

Fourier Functor and its Application to the Moduli of Bundles on an Abelian Variety

Shigeru Mukai

At the symposium we talked on the vector bundles on a $K3$ surface and applications to the geometry of a $K3$ surface. Most content of our talk is contained in the paper "On the moduli space of bundles on $K3$ surfaces, I" to appear in the proceeding of the symposium on vector bundles at Tata Institute in 1984. In this article we discuss the vector bundles on an abelian variety instead.

In [12], we have defined the Fourier functor and shown its basic properties. This functor is a powerful tool for investigating the vector bundle (or coherent sheaves, more generally) on an abelian variety as we have shown for the Picard bundles in [12]. In this article, generalizing the results in [12], we shall show that a sheaf and its Fourier transform have the same local (in the Zariski topology) moduli space and apply this to the study of the moduli space of vector bundles on an abelian variety X . In Section 1, we shall show that the moduli space of the Picard bundles is non-reduced in the case X is the Jacobian variety of a hyperelliptic curve of genus ≥ 3 . In the remaining sections, we shall mainly study the sheaves of U -type, which were first studied in [20] over an abelian surface.

Definition 0.1. Let (X, ℓ) be a principally polarized abelian variety, i.e., ℓ is an algebraic equivalence class of ample line bundles with Euler Poincaré characteristic 1. A sheaf E on X is of U -type if there exists a homomorphism $f: L^{-1} \rightarrow H$ from a line bundle L^{-1} in the class $-\ell$ to a homogeneous vector bundle H such that $\text{Hom}(f, P): \text{Hom}_{\sigma_x}(H, P) \rightarrow \text{Hom}_{\sigma_x}(L^{-1}, P)$ is injective for every $P \in \text{Pic}^0 X$ and E is isomorphic to the cokernel of f . (A vector bundle H is homogeneous if and only if there exists a filtration $0 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$ such that $H_i/H_{i-1} \in \text{Pic}^0 X$ for every $i = 1, \dots, n$, cf. Theorem 4.17 in [11] and Section 3 in [12].)

We shall show in Section 2 that a sheaf of U -type is simple and the isomorphism classes of rank r sheaves of U -type form an open subset isomorphic to $X \times \text{Hilb}^{r+1} X$ in the moduli space of simple sheaves, where $\text{Hilb}^{r+1} X$ is the Hilbert scheme of 0-dimensional subschemes of length

$r+1$ of X . So the moduli space of sheaves of U -type is connected by [5] but reducible if $\dim X \geq 3$ and r is sufficiently large by [7], though the rank r sheaves of U -type have the same Chern classes and are stable if X is the Jacobian variety of a curve.

In the case $\dim X = 2$, the moduli space of stable sheaves is smooth as we have shown in [13] (we shall give another proof of this fact in Section 3) and seems to be irreducible if we fix the rank and the Chern classes. Let $M(r, n\ell, \chi)$ be the moduli space of stable (with respect to L) sheaves E with rank r , $c_1(E) \cong n\ell$ and $\chi(E) = \chi$. Every connected component of $M(r, n\ell, \chi)$ is a smooth $2\lambda + 2$ dimensional variety, where $\lambda = n^2 - r\chi$. In [12], we have shown that $M(r, \ell, 0)$ is isomorphic to $X \times X$ when (X, ℓ) is not of product type. In [20], Umemura has shown that $M(r, \ell, -1)$ has a component whose general member is of U -type and which is birationally equivalent to the product $X \times S^{r+1}X$ of X and the $(r+1)$ -st symmetric product $S^{r+1}X$. In Sections 4 and 5, we shall prove the following:

Theorem 0.2. *For every principally polarized abelian surface (X, ℓ) , the moduli space $M(r, \ell, -1)$ is irreducible.*

Theorem 0.3. *Assume that a principally polarized abelian surface (X, ℓ) is not of product type. Then a sheaf belongs to $M(r, \ell, -1)$ if and only if it is of U -type. In particular, $M(r, \ell, -1)$ is isomorphic to $X \times \text{Hilb}^{r+1}X$.*

Notation. All varieties are considered to be over an algebraically closed field k . A sheaf E on X is a coherent \mathcal{O}_X -module. $r(E)$ is the rank of E at the generic point of X . $c_i(E)$ is the i -th Chern class and $\chi(E)$ is the Euler-Poincaré characteristic of E , that is, $\chi(E) = \sum_i (-1)^i \dim H^i(X, E)$. A subsheaf F of E is a *subbundle* if the quotient E/F is torsion free.

§ 1. Complement to [12]

In [12], we have defined the Fourier functor and shown that it gives an equivalence between the derived categories of coherent sheaves on an abelian variety and its dual. In this section we generalize this to an abelian scheme and study the relation between the numerical invariants of a sheaf and its Fourier transform.

Let A be an abelian scheme over a scheme T and \hat{A} its dual abelian scheme. Since A/T has a section, there exists a Poincaré line bundle \mathcal{P} on $A \times_T \hat{A}$. We assume that \mathcal{P} is normalized, i.e., both $\mathcal{P}|_{0 \times \hat{A}}$ and $\mathcal{P}|_{A \times 0}$ are trivial. Let \hat{S} be the left exact functor on \mathcal{O}_A -modules M into the category of $\mathcal{O}_{\hat{A}}$ -modules such that

$$\hat{S}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M)$$

where $\pi_A: A \times_T \hat{A} \rightarrow A$ and $\pi_{\hat{A}}: A \times_T \hat{A} \rightarrow \hat{A}$ are projections. Interchanging the role of A and \hat{A} , we obtain the functor S of $\mathcal{O}_{\hat{A}}$ -modules into the category of \mathcal{O}_A -modules. Let $RS^{\hat{}}$ and RS be the derived functors of $S^{\hat{}}$ and S , respectively. $RS^{\hat{}}$ is a functor on the derived category $D(A)$ to the derived category $D(\hat{A})$. Then we have:

Theorem 1.1. *Let $\omega_{A/T}$ (resp. $\omega_{\hat{A}/T}$) be the relative canonical line bundle of A/T (resp. \hat{A}/T). Then we have isomorphisms of functors*

$$RS \circ RS^{\hat{}} \cong (-1_A)^* \circ (\otimes \omega_{A/T}^{-1})[-g]$$

and

$$RS^{\hat{}} \circ RS \cong (-1_{\hat{A}})^* \circ (\otimes \omega_{\hat{A}/T}^{-1})[-g],$$

where g is the relative dimension of A/T .

We have proved this in the case $T = \text{Spec}(k)$ in [12] Theorem 2.2. For our purpose in this section, the case A/T is trivial (or product type) is sufficient. In this case, the theorem says that $RS \circ RS^{\hat{}} \cong (-1_A)^*[-g]$, which is a corollary of the proof of Theorem 2.2 in [12]. The above generalized form is due to Moret-Bailly. For the proof we need the following proposition which was essentially proved in Section 13 [14].

Proposition 1.2. *Let $f: X \rightarrow Y$ be a proper flat integral morphism. Let F and G be vector bundles on X and Z the maximal subscheme of Y over which F and G are isomorphic to each other (see Section 1 [11] and Section 10 [14] for the definition of Z). Assume that G is f -simple, i.e., the natural homomorphism $\mathcal{O}_Y \rightarrow f_* \mathcal{E}nd_{\mathcal{O}_X}(G)$ is universally an isomorphism, that $H^i(X_y, \mathcal{H}om_{\mathcal{O}_y}(G_y, F_y))$ vanishes for all i and $y \in Y - Z$ and that $\text{depth}_{\mathcal{O}_y} \mathcal{F}_Z = n = \dim X/Y$. Then we have*

$$R^i f_* \mathcal{H}om_{\mathcal{O}_X}(G, F) \begin{cases} = 0 & i \neq n \\ \cong \mathcal{E}xt_{\mathcal{O}_Y}^n(\mathcal{O}_Z, \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} L^{-1} & i = n \end{cases}$$

and $\mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_Z, \mathcal{O}_Y) = 0$ for every $i \neq n$, where L is a line bundle on Z such that $G_Z \cong F_Z \otimes f_{Z*} L$ and $L^{-1} \cong f_{Z,*} \mathcal{H}om_{\mathcal{O}_Z}(G_Z, F_Z)$.

Proof. By (7.7.6) EGA III, there exists a coherent \mathcal{O}_Y -module L and an isomorphism of functors on quasi-coherent \mathcal{O}_Y -modules M

$$f_* (\mathcal{H}om_{\mathcal{O}_X}(G, F) \otimes_{\mathcal{O}_Y} M) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(L, M).$$

By definition, Z is the subscheme defined by the annihilator ideal of L . Since G is f -simple, L is an invertible \mathcal{O}_Z -module (Lemma 1.6 [11]) and we

have $L^{-1} \cong f_{z,*} \mathcal{H}om_{\mathcal{O}_Z}(G_Z, F_Z)$ by putting $M = \mathcal{O}_Z$ in the above isomorphism of functors. In the case Y is affine, there exists a complex

$$K_\bullet = [0 \rightarrow K_0 \xrightarrow{\alpha_0} K_1 \rightarrow \dots \rightarrow K_n \rightarrow 0]$$

of finitely generated projective \mathcal{O}_Y -modules and an isomorphism of functors on \mathcal{O}_Y -modules M

$$R^i f_* (\mathcal{H}om_{\mathcal{O}_X}(G, F) \otimes_{\mathcal{O}_Y} M) \cong H^i(K_\bullet \otimes_{\mathcal{O}_Y} M)$$

by the theorem in Section 5 [14]. By our assumption, there is a regular sequence x_1, \dots, x_n in \mathcal{S}_Z at any point $z \in Z$ and $H^i(K_\bullet)$'s are annihilated by a power of (x_1, \dots, x_n) . Hence so are the cohomologies $H^i(K_\bullet^\vee)$'s of the dual complex K_\bullet^\vee of K_\bullet . Therefore, by the same argument as in the lemma in Section 13 [14], the following sequence is exact:

$$0 \rightarrow K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_n \rightarrow R^n f_* \mathcal{H}om_{\mathcal{O}_X}(G, F) \rightarrow 0$$

and $\mathcal{E}xt_{\mathcal{O}_Y}^i(R^n f_* \mathcal{H}om_{\mathcal{O}_X}(G, F), \mathcal{O}_Y)$ vanishes for every $i \neq n$. By the universal property of the complex K_\bullet , we have an isomorphism of coherent \mathcal{O}_Y -modules M :

$$\begin{aligned} f_*(\mathcal{H}om_{\mathcal{O}_X}(G, F) \otimes_{\mathcal{O}_Y} M) &\cong \text{Ker} [K_0 \otimes M \xrightarrow{\alpha_0 \otimes 1} K_1 \otimes M] \\ &\cong \text{Ker} [\mathcal{H}om_{\mathcal{O}_Y}(K_0^\vee, M) \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(K_1^\vee, M)] \\ &\cong \mathcal{H}om_{\mathcal{O}_Y}(\text{Coker } \alpha_0^\vee, M). \end{aligned}$$

Therefore we have an isomorphism

$$L \cong \text{Coker } \alpha_0^\vee \cong \mathcal{E}xt_{\mathcal{O}_Y}^n(R^n f_* \mathcal{H}om_{\mathcal{O}_X}(G, F), \mathcal{O}_Y).$$

Since these isomorphisms are canonical, they exist in the case Y is general, too. Since K_\bullet^\vee gives a projective resolution of L , we have

$$\mathcal{E}xt_{\mathcal{O}_Y}^i(L, \mathcal{O}_Y) \begin{cases} = 0 & i \neq n \\ \cong R^n f_* \mathcal{H}om_{\mathcal{O}_X}(G, F) & i = n. \end{cases} \quad \text{q.e.d.}$$

Proof of Theorem 1.1. The composite $RS \circ R\hat{S}$ is the integral functor defined by the kernel complex

$$Rp_{12,*}(p_{13,*} \mathcal{P} \otimes p_{23,*} \mathcal{P}) \in D(A \times_T A),$$

where p_{ij} 's are the projections of $A \times_T A \times_T \hat{A}$ onto the (i, j) -th factors. Since $p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}|_{p_{13}^{-1}(a_1, a_2)} \cong \mathcal{P}|_{\tau_Z^{-1}(a_1 + a_2)}$ for every $(a_1, a_2) \in A \times_T A$, we have $p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P} \cong (m \times 1)^* \mathcal{P} \otimes_{\mathcal{O}_T} M$ for some line bundle M on T , where m :

$A \times_T A \rightarrow A$ is the group law of A/T . Since the restrictions of $p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}$ and $(m \times 1)^* \mathcal{P}$ to $0 \times 0 \times \hat{A}$ are both trivial, the line bundle M is trivial and we have

$$\begin{aligned} R p_{12,*} (p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}) &\cong R p_{12,*} ((m \times 1)^* \mathcal{P}) \\ &\cong m^* R \pi_{A,*} \mathcal{P}. \end{aligned}$$

Since A is the neutral component of the Picard scheme of \hat{A}/T , the maximal subscheme over which \mathcal{P} is trivial is the 0-section $s(T)$ of A/T . Hence, by Proposition 1.2, we have $R^i \pi_{A,*} \mathcal{P} = 0$ for every $i \neq g$ and $R^g \pi_{A,*} \mathcal{P}$ is isomorphic to the determinant of the normal bundle N of the 0-section $s(T)$ in A . Since $s: T \rightarrow A$ is a section and since A/T is a group scheme, the relative canonical line bundle $\omega_{A/T}$ is isomorphic to the pullback of $s^* N^{-1}$. Hence we have

$$m^* R^g \pi_{A,*} \mathcal{P} \cong \mu_* \omega_{A/T}^{-1}$$

and

$$RS \circ R\hat{S} = (-1_A)^* \circ (\otimes \omega_{A/T}^{-1})[-g],$$

where

$$\mu = (1, -1): A \rightarrow A \times_T A, \mu(a) = (a, -a). \quad \text{q.e.d.}$$

The Fourier functor commutes with base change in the derived categories.

