of representatives of the irreducible modules over the Weyl group.

In this article, we give a method of computation to decompose the zero weight space into irreducible modules over the Weyl group. In this method, we employ the representations on the symmetric tensors over the natural space. We apply this method to types A, B, and C. For type D, although a similar treatment leads to quite complicated computations, one of us (S. Ariki) solved this case [1]. His results will be published elsewhere.

Moreover, we investigate the type A more closely. We provide some interpretations from the view point of representations of general linear groups and symmetric groups. One of them has already been obtained in [3]. Next we ascribe the character values on the zero weight space to those of Coxeter elements. Finally we collect some auxiliary formulas to help computation, and give a new series of examples of irreducible $SL(n, \mathbf{C})$-modules whose zero weight spaces become irreducible.

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