Proposition 1.3. *Let $f: T' \rightarrow T$ be a morphism. Then the following diagram is quasi-commutative, where L denotes the left derived functors:*

$$\begin{array}{ccc} D(\hat{A}) & \xrightarrow{\quad RS \quad} & D(A) \\ L(1_{\hat{A}} \times_T f)^* \downarrow & & \downarrow L(1_A \times_T f^*) \\ D(\hat{A} \times_T T') & \xrightarrow{\quad RS' \quad} & D(A \times_T T') \end{array}$$

Proof. Put $A' = A \times_T T$, $\hat{A}' = \hat{A} \times_T T'$, $\mathcal{P}' = \mathcal{P} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$, $f_{\hat{A}} = f \times_T \hat{A}: \hat{A}' \rightarrow \hat{A}$ and $f_A = f \times_T A: A' \rightarrow A$. Then $RS'(\?) = R\pi_{A',*}(\mathcal{P}' \otimes \pi_{\hat{A}}^* \?)$, where $\?$ is an object or morphism in $D(\hat{A})$. Hence we have

$$\begin{array}{ccc} A' & \xleftarrow{\pi_{A'}} & A' \times_{T'} \hat{A}' = (A \times_T \hat{A}) \times_T T' \\ \downarrow & & \downarrow \tau \\ A & \xleftarrow{\quad} & A \times_T \hat{A} \end{array}$$

$$\begin{aligned}
 (RS' \circ Lf_{\hat{\lambda}}^*)(?) &\cong R\pi_{A,*}(\mathcal{P}' \otimes \pi_{\hat{\lambda}}^* Lf_{\hat{\lambda}}^*?) \\
 &\cong R\pi_{A',*}(\mathcal{P}' \otimes L(f_{\hat{\lambda}} \circ \pi_{\hat{\lambda}})^*?) \\
 &\cong R\pi_{A',*}(\tau^* \mathcal{P} \otimes L\tau^*(L\pi_{\hat{\lambda}}^*?)) \\
 &\cong R\pi_{A',*}L\tau^*(\mathcal{P} \otimes \pi_{\hat{\lambda}}^*?) \\
 &\cong Lf_{\hat{\lambda}}^* R\pi_{A,*}(\mathcal{P} \otimes \pi_{\hat{\lambda}}^*?) \\
 &= (Lf_{\hat{\lambda}}^* \circ RS)(?)
 \end{aligned}$$

where $\tau = f_{A \times_T \hat{A}}$ and $?$ is an object or a morphism in $D(\hat{A})$. Therefore we have

$$RS' \circ Lf_{\hat{\lambda}}^* \cong Lf_{\hat{\lambda}}^* \circ RS. \qquad \text{q.e.d.}$$

Definition 1.4. Let M be a coherent \mathcal{O}_A -module. The *weak index theorem* (*W. I. T.* for short) holds for M and its index is equal to i_0 if $R^i \hat{S}(M) = 0$ for every $i \neq i_0$. The $\mathcal{O}_{\hat{A}}$ -module $R^i \hat{S}(M)$ is called the *Fourier transform* of M and denoted by \hat{M} .

The proof of the following proposition is similar to the case $T = \text{Spec } k$.

Proposition 1.5. *Let M be as above. Then \hat{M} is coherent, W. I. T. holds for \hat{M} and its index is $g - i_0$. Moreover, \hat{M} is isomorphic to $(-1_A)^* M \otimes \omega_{A/T}^{-1}$.*

Our next goal is to prove the following theorem.

Theorem 1.6. *Let M be a T -flat coherent \mathcal{O}_A -module. Assume that T is noetherian.*

(1) *If W. I. T. holds for $M \otimes_{\mathcal{O}_T} k(t_0)$ and its index is i_0 , then there is an open neighbourhood U of $t_0 \in T$ such that W. I. T. holds for $M \otimes_{\mathcal{O}_T} k(t)$ and its index is i_0 for every $t \in U$. In other words, "W. I. T. holds" is an open condition.*

(2) *If W. I. T. holds for all $M \otimes_{\mathcal{O}_T} k(t)$, $t \in T$, and their indices are equal to i_0 , then*

- a) *W. I. T. holds for M and its index is i_0 ,*
- b) *The Fourier transform \hat{M} is flat over T ,*
- c) *For every T -scheme T' , $M \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$ satisfies a) and b) and we have $(M \otimes_{\mathcal{O}_T} \mathcal{O}_{T'})^\wedge \cong \hat{M} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$.*

Proposition 1.7. *Let $f: X \rightarrow Y$ be a proper morphism of T -schemes and F a Y -flat coherent \mathcal{O}_X -module. Assume that Y is flat and of finite type over T and that T is noetherian.*

- (1) *The following conditions are equivalent:*
 - i) $R^i f_{i_0,*}(F \otimes_{\mathcal{O}_T} k(t_0)) = 0$ for every $i \neq i_0$,
 - ii) *There is a neighbourhood of $p^{-1}(t_0)$ over which $R^i f_* F$ is zero for every $i \neq i_0$ and T -flat for $i = i_0$, where $p: Y \rightarrow T$ is the structure morphism.*
- (2) *If the equivalent conditions i) and ii) are satisfied for every $t_0 \in T$, then $R^i f_{T',*}(F \otimes_{\mathcal{O}_{T'}})$ is zero for every $i \neq i_0$ and T' -flat for $i = i_0$ for every T -scheme T' .*

Proof. Since direct image is compatible with flat base change and since flatness is an open condition, we may assume that T and Y are spectra of local rings A and B , respectively. By virtue of EGA III (6.10.5) (or [8] Section 5), there exist a complex K' of finite free B -modules and a functorial isomorphism $H^i(Y, F \otimes_B M) \xrightarrow{\sim} H^i(K' \otimes_B M)$ for every i and B -module M . Hence by the lemma below i) and ii) are equivalent and if these equivalent conditions are satisfied, then $H^i(K')$ is A -flat for every i . Hence

$$H^i(Y, F \otimes_A C) \xrightarrow{\sim} H^i(K' \otimes_A C) \cong H^i(K') \otimes_A C \xleftarrow{\sim} H^i(Y, F) \otimes_A C$$

for every A algebra C , which implies (2).

Lemma 1.8. *Let K' be a complex of A -flat finite B -modules bounded on both sides. Let i_0 be a fixed integer. Then $H^i(K' \otimes_A k) = 0$ for every $i \neq i_0$ if and only if $H^i(K')$ is zero for every $i \neq i_0$ and A -flat for $i = i_0$.*

Artin's Lemma. *Let $A \rightarrow B$ be a homomorphism of noetherian local rings and $f: M \rightarrow N$ a homomorphism of finite B -modules with N A -flat. Then $f \otimes_A k$ is injective if and only if f is injective and the cokernel of f is A -flat.*

For the proof see Section 20 in [10] or EGA 0 III (10.2.4).

Sublemma. *Assume that $H^j(K')$ is A -flat for every $j > i$. Then*

- (1) $H^i(K' \otimes_A k) = 0$ if and only if $H^i(K') = 0$.
- (2) $H^{i-1}(K' \otimes_A k) = 0$ if and only if $H^{i-1}(K') = 0$ and $H^i(K')$ is A -flat.

Proof. Let M (resp. N) be the kernel (resp. the image) of the homomorphism $K^i \rightarrow K^{i+1}$. Both M and N are A -flat finite B -modules and we have the exact sequence $K^{i-1} \rightarrow M \rightarrow H^i(K')$. Since N is A -flat, $M \otimes_A k \rightarrow K^i \otimes_A k$ is injective. Therefore $H^i(K' \otimes_A k) \cong \text{Coker}[K^{i-1} \otimes_A k \rightarrow M \otimes_A k] \cong H^i(K') \otimes_A k$. Since $H^i(K')$ is a finite B -module, $H^i(K' \otimes_A k) = 0$ if and only if $H^i(K') = 0$ by Nakayama's lemma, which shows (1). Let L be the cokernel of $K^{i-1} \rightarrow K^i$. We have the exact sequence

$$0 \rightarrow H^{i-1}(K') \rightarrow L \xrightarrow{f} M \rightarrow H^i(K') \rightarrow 0.$$

Since $M \otimes_A k \rightarrow K' \otimes_A k$ is injective, the kernel of $f \otimes_A k$ is isomorphic to $H^{i-1}(K' \otimes_A k)$. Hence (2) follows from Artin's lemma. q.e.d.

Proof of Lemma 1.8. Assume that $H^i(K' \otimes_A k) = 0$ for every $i \neq i_0$. Let i be an index such that $H^j(K')$ is A -flat for every $j > i$. Such an i exists because K' is bounded. By the sublemma $H^i(K') = 0$ if $i \neq i_0$ and $H^{i_0}(K')$ is A -flat. Hence $H^j(K')$ is A -flat for every $j > i - 1$. By descending induction on i , $H^i(K')$ is A -flat for every i and zero for every $i \neq i_0$.

Assume that $H^i(K')$ is zero for every $i \neq i_0$ and A -flat for $i = i_0$. Then, by the sublemma, $H^i(K' \otimes_A k) = 0$ for every $i \neq i_0$. q.e.d.

Proof of Theorem 1.6. Apply Proposition 1.7 to $f: A \times_{\tau} \hat{A} \rightarrow \hat{A}$ and $F = \mathcal{P} \otimes_{e_A} M$. Since $p: \hat{A} \rightarrow T$ is proper, as a neighbourhood in the condition ii), we can take one of the form $p^{-1}(U)$, U a neighbourhood of t_0 . Hence W.I.T. holds for $M \otimes_{e_{\tau}} k(t_0)$ if and only if W.I.T. holds for M_U and $(M_U)^\wedge$ is T -flat for some neighbourhood U of t_0 . (2.c) follows from (2) of Proposition 1.7 and Proposition 1.2. The remaining part is straightforward. q.e.d.

Let X be an analytic space or a scheme over an algebraically closed field and F a coherent \mathcal{O}_X -module. Let \mathcal{C} be the category of artinian local rings (A, m) over k and define the functor \mathcal{D}_F on \mathcal{C} by

$$\mathcal{D}_F(A) = \{(\alpha, \mathcal{F}) \mid \mathcal{F} \text{ is an } A\text{-flat } \mathcal{O}_{X_A}\text{-module and } \alpha \text{ is an isomorphism between } F \text{ and } \mathcal{F} \otimes_A k\} / \sim,$$

where $k = A/m$ is the residue field and $(\alpha, \mathcal{F}) \sim (\alpha', \mathcal{F}')$ if and only if there exists an isomorphism $\varphi: \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ such that $\alpha' \circ (\varphi \otimes_A k) = \alpha$.

Proposition 1.9. *The functor \mathcal{D}_F has a pro-representable hull (R, ξ) in the sense of [17] and the Zariski tangent space of R is canonically isomorphic to $\text{Ext}_{e_X}^1(F, F)$.*

Proof. Let $A' \rightarrow A$ and $A'' \rightarrow A$ be morphisms in \mathcal{C} and consider the map

$$(*) \quad \mathcal{D}_F(A' \times_A A'') \rightarrow \mathcal{D}_F(A') \times_{\mathcal{D}_F(A)} \mathcal{D}_F(A'').$$

By Theorem 2.11 [17], it suffices to show that this map (*) is always a surjection and a bijection when $A = k$. If $(\alpha', \mathcal{F}') \in \mathcal{D}_F(A')$, $(\alpha'', \mathcal{F}'') \in \mathcal{D}_F(A'')$ and $(\alpha', \mathcal{F}') \otimes_A A = (\alpha'', \mathcal{F}'') \otimes_A A =: (\alpha, \mathcal{F})$ in $\mathcal{D}_F(A)$, then we

obtain the $\mathcal{O}_X \otimes (A' \times_A A'')$ -module $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$ and an isomorphism $\bar{\alpha}: (\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'') \otimes k \xrightarrow{\sim} F$.

Claim. $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$ is flat as an $A' \times_A A''$ -module.

Put $\tilde{A} = A' \times_A A''$ and let \tilde{M} be an \tilde{A} -module. Then the tensor product $\tilde{M} \otimes_{\tilde{A}} (\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'')$ is isomorphic to $(M' \otimes \mathcal{F}') \times_{(M \otimes \mathcal{F})} (M'' \otimes \mathcal{F}'')$, where $M' = \tilde{M} \otimes_{\tilde{A}} A'$, $M'' = \tilde{M} \otimes_{\tilde{A}} A''$ and $M = \tilde{M} \otimes_{\tilde{A}} A$. Since \mathcal{F}' and \mathcal{F}'' are flat, $\tilde{f} \otimes_{\tilde{A}} (\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'')$ is injective for every injection $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$. This shows our claim.

By the claim, the pair $(\bar{\alpha}, \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'')$ belongs to $\mathcal{D}_F(A' \times_A A'')$ and $(*)$ is a surjection. In the case $A = k$, let $(\tilde{\alpha}, \tilde{\mathcal{F}})$ be an element of $\mathcal{D}_F(A' \times_A A'')$ which is sent to $((\alpha', \mathcal{F}'), (\alpha'', \mathcal{F}''))$ by $(*)$. Then there exist isomorphisms $\tilde{\mathcal{F}} \otimes_{\tilde{A}} A' \xrightarrow{\sim} \mathcal{F}'$ and $\tilde{\mathcal{F}} \otimes_{\tilde{A}} A'' \xrightarrow{\sim} \mathcal{F}''$, which induce $q': \tilde{\mathcal{F}} \rightarrow \mathcal{F}'$ and $q'': \tilde{\mathcal{F}} \rightarrow \mathcal{F}''$. By our definition of \mathcal{D}_F the composite $\tilde{\mathcal{F}} \xrightarrow{q'} \mathcal{F}' \rightarrow \mathcal{F}' \otimes_A A'' \xrightarrow{\alpha''} F$ is equal to $\tilde{\mathcal{F}} \xrightarrow{q''} \mathcal{F}'' \rightarrow \mathcal{F}'' \otimes_A A' \xrightarrow{\alpha'} F$. Hence we have a homomorphism $\tilde{\mathcal{F}} \rightarrow \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$, which is an isomorphism since $\tilde{\mathcal{F}}$ is flat. q.e.d.

The pro-representable hulls (R, ξ) 's of \mathcal{D}_F are unique up to (non-canonical) isomorphisms. We call the formal scheme $\text{Spf } R$ the *formal Kuranishi space*. In the case X is a compact complex analytic space, the Kuranishi space exists for every coherent sheaf F on X ([18], [3], [4]). In this case the formal Kuranishi space is nothing but its formal completion. Returning to the general case, we assume that F is simple. Then $\mathcal{D}_F(A)$ is the set of isomorphism classes of A -flat coherent $\mathcal{O}_{X,A}$ -module \mathcal{F} with $\mathcal{F} \otimes_A k \cong F$ for every $A \in C$. Moreover, by Lemma 6.1 [9] (the assumption of torsion freeness in this lemma is superfluous), the natural homomorphism $A \rightarrow \text{End}_{\mathcal{O}_{X,A}}(\mathcal{F})$ is always an isomorphism. Hence, by the same argument as (3.1) in [17] and the claim in the proof of the above proposition, we have:

Proposition 1.10. *If F is simple, then the functor \mathcal{D}_F is pro-representable by the formal Kuranishi space.*

In the case F is simple, the Kuranishi space is unique up to canonical isomorphism and called the *local moduli space* of F , too.

Now we study the relation between the moduli spaces of F and its Fourier transform \hat{F} . Assume that W.I.T. holds for F and let $(\mathcal{F}, \alpha) \in \mathcal{D}_F(A)$. Then, by (2) of Theorem 1.6, W.I.T. holds for \mathcal{F} and α induces the isomorphism $\hat{\alpha}$ between \hat{F} and $\hat{\mathcal{F}} \otimes_A k$. Moreover, $\hat{\mathcal{F}}$ is A -flat and hence we have $(\hat{\mathcal{F}}, \hat{\alpha}) \in \mathcal{D}_{\hat{F}}(A)$. Therefore, the Fourier transformation gives an isomorphism between the two functors \mathcal{D}_F and $\mathcal{D}_{\hat{F}}$ on C . Hence we have:

Proposition 1.11. *Assume that W.I.T. holds for F . Then there is an isomorphism between the formal Kuranishi spaces of F and \hat{F} . If F is simple, then the isomorphism is canonical.*

Let $\mathcal{S}pl_X$ be the moduli functor of simple coherent \mathcal{O}_X -modules. By virtue of [1], its sheafification in the fppf topology is representable by an algebraic space Spl_X which may not be separated. For a simple coherent \mathcal{O}_X -module E , we denote by $M(E)$ the union of the irreducible components of Spl_X containing the point $[E]$ corresponding to E .

Proposition 1.12. *If E is a simple coherent \mathcal{O}_X -module and if W.I.T. holds for E , then $M(\hat{E})$ is birationally equivalent to $M(E)$. Precisely speaking, there is an isomorphism from a Zariski neighbourhood of $[E]$ onto that of $[\hat{E}]$.*

Proof. Let $f: U \rightarrow Spl_X$ be a neighbourhood of $[E]$ in the fppf topology such that a universal family \mathcal{E} exists on $X \times U$. By Theorem 1.5, shrinking U if necessary, W.I.T. holds for \mathcal{E} and $\hat{\mathcal{E}}$ is a deformation of \hat{E} . Hence we get a morphism $g: U \rightarrow Spl_{\hat{X}}$. Let $p_i: V = U \times_{Spl_X} U \rightarrow U$ be the projections ($i=1, 2$). Since $f \circ p_1 = f \circ p_2$, V has a covering $\{\pi_i: V_i \rightarrow V\}$ in the fppf topology such that $(1_X \times \pi_i)^*(1_X \times p_1)^*\mathcal{E} \cong (1_X \times \pi_i)^*(1_X \times p_2)^*\mathcal{E} \otimes_{\mathcal{O}_{V_i}} L_i$ for an invertible sheaf L_i on V_i for every i . Hence we have

$$(1_X \times \pi_i)^*(1_X \times p_1)^*\hat{\mathcal{E}} \cong (1_X \times \pi_i)^*(1_X \times p_2)^*\hat{\mathcal{E}} \otimes_{\mathcal{O}_{V_i}} L_i$$

which implies $g \circ p_1 \circ \pi_i = g \circ p_2 \circ \pi_i$ for every i . Since $\coprod \pi_i: \coprod V_i \rightarrow V$ is faithfully flat, we have $g \circ p_1 = g \circ p_2$. Let W be a Zariski open neighbourhood of $[E]$ contained in $f(U)$. Since $f^{-1}(W) \rightarrow W$ is faithfully flat and $g \circ p_1|_{f^{-1}(W)} = g \circ p_2|_{f^{-1}(W)}$, we get a morphism $h: W \rightarrow Spl_{\hat{X}}$ such that $g|_{f^{-1}(W)} = h \circ (f|_{f^{-1}(W)})$. Replacing X and E by \hat{X} and \hat{E} , respectively, we obtain a morphism \hat{h} from a Zariski open neighbourhood \hat{W} of $[\hat{E}]$ into Spl_X . $\hat{h} \circ h$ is defined and, by Proposition 1.5, equal to the involution $(-1_X)^*$ of Spl_X on a Zariski neighbourhood. Since it is the same for $h \circ \hat{h}$, h is an isomorphism on a Zariski neighbourhood of $[E]$. q.e.d.

Here we give examples of Proposition 1.12.

Example 1.13. *Let V be a finite dimensional vector space. The formal Kuranishi space of the trivial vector bundle $F = \mathcal{O}_X \otimes_k V$ on an abelian variety X is isomorphic to the formal completion at the origin of the cone $D = \text{Hom}_{k\text{-Lie alg}}(T^*, \mathfrak{gl}(V))$ in $\text{Hom}_k(T^*, \mathfrak{gl}(V))$, where T^* is the dual of the Lie algebra of \hat{X} and regarded as an abelian Lie algebra. In the case X is a complex torus, the germ of the cone at the origin is the usual Kuranishi space of F .*

Proof. Since \mathcal{O}_x is isomorphic to the one dimensional skyscraper sheaf $k(\hat{0})$ supported by the origin $\hat{0} \in \hat{X}$, the (formal) Kuranishi space of F is isomorphic to that of $\hat{F} \cong k(\hat{0}) \otimes_k V$. Giving an \mathcal{O}_X -module structure to V is equivalent, modulo the action of lattice, to giving a k -Lie algebra homomorphism $T^* \rightarrow \mathfrak{gl}(V)$, i.e., an ordered set of mutually commutative endomorphisms of V . Hence there is a natural family of deformations of \hat{F} , (D, \mathcal{F}, τ) , and for every small deformation (S, \mathcal{E}, ρ) of \hat{F} there is a morphism $f: S \rightarrow D$ such that $f(s_0) = 0$ and (S, \mathcal{E}, ρ) is equivalent to $f^*(D, \mathcal{F}, \tau)$, that is, the family (D, \mathcal{F}, τ) is complete. Since D contains the cone $\{f \in \text{Hom}_k(T^*, \mathfrak{gl}(V)); \text{rank } f \leq 1\}$, the tangent space $t_{D,0}$ is isomorphic to $\text{Hom}_k(T^*, \mathfrak{gl}(V))$. It is easily seen that the Kodaira Spencer map $T_{D,0} \rightarrow \text{Ext}_{\mathcal{O}_X}^1(\hat{F}, \hat{F}) \cong t_{\hat{X},0} \otimes_k \mathfrak{gl}(V)$ is an isomorphism. Hence the family (D, \mathcal{F}, τ) is effective. q.e.d.

Remark 1.14. Example 1.13 says that the cone $\{\alpha \in H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F)); \alpha \cup \alpha = 0\}$ in the vector space $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F))$ is just the Kuranishi space of the trivial bundle F . If $g = \dim X \geq 2$ and $\dim V \geq 2$, then $D \neq \text{Hom}_k(T, \mathfrak{gl}(V))$. Hence D is singular at 0. This fact was shown in the case $k = \mathbb{C}$ and $\dim V = 2$ by Kodaira-Spencer [8] Section 16 as an example of a complex manifold with the singular Kuranishi space. The fact that D is exactly the Kuranishi space is also mentioned in [15].

Here is an example of a simple vector bundle whose local moduli (=Kuranishi space) is not reduced even in characteristic zero.

Example 1.15. Let X be the Jacobian variety of a nonsingular curve C of genus $g \geq 2$ and E a Picard bundle on X . Then $M(E)_{\text{red}}$ is isomorphic to $X \times X$. In the case C is hyperelliptic and $g \geq 3$, the dimension of the tangent space of $M(E)$ is $3g - 2$ and greater than $2g = \dim X \times X$ at every point. Hence $M(E)$ is not reduced.

Proof. For the proof of the latter part, see [5] Section 4. By the definition of E , E is the Fourier transform of an invertible sheaf ξ on C (regard ξ as an \mathcal{O}_X -module via the natural embedding $C \rightarrow X$). So it suffices to show that $M(\xi)_{\text{red}}$ is isomorphic to $X \times X$. Let ξ' be a small deformation of ξ . Then ξ' is an invertible sheaf on a nonsingular curve in X . By the Torelli theorem, the support of ξ' is a translate of C . Hence the morphism $\varphi: X \times \text{Pic}^0 X \rightarrow M(\xi)$, $(x, P) \rightarrow [T_x^* \xi \otimes P]$, is a surjection near the point $[\xi]$. Let f be a morphism from a nonsingular curve D into $M(\xi)$ such that $\text{Im } f \cap \text{Im } \varphi \neq \emptyset$. Since $f^{-1}(\text{Im } \varphi)$ contains a nonempty open set and $X \times \text{Pic}^0 X$ is complete, $f|_{f^{-1}(\text{Im } \varphi)}$ is extended to a morphism $\tilde{f}: D \rightarrow X \times \text{Pic}^0 X$. The translate g of f by \tilde{f} ($X \times \text{Pic}^0 X$ acts on $M(\xi)$ and hence the set of D -valued points acts on that of $M(\xi)$) maps $f^{-1}(\text{Im } \varphi)$ to the point

[ξ]. Hence g is a constant map which means that $f(D)$ is contained in $\text{Im } \varphi$. Therefore φ is a surjection. φ is a closed immersion by [12], Section 4. Hence $M(\xi)_{\text{red}}$ is isomorphic to $X \times \text{Pic}^0 X$. q.e.d.

Next we consider the relation between numerical invariants of a coherent sheaf F and its Fourier transform. We first consider the case $k = \mathbb{C}$. Let X be a complex torus of dimension g . Since the tangent bundle of a torus is trivial, the following diagram is commutative by the Grothendieck-Riemann-Roch theorem:

$$(1.16) \quad \begin{array}{ccc} K(X) & \longrightarrow & K(\hat{X}) \\ \text{ch} \downarrow & & \text{ch} \downarrow \\ H^*(X, \mathbb{Q}) & \xrightarrow{s} & H^*(\hat{X}, \mathbb{Q}) \end{array}$$

where $K(X)$ is the Grothendieck K -group of X and ch is the Chern character. The correspondence of the above first row is $[F] \rightarrow \sum_i (-1)^i [R^i \hat{S}(F)]$ for every coherent \mathcal{O}_X -module F and the second $s(x) = \pi_{X,*}(e' \cup \pi_X^* x)$ for every $x \in H^*(X, \mathbb{Q})$, where $f \in H^2(X \times \hat{X}, \mathbb{Z})$ is the Chern class of a Poincaré bundle on $X \times \hat{X}$ and \cup is the cup product. Since $H_1(X, \mathbb{Z})$ is the lattice of X , $H_1(X, \mathbb{Z})$ and $H_1(\hat{X}, \mathbb{Z})$ are canonically dual to each other. Since $H^*(X, \mathbb{Z})$ is isomorphic to the exterior algebra of $H^1(X, \mathbb{Z})$, $H^n(X, \mathbb{Z})$ and $H^n(\hat{X}, \mathbb{Z})$ are canonically dual to each other.

Proposition 1.17. *If $x \in H^p(X, \mathbb{Z})$, then $s(x) = (-1)^{p(p+1)/2+g} \alpha^p(x)$, where $\alpha^p: H^p(X, \mathbb{Z}) \rightarrow H^{2g-p}(X, \mathbb{Z})^* \cong H^{2g-p}(\hat{X}, \mathbb{Z})$ is the Poincaré duality.*

Proof. Let e_1, \dots, e_{2g} be a basis of $H^1(X, \mathbb{Z})$ and e_1^*, \dots, e_{2g}^* the dual basis of $H^1(\hat{X}, \mathbb{Z})$. Then f is equal to $\sum_{i=1}^{2g} e_i \wedge e_i^*$ ([14] Section 9). Hence the n -fold wedge product is $f \wedge \dots \wedge f = (-1)^{n(n-1)/2} n! \sum_{i_1 < \dots < i_n} e_{i_1} \wedge \dots \wedge e_{i_n} \wedge e_{i_1}^* \wedge \dots \wedge e_{i_n}^*$, that is, $e' = \sum_{i=0}^{2g} (-1)^{n(n-1)/2} \delta_n$, where $\delta_n \in H^n(X, \mathbb{Z}) \otimes H^n(\hat{X}, \mathbb{Z})$ is the Kronecker's delta. Hence $e' \cup \pi_X^*(x) = \sum_{i=0}^{2g} (-1)^{n(n-1)/2} \delta_n \wedge \pi_X^*(x)$ and contained in $\bigoplus_{i=0}^{2g} H^{p+n}(X, \mathbb{Z}) \otimes H^n(\hat{X}, \mathbb{Z})$. $\pi_{X,*}$ is the composite of the natural projection $H^*(X \times \hat{X}, \mathbb{Z}) \rightarrow \bigoplus_i H^{2g}(X, \mathbb{Z}) \otimes H^i(\hat{X}, \mathbb{Z})$ and

$$\bigoplus_i H^{2g}(X, \mathbb{Z}) \otimes H^i(\hat{X}, \mathbb{Z}) \xrightarrow{\varepsilon \otimes 1} \bigoplus_i \mathbb{Z} \otimes H^i(\hat{X}, \mathbb{Z}) \cong H^*(\hat{X}),$$

where ε is the orientation of X . Hence

$$s(x) = \pi_{X,*}(e' \cup \pi_X^*(x)) = (-1)^{(2g-p)(2g-p-1)/2} \pi_{X,*}(\delta_{2g-p} \wedge \pi_X^*(x))$$

and is equal to $(-1)^{p(p+1)/2+g} \alpha^p(x)$ because if $x = e_{i_1} \wedge \dots \wedge e_{i_p}$, then

$$\pi_{X,*}(\delta_{2g-p} \wedge \pi_X^*(x))$$

$$\begin{aligned}
 &= \pi_{X,*}((e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_{2g-p}}) \wedge (e_{j'_1}^* \wedge \cdots \wedge e_{j'_{2g-p}}^*)) \\
 &= \varepsilon(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{2g-p}}) e_{j'_1}^* \wedge \cdots \wedge e_{j'_{2g-p}}^* \\
 &= \alpha(x).
 \end{aligned}$$

q.e.d.

Corollary 1.18. *If W.I.T. holds for a coherent \mathcal{O}_X -module F and its index is equal to i , then $\text{ch}^p(\hat{F}) = (-1)^{i+p} \alpha^{2g-2p} (\text{ch}^{g-p}(F))$ in $H^{2p}(\hat{X}, \mathbb{Z})$.*

The proof is immediate from (1.16) and the proposition. Now let us consider the case X is an abelian variety over an algebraically closed field. Let $A'(X)$ be the Chow ring modulo numerical equivalence of X and define \hat{s} and s by $\hat{s}(x) = \pi_{\hat{X},*}(e^f \cdot \pi_X^*(x))$ and $s(y) = \pi_{X,*}(e^f \cdot \pi_{\hat{X}}^*(y))$ for every $x \in A'(X)$ and $y \in A'(\hat{X})$, respectively, where $f \in A^1(X \times \hat{X})$ is the Chern class of a Poincaré bundle. Then, by the Grothendieck-Riemann-Roch theorem, we have the commutative diagram:

$$\begin{array}{ccc}
 K'(X) & \longrightarrow & K'(\hat{X}) \\
 \downarrow & & \downarrow \\
 A'(X)_{\mathbb{Q}} & \xrightarrow{\hat{s}} & A'(\hat{X})_{\mathbb{Q}}
 \end{array}$$

(1.19)

where the above first row is the same as in (1.16). For $x = \sum_{i=0}^g x^i$, $x^i \in A^i(X)$, we denote $\sum_{i=0}^g (-1)^i x^i$ by x^\vee . $\langle x, y \rangle$ is the intersection number of x and y . Put $\langle x, y \rangle = \langle x^\vee, y \rangle$. $\langle \cdot, \cdot \rangle$ is a bilinear form on $A'(X)$ and symmetric (resp. skew-symmetric) if g is even (resp. odd). By the Riemann-Roch theorem, we have

$$(1.20) \quad \sum (-1)^i \dim \text{Ext}_{\mathcal{O}_X}^i(E, F) = \langle \text{ch}(E) \cdot \text{ch}(F) \rangle$$

for every pair of coherent \mathcal{O}_X -modules E and F .

- Proposition 1.21.**
- (1) $s \circ \hat{s}(x) = (-1)^g (-1_x)^*(x)$ for every $x \in A'(x)$.
 - (2) $\hat{s}(x * x') = \hat{s}(x) \cdot \hat{s}(x')$, where $*$ is the Pontrjagin product.
 - (3) $\hat{s}(x^\vee) = (-1)^g (-1_x)^* \hat{s}(x)^\vee$ for every $x \in A'(x)$.
 - (4) $\langle \hat{s}(x), \hat{s}(x') \rangle = \langle x, x' \rangle$ for every $x, x' \in A'(X)$.
 - (5) $\langle x, s(y) \rangle = \langle \hat{s}(x), y \rangle$ for every $x \in A'(X)$ and $y \in A'(\hat{X})$.

Proof. Since $\text{ch}: K(X) \otimes \mathbb{Q} \rightarrow A'(X)_{\mathbb{Q}}$ is surjective, we may assume that x, x' and y are Chern characters of coherent modules. Hence (1), (2) and (3) follow from (2.2), (3.7) and (3.8) in [12], respectively. For example, since there is a spectral sequence $R^i S(R^j \hat{S}(F)) \Rightarrow (-1_x)^*(F)$ ($i+j=g$), 0 ($i+j \neq g$), we have

$$\begin{aligned}
 s \circ \hat{s}(\text{ch}(F)) &= s(\sum (-1)^i \text{ch}(R^i \hat{S}(F))) \\
 &= \sum (-1)^{i+j} \text{ch}(R^i S(R^j \hat{S}(F))) \\
 &= (-1)^g \text{ch}((-1_x)^* F).
 \end{aligned}$$

Let E and F be coherent \mathcal{O}_X -modules. By Theorem 2.2 in [12], we have $\text{Ext}_{\mathcal{O}_X}^i(E, F) \cong \text{Ext}_{D(X)}^i(E, F) \cong \text{Ext}_{D(X)}^i(R\hat{S}(E), R\hat{S}(F))$. Since the i -th cohomology of $R\hat{S}(E)$ is $R^i\hat{S}(E)$, there is a spectral sequence $\text{Ext}_{\mathcal{O}_X}^i(R^j\hat{S}(E), R^k\hat{S}(F)) \Rightarrow \text{Ext}_{D(X)}^{i+j+k}(R\hat{S}(E), R\hat{S}(F))$. Hence (4) follows from (1.20):

$$\begin{aligned} & \langle \hat{s}(\text{ch}(E)), \hat{s}(\text{ch}(F)) \rangle \\ &= \sum (-1)^{j+k} \langle \text{ch}(R^j\hat{S}(E)), \text{ch}(R^k\hat{S}(F)) \rangle \\ &= \sum (-1)^{i+j+k} \dim \text{Ext}_{\mathcal{O}_X}^i(R^j\hat{S}(E), R^k\hat{S}(F)) \\ &= \sum (-1)^i \dim \text{Ext}_{D(X)}^i(R\hat{S}(E), R\hat{S}(F)) \\ &= \sum (-1)^i \dim \text{Ext}_{\mathcal{O}_X}^i(E, F) \\ &= \langle \text{ch}(E), \text{ch}(F) \rangle. \end{aligned}$$

(5) follows from (1), (3) and (4) because

$$\begin{aligned} (x, s(y)) &= \langle x^\vee, s(y) \rangle \\ &= \langle \hat{s}(x^\vee), \hat{s}(y) \rangle \\ &= \langle (-1)^g (-1)_\beta^* \hat{s}(x)^\vee, (-1)^g (-1)_\beta^* y \rangle \\ &= \langle \hat{s}(x)^\vee, y \rangle \\ &= (\hat{s}(x), y) \end{aligned} \quad \text{q.e.d.}$$

Let κ (resp. μ) be the composite of the natural projection from $A'(X)$ onto $A_0(X) = A^g(X)$ (resp. $A^0(X)$) and $\text{deg}: A_0(X) \xrightarrow{\sim} \mathbb{Z}$ (resp. the isomorphism $A^0(X) \xrightarrow{\sim} \mathbb{Z}$ which maps $1 = [X]$ to 1).

Proposition 1.22. (1) \hat{s} maps $A^0(X)$ onto $A_0(\hat{X})$ and $A_0(X)$ onto $A^0(\hat{X})$.
 (2) $\kappa(\hat{s}(x)) = (-1)^g \mu(x)$ and $\mu(\hat{s}(x)) = \kappa(x)$ for every $x \in A'(X)$.

Proof. W.I.T. holds for \mathcal{O}_X , $\hat{\mathcal{O}}_X \cong k(\hat{0})$, $\hat{0}$ is the origin of \hat{X} and $i(\mathcal{O}_X) = g$ Example 2.6 [12]. Hence $\hat{s}(1) = \hat{s}(\text{ch}(\mathcal{O}_X)) = (-1)^g(\hat{0})$. Therefore \hat{s} maps $A^0(X)$ bijectively onto $A_0(\hat{X})$. The latter half of (1) follows from the duality. (2) follows from (5) of Proposition 1.22, because $\kappa(\hat{s}(x)) = (1, s(x)) = (s(1) \cdot x) = (-1)^g \mu(x)$. q.e.d.

Corollary 1.22. \hat{s} maps $\bigoplus_{i=1}^{g-1} A^i(X)_{\mathbb{Q}}$ onto $\bigoplus_{i=1}^{g-1} A_i(\hat{X})_{\mathbb{Q}}$.

Let D be a nondegenerate divisor on X , i.e., $\chi(D) = (D^g)/g! \neq 0$. By the vanishing theorem ([14] Section 16), W.I.T. holds for $L = \mathcal{O}_X(D)$ and $E = \hat{L}$ is locally free. By [12] (3.1), we have $E \otimes P_x = \hat{L} \otimes P_x \cong (T_{-x}^* L)^\wedge \cong (L \otimes P_{-\phi_D(x)})^\wedge \cong T_{-\phi_D(x)}^* \hat{L} = T_{\phi_D(x)}^* E$ for every $x \in X$. Hence $(\det E) \otimes P_x \cong T_{\phi_D(x)}^* \det E$, where r is the rank of E and equal to $|\chi(D)|$. Let \hat{D} be the codimension one part of $\hat{s}(D)$. Then this means that $-\phi_{\hat{D}}(\phi_D(x)) =$

$(-1)^{i(D)} r_X$. In the case X is an abelian surface, $\hat{s}(D) = \hat{D}$ by Corollary 1.22. Hence we have:

Proposition 1.23. *If X is an abelian surface and D is a nondegenerate divisor, then $\hat{s}(D) \in A^1(\hat{X})$ and $\phi_{\hat{s}(D)} \circ \phi_D = \chi(D)_X$ and $\chi(D) = \chi(\hat{s}(D))$. In particular, if D is a principal polarization, then we have $\hat{s}(D) = D$.*

Since $A^1(X)$ is generated by ample divisors, we have:

Corollary 1.24. *\hat{s} maps $A^1(X)$ onto $A^1(\hat{X})$ if X is an abelian surface.*

Problem 1.25. *Does \hat{s} map $A^i(X)$ (resp. $A^i(X)_0$) onto $A^{g-i}(\hat{X})$ (resp. $A^{g-i}(\hat{X})_0$) for every i ?*

§ 2. Sheaves of U -type

We study some properties of the sheaves of U -type on an abelian variety (Definition 0.1). Let (X, ℓ) be a principally polarized abelian variety, that is, ℓ is an algebraic equivalence class of an ample line bundle L with $\chi(L) = 1$. We identify the dual abelian variety \hat{X} with X by the isomorphism ϕ_L ([14]).

Proposition 2.1. *Let $f: L^{-1} \rightarrow H$ be a nonzero homomorphism from a line bundle L^{-1} in the class $-\ell$ to a homogeneous vector bundle H and E the cokernel of f . Then the following conditions are equivalent:*

- (1) E is simple.
- (2) E is of U -type, that is, $\text{Hom}_{e_X}(E, P) = 0$ for every line bundle $P \in \text{Pic}^0 X$,
- (3) *W.I.T.* holds for E (cf. Definition 1.4), and
- (4) *W.I.T.* holds for E , its index is equal to $g - 1$ and \hat{E} is isomorphic to $(-1_X)^* L \otimes \mathcal{I}$ for an ideal \mathcal{I} of \mathcal{O}_X of colength $r(H)$.

Proof. We prove the proposition following the diagram

$$(1) \implies (2) \implies (4) \implies (1) \quad \text{and} \quad (2) \iff (3).$$

Suppose $\text{Hom}_{e_X}(E, P) \neq 0$, for some $P \in \text{Pic}^0 X$. Since H is homogeneous, $\text{Hom}(P, H) \neq 0$ by Proposition 4.18 in [11]. Since $\text{Hom}(P, L^{-1}) = 0$, we have $\text{Hom}(P, E) \neq 0$, by the exact sequence $0 \rightarrow L^{-1} \xrightarrow{f} H \rightarrow E \rightarrow 0$. Hence if $r(E) \neq 1$, then E is not simple. In the case $r(E) = 1$, E is not simple, either because E is not isomorphic to P . Hence (1) implies (2). Operating the functor S to the above exact sequence, we have a long exact sequence. Since $R^i S(L^{-1})$ is zero for $i \neq g$ and isomorphic to $(-1_X)^* L$ for $i = g$ ([12] Proposition 3.11) and since $R^i S(H)$ is zero for $i \neq g$ ([12] Proposition 3.2),

we have the exact sequence

$$(2.2) \quad 0 \longrightarrow R^{r-1}S(E) \longrightarrow R^rS(L^{-1}) \xrightarrow{\hat{f}} R^rS(H) \longrightarrow R^rS(E) \longrightarrow 0.$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad (-1_x)^*L$$

By the duality, the condition (2) is equivalent to that $\hat{g} \circ \hat{f}$ is not zero for every nonzero homomorphism $\hat{g}: R^rS(H) \rightarrow k(x)$, that is, $R^rS(L^{-1}) \otimes k(x) \rightarrow R^rS(H) \otimes k(x)$ is surjective for every $x \in X$. Hence (2) and (3) are equivalent to each other by Nakayama's lemma. By (2.2), if these equivalent conditions are satisfied, then the index of E is equal to $g-1$ and $\hat{E} \otimes (-1_x)^*L^{-1}$ is isomorphic to the ideal $\mathcal{I} = \text{Ker}(\hat{f} \otimes (-1_x)^*L^{-1})$. Since $\text{length}(R^rS(H))$ is equal to $r(H)$, $\text{colength}(\mathcal{I})$ is equal to $r(H)$. Hence (2) implies (4). Since $\text{End}_{\mathcal{O}_x}(\mathcal{I}) \cong k$, (4) implies that \hat{E} is simple and hence E is simple ([12] Corollary 2.5). q.e.d.

Corollary 2.3. *The following conditions are equivalent to each other for a homogeneous vector bundle H :*

- (1) *There is a nonzero homomorphism $f: L^{-1} \rightarrow H$ whose cokernel is simple,*
- (2) *$\dim \text{Hom}_{\mathcal{O}_x}(H, P) \leq 1$ for every $P \in \text{Pic}^0 X$, and*
- (3) *$\hat{H} \cong \mathcal{O}_x / \mathcal{I}$ for an ideal \mathcal{I} of \mathcal{O}_x .*

Proof. By the proposition, (1) implies that there is a surjection $(L^{-1})^\wedge \rightarrow \hat{H}$. Since $(L^{-1})^\wedge$ is an invertible sheaf, we have (3). Conversely, if (3) holds, then there is a surjection $h: L \rightarrow (-1_x)^*\hat{H} \cong \mathcal{O}_x / (-1_x)^*\mathcal{I}$. Let f be the Fourier transform of h . Then f is a homomorphism from L^{-1} into H and $\hat{f} = (-1_x)^*h$ is a surjection. Hence we have (1). (2) is equivalent to that $\dim \text{Hom}_{\mathcal{O}_x}(\hat{H}, k(x)) \leq 1$ for every $x \in X$. Hence, by Nakayama's lemma, it is equivalent to (3). q.e.d.

By the proposition and the corollary, if f is simple, then it is isomorphic to $(-1_x)^*((-1_x)^*(L \otimes \mathcal{I})^\wedge)$, where \mathcal{I} is the ideal defining $\text{Spec } \hat{H}$, and hence f is independent of the choice of f .

Definition 2.4. For a homogeneous vector bundle H satisfying the equivalent conditions of Corollary 2.3 and a line bundle L belonging to ℓ , $U(L, H)$ is the cokernel of a nonzero homomorphism $f: L^{-1} \rightarrow H$ satisfying the equivalent conditions of Proposition 2.2.

By definition, a coherent \mathcal{O}_x -module is of U -type if and only if it is isomorphic to $U(L, H)$ for some L and H .

Example 2.5 ([20]). Let $P_0, \dots, P_r \in \text{Pic}^0 X$ and assume that they are

mutually distinct. Let $f_i: L^{-1} \rightarrow P_i$ be a nonzero homomorphism. Then $U(L, \bigoplus_{i=0}^r P_i) = \text{Coke}[f = (f_0, \dots, f_r): L^{-1} \rightarrow \bigoplus_{i=0}^r P_i]$ is simple and independent of the choice of $\{f_i\}$.

Let us consider the moduli of sheaves of U -type.

Proposition 2.6. *The property that a coherent sheaf E is of U -type is an open condition. Precisely speaking, let E be an S -flat coherent $\mathcal{O}_{X \times S}$ -module. Then if $E|_{X \times s_0}$ is of U -type, then so is $E|_{X \times s}$ for every point s near to s_0 .*

Proof. By Theorem 1.5 and Proposition 2.1, it suffices to show that the property “ F is isomorphic to $L \otimes \mathcal{I}$ for a line bundle L and an ideal \mathcal{I} of finite colength” is an open condition. Let \mathcal{F} be an S -flat coherent $\mathcal{O}_{X \times S}$ -module such that $\mathcal{F}|_{X \times s_0} \cong L \otimes \mathcal{I}$. Let $0 \rightarrow \mathcal{E}_n \rightarrow \mathcal{E}_{n-1} \rightarrow \dots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$ be a resolution of \mathcal{F} by locally free sheaves and put $\det \mathcal{F} = \bigotimes_{i=1}^n (\det \mathcal{E}_i)^{(-1)^i}$. Then $\det \mathcal{F}$ is a line bundle on $X \times S$ and there is a natural homomorphism $f: \mathcal{F} \rightarrow \det \mathcal{F}$. Since $\mathcal{F}|_{X \times s_0}$ is torsion free, f is injective on $X \times s_0$. Hence f is injective on $X \times U$ for a neighbourhood U of s_0 . Therefore $\mathcal{F}|_{X \times s}$ is isomorphic to $L' \otimes \mathcal{I}'$ for every $s \in U$, where $L' = (\det \mathcal{F})|_{X \times s}$. Since \mathcal{F} is S -flat, the colength of \mathcal{I}' is finite and equal to that of \mathcal{I} . q.e.d.

In the case $g \geq 2$, the moduli of sheaves of the form $L \otimes \mathcal{I}$ is the product of the moduli of L , which is isomorphic to $\text{Pic}^0 X \cong X$, and the moduli of the ideal \mathcal{I} , which we denote by $\text{Hilb}^n X$, where n is the colength of \mathcal{I} . Hence, by Proposition 1.12, Proposition 2.1 and Proposition 2.6, we have:

Theorem 2.7. *Assume that $g \geq 2$. The moduli of sheaves of U -type of rank r is an open set of Spl_X and isomorphic to $X \times \text{Hilb}^{r+1} X$.*

Remark 2.8. A sheaf of U -type in Example 2.5 corresponds to a point $(*, \mathcal{I}) \in X \times \text{Hilb}^{r+1} X$ such that $\text{Supp } \mathcal{O}_X/\mathcal{I}$ is a set of $r+1$ mutually distinct points.

Remark 2.9. In the case $g=1$, a simple sheaf E is of U -type if and only if $\text{deg } E=1$. Hence $U(L, H) \cong U(L', H')$ if and only if their ranks and determinants are the same, respectively (cf. [2], [16]).

Next we study the stability of $U(L, H)$.

Proposition 2.10. *Let Θ be the zero locus of a nonzero section of L . If Θ is irreducible, then $U(L, H)$ is torsion free.*

Proof. By the definition of $U(L, H)$, there is an exact sequence $0 \rightarrow$

$L^{-1} \rightarrow H \xrightarrow{\varphi} U(L, H) \rightarrow 0$. Let T be the torsion part of $U(L, H)$. Then $\varphi^{-1}(T)$ is a subsheaf of H containing L^{-1} and its rank is equal to 1. Since $H/\varphi^{-1}(T)$ is torsion free, $\varphi^{-1}(T)$ is locally free by the lemma below. Let $g: H \rightarrow P$ be a nonzero homomorphism from H into a line bundle $P \in \text{Pic}^0 X$. By Proposition 2.1, $g|_{L^{-1}}$ is nonzero. Since $g(L^{-1}) \subseteq g(\varphi^{-1}(T)) \subseteq P$ and since $P^{-1} \otimes \text{Coker}(g|_{L^{-1}})$ is isomorphic to the structure sheaf of a translate of Θ by some $x \in X$, we have by the irreducibility of H either $\varphi^{-1}(T) = L^{-1}$ or $g|_{\varphi^{-1}(T)}$ is an isomorphism. Since $\varphi^{-1}(T) = L^{-1}$ implies that $T = 0$, it suffices to show the latter case never happens. In the latter case, H is the direct sum of $\varphi^{-1}(T)$ and $\text{Ker } g$. Hence $U(L, H)$ is isomorphic to $(\varphi^{-1}(T)/L^{-1}) \oplus \text{Ker } g$, which contradicts the simpleness of $U(L, H)$. *q.e.d.*

Lemma 2.11. *Let M be a subsheaf of a locally free sheaf H on a smooth variety X . If $r(M) = 1$ or $\dim X = 2$, then there is a locally free subsheaf \tilde{M} of H such that $M \subset \tilde{M}$ and $\text{codim Supp } \tilde{M}/M \geq 2$.*

Proof. By assumption, there is a locally free sheaf \tilde{M} containing M such that $\text{codim Supp } \tilde{M}/M \geq 2$: In the case $r(M) = 1$, \tilde{M} is $\det M$ and in the case $\dim X = 2$, $\tilde{M} = M^{\vee}$. Put $U = X - \text{Supp } \tilde{M}/M$ and let $i: U \hookrightarrow X$ be the canonical inclusion. Since $\text{codim } X - U \geq 2$, $i_*(H|_U)$ is isomorphic to H and $i_*(M|_U) = i_*(\tilde{M}|_U)$ is isomorphic to \tilde{M} (EGA IV (5.10.5)). Therefore the canonical inclusion $\alpha: M \hookrightarrow H$ extends to an injection $i_*(\alpha|_U): \tilde{M} \rightarrow H$. *q.e.d.*

To show the stability of $U(L, H)$ we need an elementary fact on homogeneous vector bundles.

Proposition 2.12. *Let F be a nonzero quotient of a homogeneous vector bundle H . Then $c_1(F) \geq 0$, i.e., $\det F$ is algebraically equivalent to an effective divisor or a zero divisor. Moreover if $c_1(F) = 0$, i.e., $\det F$ is algebraically equivalent to zero and if F is torsion free, then F is a homogeneous vector bundle.*

Proof. We prove the proposition by induction on $r(H)$. In the case $r(H) = 1$ our assertion is clear because every nonzero quotient is a torsion sheaf or H itself. Assume that $r(H) \geq 2$. There are an exact sequence $0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0$ such that both H' and H'' are nonzero homogeneous vector bundle and an exact sequence $0 \rightarrow F' \rightarrow F \xrightarrow{\varphi} F'' \rightarrow 0$ such that F' and F'' are quotients of H' and H'' , respectively. By induction hypothesis, we have $c_1(F') \geq 0$ and $c_1(F'') \geq 0$ and hence $c_1(F) = c_1(F') + c_1(F'') \geq 0$. Assume that F is torsion free and that $c_1(F) = 0$. Let T be the torsion part of F'' . Since F''/T is a quotient of H , $c_1(F''/T) \geq 0$. Since $0 = c_1(F) =$

$c_1(F') + c_1(T) + c_1(F''/T)$, we have $c_1(F') = c_1(T) = c_1(F''/T) = 0$. Hence both F' and F''/T are homogeneous vector bundles by induction hypothesis and $\text{codim Supp } T \geq 2$. The latter implies that the exact sequence $0 \rightarrow F' \rightarrow \varphi^{-1}(T) \rightarrow T \rightarrow 0$ splits since F' is locally free. Since $\varphi^{-1}(T)$ is contained in F and F is torsion free, $\varphi^{-1}(T)$ is torsion free and hence $T = 0$. Therefore F'' is a homogeneous vector bundle. Since F is an extension of F'' by a homogeneous vector bundle, F is also a homogeneous vector bundle. (Recall that F is homogeneous if and only if it has a filtration whose successive quotients are line bundles algebraically equivalent to zero.) q.e.d.

The following is the dual form and an immediate consequence of the proposition.

Proposition 2.13. *Let E be a subsheaf of a homogeneous vector bundle H . Then $c_1(E) \leq 0$ and if $c_1(E) = 0$, then there exists a homogeneous sub-bundle \tilde{E} of H such that $\tilde{E} \supseteq E$ and $\text{codim Supp } \tilde{E}/E \geq 2$.*

Let Θ be as in Proposition 2.10.

Theorem 2.14. *Assume that $(\Theta^{g-1}.D) \geq (\Theta^g) = g!$ for every effective divisor D . Then $U(L, H)$ is μ -stable (with respect to L , see Definition 3.9). The assumption is satisfied if (X, Θ) is the principally polarized Jacobian variety of a nonsingular curve C and if Θ does not contain an abelian sub-variety.*

Proof. By assumption, it is easily checked that Θ is irreducible. Hence $U = U(L, H)$ is torsion free by Proposition 2.10. Let F be a torsion free quotient different from U itself. By Proposition 2.1, $\text{Hom}_{\mathcal{O}_X}(F, P) = 0$ for every $P \in \text{Pic}^0 X$, in particular, F is not homogeneous. Hence, by Proposition 2.12, since F is also a quotient of a homogeneous vector bundle H , $c_1(F)$ is algebraically equivalent to an effective divisor D . Therefore, $\mu(F) = (D.\Theta^{g-1})/r(F) < (\Theta^g)/r(U)\mu(U)$, that is, U is μ -stable. Let us prove the second half of the proposition. Since Θ^{g-1} is numerically equivalent to $(g-1)! C$, it suffices to show the inequality $(D.C) \geq (\Theta.C) = g$ for every effective divisor D . Let D_x be the translate of D by x . If $D_x \not\subset C$, $D_x.C$ is an effective divisor on C of degree $n = (C.D)$. Define the rational map $\alpha: X \rightarrow \text{Sym}^n C$ by $\alpha(x) = D_x.C$ and the morphism $f: \text{Sym}^n C \rightarrow \text{Pic}^0 C$ so that $f(y)$ for $y \in \text{Sym}^n C$ is the divisor class of $y - D|_C$. Then the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & \text{Sym}^n C \\
 \phi_D \downarrow & & \downarrow f \\
 \text{Pic}^0 X & \xrightarrow[\sim]{\text{restriction}} & \text{Pic}^0 C
 \end{array}$$

Since ϕ_D is nonzero, $f(\text{Sym}^n C)$ contains an abelian subvariety. Since θ is a translate of $f(\text{Sym}^{g-1} C)$, we have $n \geq g$ by our assumption. q.e.d.

We consider two variants of sheaves of U -type.

Let M be a line bundle on X whose index is 1 and Euler Poincaré characteristic is -1 , i.e., $h^i(M) = 0$ ($i \neq 1$) and $h^1(M) = 1$. From M and a homogeneous vector bundle H , we can construct a simple vector bundle $V(M, H)$ in a manner similar to $U(L, H)$. Let E be an extension of M^{-1} by H .

$$0 \longrightarrow H \longrightarrow E \longrightarrow M^{-1} \longrightarrow 0$$

Proposition 2.15. *In the above situation, the following are equivalent to each other:*

- (1) E is simple,
- (2) $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for every $P \in \text{Pic}^0 X$,
- (3) *W.I.T.* holds for E , and
- (4) *W.I.T.* holds for E , its index is equal to $g - 1$ and \hat{E} is isomorphic to $(-1_X)^* M \otimes \mathcal{I}$, where \mathcal{I} is an ideal of \mathcal{O}_X of colength $r(H)$ and \hat{X} is identified with X by the isomorphism ϕ_X .

Corollary 2.16. *Let H be a homogeneous vector bundle on X . Then the following are equivalent to each other:*

- (1) *There is an extension of M^{-1} by H which is simple,*
- (2) $\dim \text{Hom}_{\mathcal{O}_X}(H, P) \leq 1$ for every $P \in \text{Pic}^0 X$, and
- (3) $\hat{H} \cong \mathcal{O}_X / \mathcal{I}$ for an ideal \mathcal{I} of \mathcal{O}_X .

Moreover, if these equivalent conditions are satisfied, then every extension of M^{-1} by H which is simple is isomorphic to $(-1_X)^((-1_X)^* M \otimes \mathcal{I})^\wedge$ and hence isomorphic to each other. We denote it by $V(M, H)$.*

Theorem 2.17. *Assume that X is an abelian variety and that $g \geq 2$. The moduli of vector bundles $V(M, H)$ of rank r is an open set of Spl_X and isomorphic to $X \times \text{Hilb}^{r-1} X$.*

The proofs are similar to the case of $U(L, H)$.

Let X be a complex torus or an abelian variety and H be a homogeneous vector bundle on X . Let $f: H \rightarrow k(x)$ ($x \in X$) be a nonzero homomorphism and E the kernel of f .

Proposition 2.18. *In the above situation, the following are equivalent to each other:*

- (1) E is simple,
- (2) $\text{Hom}_{\mathcal{O}_X}(P, E) = 0$ for every $P \in \text{Pic}^0 X$,

(3) $\Delta_x(R\hat{S}(E))$ is a sheaf, i.e., its cohomology groups are zero except in one place, where $\Delta_x(?) = R\text{Hom}_{\mathcal{O}_x}(?, \mathcal{O}_x)[g]$, the dualizing functor of \hat{X} , and

(4) $\Delta_x(R\hat{S}(E)) \cong (-1_x)^*(P \otimes \mathcal{I})[g+1]$ for some $P \in \text{Pic}^0 \hat{X}$, where \mathcal{I} is an ideal of \mathcal{O}_x of colength $r(H)$ and $(H^\vee)^\wedge \cong \mathcal{O}_x/\mathcal{I}$.

Proof. If $\text{Hom}_{\mathcal{O}_x}(P, E) \neq 0$, then $\text{Hom}_{\mathcal{O}_x}(P, H) \neq 0$. Since H is homogeneous, $\text{Hom}_{\mathcal{O}_x}(H, P) \neq 0$. Hence $\text{Hom}_{\mathcal{O}_x}(E, P) \neq 0$. Since E is not isomorphic to P , E is not simple, which shows that (1) implies (2). Put $K' = \Delta_x(R\hat{S}(E))$. K' is a complex of \mathcal{O}_x -modules. There is a spectral sequence $E_2^{p,q} = \mathcal{E}_{xt}^p(R^q\hat{S}(E)) \Rightarrow H^{p-q-s}(K')$. By the exact sequence $0 \rightarrow E \rightarrow H \xrightarrow{f} k(x) \rightarrow 0$, $R^q\hat{S}(E)$ is zero if $q \neq 1, g$, isomorphic to P_x if $q=1$ and isomorphic to \hat{H} if $q=g$. Hence $E_2^{p,q}$'s are zero except $E_2^{0,1} \cong P_x^{-1}$ and $E_2^{g,g} \cong \mathcal{E}_{xt}^g(\hat{H}, \mathcal{O}_x)$. Hence we have an exact sequence

$$0 \rightarrow H^{-1-s}(K') \rightarrow E_2^{0,1} \xrightarrow{\delta} E_2^{g,g} \rightarrow H^{-s}(K') \rightarrow 0$$

and $H^i(K') = 0$ for every $i \neq -1-g, -g$. Since $H^{-1-s}(K')$ is always nonzero, K' is a sheaf if and only if $H^{-s}(K') = 0$. Hence if K' is a sheaf, $H^{-1-s}(K') \cong P_x^{-1} \otimes \mathcal{I}$, where \mathcal{I} is an ideal of \mathcal{O}_x and $\mathcal{O}_x/\mathcal{I} \cong \mathcal{E}_{xt}^g(\hat{H}, \mathcal{O}_x)$. Since $\Delta_x \circ R\hat{S} \cong ((-1_x)^* \circ R\hat{S} \circ \Delta_x)[g]$, $\mathcal{E}_{xt}^g(\hat{H}, \mathcal{O}_x)$ is isomorphic to $(-1_x)^*(H^\vee)^\wedge$. Hence (3) implies (4). Since $\Delta_x \circ R\hat{S}$ is an anti-equivalence between the categories $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ and since $K' = (-1_x)^*(P_x \otimes \mathcal{I})[g+1]$ is simple, E is simple. Hence (4) implies (1). Lastly we show that (2) implies (3). Since $E_2^{g,g}$ is an \mathcal{O}_x -module of finite length, $H^{-s}(K')$ is zero if and only if $\text{Hom}_{\mathcal{O}_x}(H^{-s}(K'), k(\hat{x})) = 0$ for every point $\hat{x} \in \hat{X}$. Since $H^i(K')$ is zero for every $i > -g$, $\text{Hom}_{\mathcal{O}_x}(H^{-s}(K'), k(\hat{x}))$ is isomorphic to $\text{Hom}_{\mathcal{D}(X)}(K', k(\hat{x})[g])$. Since $k(\hat{x})[g] \cong \Delta(k(\hat{x})[-g]) \cong \Delta(R\hat{S}(P_{-\hat{x}}))$ and since $\Delta \circ R\hat{S}$ is an anti-equivalence, $\text{Hom}_{\mathcal{D}(X)}(K', k(\hat{x})[g])$ is isomorphic to $\text{Hom}_{\hat{\mathcal{D}}(X)}(P_{-\hat{x}}, E) \cong \text{Hom}_{\mathcal{O}_x}(P_{-\hat{x}}, E)$. Therefore (2) implies (3). q.e.d.

Corollary 2.19. *The following are equivalent to each other for a homogeneous vector bundle H on X ;*

- (1) *There is a nonzero homomorphism $f: H \rightarrow k(x)$ for a point $x \in X$ whose kernel is simple,*
- (2) *$\dim \text{Hom}_{\mathcal{O}_x}(P, H) \leq 1$ for every $P \in \text{Pic}^0 X$, and*
- (3) *$(H^\vee)^\wedge \cong \mathcal{O}_x/\mathcal{I}$ for an ideal \mathcal{I} of \mathcal{O}_x .*

Moreover, if these equivalent conditions are satisfied, then every kernel of a nonzero homomorphism from H into $k(x)$ which is simple is isomorphic to $R\hat{S}(\Delta_x(P_x \otimes \mathcal{I}))[1]$. In particular, $\text{Ker } f$ is independent of f . We denote it by $W(x, H)$.

It is easily verified that Theorem 1.6 and Proposition 1.12, etc. are also true, if we replace RS by $\Delta \circ RS$ and “W.I.T. holds for E ” by “ $\Delta(RS(E))$ is a sheaf”. Hence we have:

Theorem 2.20. *Assume that X is an abelian variety and that $g \geq 2$. The moduli of $W(x, H)$'s of rank r is an open set of Spl_x and isomorphic to $X \times \text{Hilb}^r \hat{X}$.*

On the stability of $W(x, H)$, we have:

Proposition 2.21. *$W(x, H)$ is stable with respect to an arbitrary ample line bundle $\mathcal{O}_X(1)$.*

Proof. Let E be a nonzero subsheaf of $W = W(x, H)$ such that $r(E) < r(W)$. By Proposition 2.13, we have $c_1(E) \leq 0$. If $-c_1(E)$ is algebraically equivalent to an effective divisor D , then $\mu(E) = (-D \cdot \mathcal{O}_X(1)^{g-1})/r(E) < 0 = \mu(W)$. Hence $\chi(E(n))/r(E) < \chi(W(n))/r(W)$ for $n \gg 0$. If $c_1(E) = 0$, then there is a homogeneous subbundle \tilde{E} containing E by Proposition 2.13. By Proposition 2.18, $\text{Hom}_{\mathcal{O}_X}(P, W) = 0$ and hence $\text{Hom}_{\mathcal{O}_X}(P, E) = 0$ for every $P \in \text{Pic}^0 X$. In particular, we have $E \neq \tilde{E}$. Therefore we have

$$\frac{\chi(E(n))}{r(E)} \leq \frac{\chi(\tilde{E}(n)) - 1}{r(E)} = \frac{\chi(\tilde{E}(n))}{r(E)} - \frac{1}{r(E)} < \frac{\chi(H(n))}{r(H)} - \frac{1}{r(H)} = \frac{\chi(W(n))}{r(W)}$$

for $n \gg 0$.

q.e.d.

§ 3. Generalities on sheaves on an abelian surface

In this section we prove some basic facts on sheaves on X , an abelian surface or a complex torus of dimension 2. Let E and F be sheaves on X . By the Riemann-Roch theorem, we have

$$(3.1) \quad \begin{aligned} \chi(E, F) &:= \sum_{i=0}^2 (-1)^i \dim \text{Ext}_{\mathcal{O}_X}^i(E, F) \\ &= r(E)\chi(F) - (c_1(E) \cdot c_1(F)) + \chi(E)r(F). \end{aligned}$$

By the Serre duality, we have

$$(3.2) \quad \dim \text{Ext}_{\mathcal{O}_X}^i(E, F) = \dim \text{Ext}_{\mathcal{O}_X}^{2-i}(F, E)$$

for every i .

Definition 3.3. $\lambda(E) = \frac{1}{2}(c_1(E)^2) - r(E)\chi(E)$

Putting $E = F$ in (3.1) and (3.2), we have

$$(3.4) \quad \dim \text{Ext}_{\mathcal{O}_X}^1(E, E) = 2\lambda(E) + 2 \dim \text{End}_{\mathcal{O}_X}(E).$$

In particular, we have $\lambda(E \otimes L) = \lambda(E)$ for every line bundle L .

Let E be a simple sheaf on X , that is, $\text{End}_{\mathcal{O}_X}(E) \cong k$. There exists the formal moduli of deformations of E .

Proposition 3.5. *The formal moduli of deformations of a simple sheaf E on X is smooth and of dimension $2\lambda(E) + 2$ if either E is locally free or X is an abelian surface.*

Proof. When E is locally free, let α be the natural morphism from the functor of deformations of E to $\text{Pic } X$ which assigns $\det F$ to a vector bundle F . Let $\theta \in H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E))$ be an obstruction for the formal moduli to be smooth. Then $\alpha(\theta) \in H^2(X, \mathcal{O}_X)$ is an obstruction for $\text{Pic } X$ to be smooth. Since $\text{Pic } X$ is smooth, $\alpha(\theta)$ is zero. On the other hand, it is easy to see that $\alpha(\theta) = H^2(\text{Tr})(\theta)$ for the trace homomorphism $\text{Tr}: \mathcal{E}nd_{\mathcal{O}_X}(E) \rightarrow \mathcal{O}_X$. Since E is simple, $H^0(i)$ is surjective for the natural homomorphism $i: \mathcal{O}_X \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(E)$. Hence, by the Serre duality, $H^2(\text{Tr})$ is injective. It follows that θ is zero, which implies the local moduli is smooth.

When X is an abelian surface, take a sufficiently ample line bundle L so that $H^i(X, E \otimes L \otimes P) = 0$ for every $i > 0$ and $P \in \text{Pic}^0 X$. By the base change theorem, $\hat{S}(E \otimes L)$ is a vector bundle and $R^i \hat{S}(E \otimes L)$ is zero for every $i > 0$. By Proposition 1.12, the formal moduli of $E \otimes L$ is isomorphic to that of its Fourier transform $(E \otimes L)^\wedge = \hat{S}(E \otimes L)$, which is smooth since $(E \otimes L)^\wedge$ is locally free. Obviously the formal moduli of E is isomorphic to that of $E \otimes L$ and hence it is smooth.

The second half of the proposition follows from (3.4) and the well-known fact that the tangent space of the local moduli of E is canonically isomorphic to $\text{Ext}_{\mathcal{O}_X}^1(E, E)$. q.e.d.

Let M_L be the moduli space of stable (with respect to an ample line bundle L on X) sheaves on an abelian surface X . Since the stability is an open condition and stable sheaves are simple, the formal neighbourhood of M_L at $[E]$ is isomorphic to the formal moduli of E for every stable sheaf E . Hence by the proposition, we have:

Corollary 3.6. *The moduli space M_L of stable sheaves on an abelian surface X is smooth and $\dim_{\mathbb{C}} M_L = 2\lambda(E) + 2$ for every stable sheaf E on X .*

Proposition 3.7. *Assume that both E and F satisfy W.I.T. and let $i(E)$ and $i(F)$ be their indices (Definition 1.4). Then we have*

- 1) $\chi(E, F) = (-1)^{i(E) + i(F)} \chi(\hat{E}, \hat{F})$.
- 2) $\lambda(E) = \lambda(\hat{E})$.
- 3) $(c_i(E), c_i(F)) = (-1)^{i(E) + i(F)} (c_i(\hat{E}), c_i(\hat{F}))$.

Proof. By Corollary 2.5 [12], $\text{Ext}_{\mathcal{O}_X}^i(E, F)$ is isomorphic to $\text{Ext}_{\mathcal{O}_X}^{i+\iota(E)-\iota(F)}(\hat{E}, \hat{F})$ from which 1) follows immediately. By 1) and (3.1), we have the equality

$$\begin{aligned} r(E)\chi(F) - (c_1(E).c_1(F)) + \chi(E)r(F) \\ = (-1)^{\iota(E)+\iota(F)}\{r(\hat{E})\chi(\hat{F}) - (c_1(\hat{E}).c_1(\hat{F})) + \chi(\hat{E})r(\hat{F})\}. \end{aligned}$$

Putting $E=F$ in this equality, we have 2). By Corollary 2.8 [12], $\chi(F)$ is equal to $(-1)^{\iota(F)}r(\hat{F})$ and $r(E)$ is equal to $(-1)^{\iota(E)}\chi(\hat{E})$. Hence $r(E)\chi(F)$ is equal to $(-1)^{\iota(E)+\iota(F)}\chi(\hat{E})r(\hat{F})$. Hence by the above equality we have 3). q.e.d.

By (3.4), $\lambda(E \otimes L) = \lambda(E)$ for every line bundle L . $\lambda(E)$ is a very important invariant of sheaves on X .

Proposition 3.8. *Let E be a simple sheaf on X . Then we have*

- 1) $\lambda(E) \geq 0$.
- 2) *If $\lambda(E) = 0$ and E is torsion free, then E is a semi-homogeneous vector bundle, i.e., for every $x \in X$ there is a $P \in \text{Pic}^0 X$ such that $T_x^* E \cong E \otimes P$.*

Proof. 1) Since E has the formal moduli of its deformations (Proposition 1.10), we have $\lambda(E) = \dim \text{Ext}^1(E, E) - 2 \geq 0$ by the same argument as in Proposition 3.16 [11].

2) The formal moduli of deformations of E exists and has dimension 2. Hence the subscheme $S = \{(x, P) | T_x^* E \otimes P \cong E\}$ of $X \times \text{Pic}^0 X$ has dimension ≥ 2 and the restriction $S \rightarrow X$ of the projection π_X to S is surjective. Hence for every $x \in X$, there is a $P \in \text{Pic}^0 X$ such that $T_x^* E \cong E \otimes P$. q.e.d.

Next we show 1) of the proposition for μ -semi-stable sheaves. Let M be an ample line bundle on X or a Kähler form of X .

Definition 3.9. E is μ -stable (resp. μ -semi-stable) with respect to M if E is torsion free and

$$\mu(F) = \frac{(c_1(F).M)}{r(F)} < \frac{(c_1(E).M)}{r(E)} = \mu(E) \quad (\text{resp. } \leq)$$

for every nonzero proper subsheaf F of E .

Every μ -stable sheaf is simple and if E is μ -stable (resp. μ -semi-stable) then so are E^\sim and $E \otimes L$ where L is an arbitrary line bundle on X . When we fix M , there is the following implication: μ -stable \Rightarrow stable \Rightarrow semi-stable $\Rightarrow \mu$ -semi-stable. If E is μ -semi-stable, then E has a filtration $0 = E_0 \subset E_1$

$\subset \dots \subset E_n = E$ such that F_i is μ -stable and $\mu(F_i) = \mu(E)$ for every $i = 1, \dots, n$, where $F_i = E_i/E_{i-1}$, $i = 1, \dots, n$.

Lemma 3.10. *Let E be a μ -semi-stable sheaf and F_i 's as above. Then $\lambda(E)/r(E) \geq \sum_{i=1}^n \lambda(F_i)/r(F_i)$ and the equality holds if and only if $c_1(E)/r(E) \approx c_1(F_i)/r(F_i)$ for every $i = 1, \dots, n$.*

Proof. It suffices to show the assertion in the case $n = 2$. Since $r(E) = r(F_1) + r(F_2)$, $c_1(E) \approx c_1(F_1) + c_1(F_2)$ and $\lambda(E) = \lambda(F_1) + \lambda(F_2)$, we have

$$\begin{aligned} \frac{\lambda(E)}{r(E)} - \frac{\lambda(F_1)}{r(F_1)} - \frac{\lambda(F_2)}{r(F_2)} &= \frac{(c_1(F_1) + c_1(F_2))^2}{r(F_1) + r(F_2)} - \frac{c_1(F_1)^2}{r(F_1)} - \frac{c_1(F_2)^2}{r(F_2)} \\ &= \frac{-(D^2)}{r(F_1)r(F_2)r(E)}, \end{aligned}$$

where $D = r(F_2)c_1(F_1) - r(F_1)c_1(F_2)$. Since $(D.M) = 0$, we have $(D^2) \geq 0$ and $(D^2) = 0$ if and only if $D \approx 0$, by the Hodge index theorem. q.e.d.

By Propositions 3.8 and 3.10 we have:

Proposition 3.11. *If E is μ -semi-stable, then $\lambda(E) \leq 0$. The equality holds if and only if E has a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ such that E_i/E_{i-1} is semi-homogeneous and*

$$\frac{c_1(E_i/E_{i-1})}{r(E_i/E_{i-1})} \approx \frac{c_1(E)}{r(E)}$$

for every $i = 1, \dots, n$.

Remark 3.12. By the result in Section 6 [11], E has a filtration as in the proposition if and only if E is semi-homogeneous in the case $\text{char.} = 0$ or $\text{char.} = p > 0$ and the p -rank of X is maximal.

For later use, we prove some properties of quotient sheaves of a homogeneous vector bundle.

Proposition 3.13. *Let E be a quotient sheaf of a homogeneous vector bundle H on X . If $(c_1(E)^2) = 0$, then $\lambda(E) \geq 0$.*

Proof. H has a filtration $0 = H_0 \subset H_1 \subset \dots \subset H_n = H$ such that $P_i := H_i/H_{i-1} \in \text{Pic}^0 X$ for every $i = 1, \dots, n$. Hence E has a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ such that E_i/E_{i-1} is a quotient of P_i . Since $c_1(P_i) \approx 0$, $c_1(E_i/E_{i-1})$ is algebraically equivalent to an effective divisor for every $i = 1, \dots, n$. If $(c_1(E)^2) = 0$, then $c_1(E) = \sum_{i=1}^n c_1(E_i/E_{i-1})$ is algebraically equi-

valent to mC for some $m > 0$ and a smooth elliptic curve C on X . Therefore there are integers $m_1, \dots, m_n \geq 0$ such that $c_1(E_i/E_{i-1})$ is algebraically equivalent to $m_i C$ for every $i = 1, \dots, n$ and $\sum_{i=1}^n m_i = m$. Let R_i be the kernel of the natural homomorphism $P_i \rightarrow E_i/E_{i-1}$. Since $c_1(R_i) \approx -m_i C$ and R_i is subsheaf of $\det R_i$ of finite colength, $\chi(R_i) \leq \chi(\mathcal{O}_X(-m_i C)) = 0$. Hence $\chi(E) = \sum_{i=1}^n \chi(E_i/E_{i-1}) = \sum_{i=1}^n (\chi(P) - \chi(R_i)) \geq 0$.
q.e.d.

The following is a standard technique for showing the stability.

Lemma 3.14. *Let E be a torsion free sheaf on X .*

1) E is μ -stable (resp. μ -semi-stable) if (*) $\mu(F) > \mu(E)$ (resp. \geq) for every μ -stable proper quotient sheaf F of E .

2) E is stable if (**) either $\mu(F) > \mu(E)$ or $\mu(F) = \mu(E)$ and $\chi(F)/r(F) > \chi(E)/r(E)$ for every stable quotient sheaf F of E .

Proof. 1) Put $\mu_0 = \min \{\mu(G) \mid G \text{ is a nontorsion quotient sheaf of } E\}$. Let F' be a nontorsion quotient sheaf of E with $\mu(F') = \mu_0$ and minimizing $r(F')$ among those sheaves. Then the quotient F of F' by its torsion part is μ -stable and $\mu(F) \leq \mu(E)$. Hence if (*) holds, then $F = E$ (resp. $\mu(E) = \mu(F) = \mu_0$). It follows that E is μ -stable (resp. μ -semi-stable).

2) Let μ_0 be as in 1). Put $\mu_1 = \min \{\chi(G)/r(G) \mid G \text{ is a nontorsion quotient sheaf of } E \text{ with } \mu(G) = \mu_0\}$. Let F be a nontorsion quotient sheaf of E with $\mu(F) = \mu_0$ and $\chi(F)/r(F) = \mu_1$ and minimizing $r(F)$ among those quotient sheaves. Then F is stable, $\mu(F) \leq \mu(E)$ and if $\mu(F) = \mu(E)$, then $\chi(F)/r(F) \leq \chi(E)/r(E)$. Hence if (**) holds, then $F = E$. It follows that E is stable.
q.e.d.

§ 4. Characterisations of sheaves of U -type and W -type

First we give a preliminary characterization of sheaves of U -type.

Proposition 4.1. *Let E be a sheaf on a principally polarized abelian surface (X, C) with $c_1(E) \approx C$ and $\chi(E) = -1$. Then the following are equivalent:*

- 1) E is of U -type.
- 2) $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for every $P \in \text{Pic}^0 X$ and the set $\Phi = \{P \in \text{Pic}^0 X \mid \text{Hom}_{\mathcal{O}_X}(P, E) \neq 0\}$ is finite.
- 3) *W.I.T.* holds for E , its index is equal to 1 and the Fourier transform \hat{E} is torsion free.

Proof. 1) \Rightarrow 2) Let E be a sheaf of U -type. By Definition 0.1, E is isomorphic to the cokernel of a homomorphism $\varphi: L^{-1} \rightarrow H$ such that

$\text{Hom}_{\mathcal{O}_X}(\varphi, P)$ is injective for every $P \in \text{Pic}^0 X$. Hence $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for every $P \in \text{Pic}^0 X$. Since $\text{Hom}_{\mathcal{O}_X}(P, L^{-1}) = \text{Ext}_{\mathcal{O}_X}^1(P, L^{-1}) = 0$ for every $P \in \text{Pic}^0 X$, the set Φ coincides with $\{P \in \text{Pic}^0 X \mid \text{Hom}_{\mathcal{O}_X}(P, H) \neq 0\}$. Hence the cardinality of Φ is at most $r(H)$ (Definition 0.1).

2) \Rightarrow 3) By the Serre duality, $h^2(E \otimes P) = \dim \text{Hom}_{\mathcal{O}_X}(E, P^{-1}) = 0$ for every $P \in \text{Pic}^0 X$. Hence by the base change theorem, $R^2 \hat{S}(E) = R^2 \pi_{X,*}(\pi_X^* E \otimes P) = 0$. The second half of 2) implies that $\hat{S}(E) = \pi_{X,*}(\pi_X^* E \otimes P)$ is supported by the finite subset Φ of \hat{X} . By Theorem 2.2 [12], there is a spectral sequence

$$R^i S(R^j \hat{S}(E)) \Rightarrow \begin{cases} (-1_X)^* E, & i+j=2 \\ 0 & \text{otherwise} \end{cases}$$

and we have $S(\hat{S}(E)) = 0$. Hence $\hat{S}(E)$ is zero. Therefore W.I.T. holds for E and its index is equal to 1. Since $\chi(E) = -1$, $h^1(E \otimes P) = 1$ for all $P \in \text{Pic}^0 X - \Phi$. Hence $\hat{E} = R^1 \hat{S}(E)$ is an invertible sheaf outside a finite set of points. In particular, the torsion part T of \hat{E} is of finite length. $S(T)$ is contained in $S(\hat{E})$ and its rank is equal to the length of T . Since W.I.T. holds for \hat{E} and its index is 1 (Corollary 2.4 [12]), $S(\hat{E})$ is zero. Hence $T = 0$, that is, \hat{E} is torsion free.

3) \Rightarrow 1) By Corollary 2.8 [12] and Proposition 1.23, we have $r(\hat{E}) = -\chi(E) = 1$, $\chi(\hat{E}) = -r(E)$ and $c_1(\hat{E}) \approx C$. Since \hat{E} is torsion free, it is isomorphic to $L \otimes \mathcal{I}$, where L is a line bundle algebraically equivalent to $\mathcal{O}_X(C)$ and \mathcal{I} is an ideal of \mathcal{O}_X of colength $r(E) + 1$. We have the exact sequence $0 \rightarrow \hat{E} \rightarrow L \rightarrow \mathcal{O}_X/\mathcal{I} \rightarrow 0$. $H = S(\mathcal{O}_X/\mathcal{I})$ is a homogeneous vector bundle because $\mathcal{O}_X/\mathcal{I} \otimes P \cong \mathcal{O}_X/\mathcal{I}$ for every $P \in \text{Pic}^0 X$. By Proposition 3.11 [12], $R^1 S(L) = 0$ and $S(L)$ is isomorphic to L^{-1} by the natural identification between X and \hat{X} . By the duality (Theorem 2.2 [12]), $S(\hat{E}) = 0$ and $R^1 S(\hat{E}) = \hat{E} \cong (-1_X)^* E$. Hence we have the exact sequence

$$0 \longrightarrow L^{-1} \xrightarrow{\varphi} H \longrightarrow (-1_X)^* E \longrightarrow 0.$$

On the other hand, since $R^2 \hat{S}(E) = 0$, $H^2(X, E \otimes P)$ is zero for every $P \in \text{Pic}^0 X$. Hence $\text{Hom}_{\mathcal{O}_X}(E, P)$ is zero and $\text{Hom}_{\mathcal{O}_X}(\varphi, P)$ is injective for every $P \in \text{Pic}^0 X$. Therefore $E = \text{Coker}(-1_X)^* \varphi$ is a sheaf of U -type. q.e.d.

Let X be an arbitrary abelian variety or complex torus.

Definition 4.2. A sheaf E on X is of W -type if it is isomorphic to the kernel of a homomorphism $\varphi: H \rightarrow k(x)$ from a homogeneous vector bundle H onto the one dimensional skyscraper sheaf $k(x)$ supported by $x \in X$ such that $\text{Hom}_{\mathcal{O}_X}(P, \varphi): \text{Hom}_{\mathcal{O}_X}(P, H) \rightarrow \text{Hom}_{\mathcal{O}_X}(P, k(x))$ is injective for every $P \in \text{Pic}^0 X$.

A sheaf E of W -type is torsion free but not locally free and satisfies $c_1(E) \approx 0$ and $\chi(E) = -1$.

Theorem 4.3. *Let E be a sheaf on X , an abelian surface or a complex torus of dimension 2, with $c_1(E) \approx 0$ and $\chi(E) = -1$. Then the following are equivalent:*

1) *There is an exact sequence*

$$0 \longrightarrow H' \longrightarrow E \longrightarrow H'' \longrightarrow k(x) \longrightarrow 0$$

for homogeneous vector bundles H' and H'' and a point $x \in X$.

2) $\text{Hom}_{\mathcal{O}_X}(P, E) = \text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for all but a finite number of $P \in \text{Pic}^0 X$.

Let L be an arbitrary ample line bundle on X or a Kähler form of X . Then the following is also equivalent to 1) and 2):

3) E is μ -semi-stable with respect to L .

Proof. Since every complex torus is Kähler, it suffices to prove

$$2) \implies 1) \implies 3) \implies 2).$$

$2) \implies 1)$ By assumption, $h^0(E \otimes P) = h^1(E \otimes P) = 0$ for all but a finite number of $P \in \text{Pic}^0 X$. Since $\chi(E \otimes P) = \chi(E) = -1$ for every $P \in \text{Pic}^0 X$, $h^1(E \otimes P) = 1$ for all but a finite number of $P \in \text{Pic}^0 X$. Hence $\hat{S}(E)$ and $R^2 \hat{S}(E)$ are of finite length and $R^1 \hat{S}(E)$ is an invertible sheaf on \hat{X} outside a finite set of points. By the spectral sequence $R^i S(R^j \hat{S}(E)) \Rightarrow (-1_X)^* E$, $i+j=2$ and 0, otherwise, we have $S(\hat{S}(E)) = 0$, $S(R^1 \hat{S}(E)) = 0$ and the exact sequence

$$0 \longrightarrow R^1 S(R^1 \hat{S}(E)) \longrightarrow (-1_X)^* E \longrightarrow S(R^2 \hat{S}(E)) \longrightarrow R^2 S(R^1 \hat{S}(E)) \longrightarrow 0.$$

Since $\hat{S}(E)$ is of finite length and $S(\hat{S}(E)) = 0$, $\hat{S}(E)$ is zero. Since the torsion part of $R^1 \hat{S}(E)$ is of finite length and $S(R^1 \hat{S}(E)) = 0$, $R^1 \hat{S}(E)$ is torsion free. Hence $R^1 \hat{S}(E)$ is isomorphic to $Q \otimes \mathcal{I}$, where Q is a line bundle and \mathcal{I} is an ideal of $\mathcal{O}_{\hat{X}}$ of finite colength. By Proposition 1.17, $c_1(Q)$ is algebraically equivalent to 0, i.e., $Q \in \text{Pic}^0 \hat{X}$. Hence we have $S(Q) = R^1 S(Q) = 0$ and $R^2 S(Q) \cong k(q)$, where q is the point of $X = \hat{X}$ corresponding to Q . From the exact sequence $0 \rightarrow R^1 \hat{S}(E) \rightarrow Q \rightarrow \mathcal{O}_{\hat{X}}/\mathcal{I} \rightarrow 0$, we have two isomorphisms $S(\mathcal{O}_{\hat{X}}/\mathcal{I}) \xrightarrow{\sim} R^1 S(R^1 \hat{S}(E))$ and $R^2 S(R^1 \hat{S}(E)) \xrightarrow{\sim} R^2 S(Q) \cong k(q)$. Therefore, we have the exact sequence

$$0 \longrightarrow H' \longrightarrow (-1_X)^* E \longrightarrow H'' \longrightarrow k(q) \longrightarrow 0,$$

where $H' = S(\mathcal{O}_{\hat{X}}/\mathcal{I})$ and $H'' = S(R^2 \hat{S}(E))$ are homogeneous vector bundles.

$1) \implies 3)$ Since H' and the kernel K of the homomorphism $H'' \rightarrow k(x)$

are μ -semi-stable and $\mu(H') = \mu(K)$, so is the extension E of K by H' .

3) \Rightarrow 2) Let P be a line bundle algebraically equivalent to \mathcal{O}_X . Since E is μ -semi-stable and $\mu(E) = \mu(P) = 0$, the cokernel E' of a nonzero homomorphism $f: P \rightarrow E$ is μ -semi-stable and $c_1(E') = 0$. Hence, by induction on $r(E)$, $\text{Hom}_{\mathcal{O}_X}(P, E) \neq 0$ for at most $r(E)$ $P \in \text{Pic}^0 X$. By a similar argument, we have $\text{Hom}_{\mathcal{O}_X}(E, P) \neq 0$ for at most $r(E)$ $P \in \text{Pic}^0 X$. q.e.d.

Corollary 4.4. *In the same situation as in the theorem, the following are equivalent:*

- 1) E is of W -type.
- 2) $\text{Hom}_{\mathcal{O}_X}(P, E) = 0$ for every $P \in \text{Pic}^0 X$ and $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for all but a finite number of $P \in \text{Pic}^0 X$.

Let L be an arbitrary ample line bundle on X or a Kähler form of X . Then the following is also equivalent to 1) and 2).

- 3) E is stable with respect to L .

Proof. We have shown 1) \Rightarrow 3) in Proposition 2.22. 3) \Rightarrow 2) Since E is μ -semi-stable, $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for all but a finite number of $P \in \text{Pic}^0 X$ by the above theorem. Since $\mu(E) = \mu(P)$ and $\chi(E)/r(E) < \chi(P)/r(P)$, $\text{Hom}_{\mathcal{O}_X}(P, E) = 0$ for every $P \in \text{Pic}^0 X$ by the definition of stability.

2) \Rightarrow 1) By the above theorem, E has an exact sequence

$$0 \longrightarrow H' \longrightarrow E \longrightarrow H'' \xrightarrow{\varphi} k(x) \longrightarrow 0.$$

Since $\text{Hom}_{\mathcal{O}_X}(P, E) = 0$, we have $H' = 0$ and $\text{Hom}_{\mathcal{O}_X}(P, \varphi)$ is injective for every $P \in \text{Pic}^0 X$. Hence E is of W -type. q.e.d.

By the above corollary and Theorem 2.21, we have:

Corollary 4.5. *Let X be an abelian surface or a complex torus of dimension 2 and L an ample line bundle or a Kähler form of X . Then the moduli space $M_L(r, 0, -1)$ of sheaves E on X with $r(E) = r$, $c_1(E) \approx 0$, $\chi(E) = -1$ and which are stable with respect to L is isomorphic to $X \times \text{Hilb}^r \hat{X}$.*

§ 5. $M(r, \ell, -1)$ in the case C is irreducible

In this section we prove Theorem 0.3. Let (X, C) be a principally polarized abelian surface such that C is irreducible. C is a smooth curve of genus 2 and X is the Jacobian variety of C . We denote the algebraic equivalence class of C by ℓ .

Lemma 5.1. *Let a and b be two points of X (b may be an infinitely near point of a). Then the cardinality of the set $\{x \in X \mid \text{the translation } C_x \text{ of } C \text{ by } x \text{ passes through } a \text{ and } b\}$ is at most two.*

Proof. When a and b are ordinary points, C_x passes through a and b if and only if $a-x$ and $b-x$ are contained in C . In other words, x is contained in the intresection $(a-C) \cap (b-C)$. Since C is irreducible and $a \neq b$, $a-C$ and $b-C$ have no common components. Hence the number of points in the intersection is at most the intersection number $(a-C, b-C) = (C^2) = 2$. When b is infinitely near to a , C_x passes through a and b if and only if $x \in a-C$ and the tangent direction of C_x at a is b . The Gauss map $\varphi: C \rightarrow P(T_{X,a})$ for which $\varphi(y) =$ "the tangent direction of C_{a-y} at a " is just the canonical map of C . Since the genus of C is equal to 2, φ is finite and its degree is 2. Hence the cardinality of the set $\{x \in X \mid C_x \text{ passes through } a \text{ and } b\} \approx \{a-y \mid \varphi(y) = b\}$ is at most two.

q.e.d.

Lemma 5.2. *Let E be a torsion free sheaf on X with $r(E) = r$, $c_1(E) \approx C$ and $\chi(E) < 0$. Then there are at most $r+1$ $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(P, E) \neq 0$ if E satisfies the following condition:*

(*) *For every homogeneous vector bundle H contained in E and with $r(H) < r$, the quotient E/H is torsion free.*

Proof. We prove the lemma by induction on r . In the case $r=1$, E is isomorphic to $\mathcal{O}_X(C_y) \otimes \mathcal{I}$ for a point $y \in X$ and an ideal \mathcal{I} of \mathcal{O}_X of colength ≥ 2 . For every $P \in \text{Pic}^0 X$, there is a unique $x \in X$ such that $\mathcal{O}_X(C_y) \otimes P^{-1} \cong \mathcal{O}_X(C_{x+y})$. Hence the set $\{P \in \text{Pic}^0 X \mid \text{Hom}_{\mathcal{O}_X}(P, E) \neq 0\}$ is isomorphic to $\{x \in X \mid C_{x+y} \text{ passes through } \text{Spec } \mathcal{O}_X/\mathcal{I}\}$. Hence its cardinality is at most two by Lemma 5.1. Assume that $r \geq 2$. We may assume that there is a nonzero homomorphism $f: P \rightarrow E$ for some $P \in \text{Pic}^0 X$. By the condition (*), $E' = \text{Coker } f$ is torsion free and also satisfies (*) (cf. Definition 0.1). By induction hypothesis, there are at most r $Q \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(Q, E') \neq 0$. If $Q \in \text{Pic}^0 X$ and $\text{Hom}_{\mathcal{O}_X}(Q, E) \neq 0$, then either $Q \cong P$ or $\text{Hom}_{\mathcal{O}_X}(Q, E') \neq 0$. Hence there are at most $r+1$ such Q 's for E .

q.e.d.

Proof of Theorem 0.3. Every sheaf of U -type is stable by Theorem 2.15. Let E be a stable sheaf with $c_1(E) \approx C$ and $\chi(E) = -1$. We show that E satisfies 2) of Proposition 4.1. Since E is stable and $\mu(E) = (c_1(E) \cdot C)/r(E) > 0 = \mu(P)$, $\text{Hom}_{\mathcal{O}_X}(E, P) = 0$ for every $P \in \text{Pic}^0 X$. Let H be a homogeneous vector bundle contained in E and with $r(H) < r(E)$ and \tilde{H} the inverse image of the torsion part T of E/H by the natural homomorphism $E \rightarrow E/H$. Since E is stable, $\mu(E) = 2/r(E) \geq \mu(\tilde{H}) = (c_1(H) + c_1(T) \cdot C)/r(\tilde{H}) = (c_1(T) \cdot C)/r(H)$. Since $r(H) < r(E)$, we have $(c_1(T) \cdot C) < 2$. Since $c_1(T)$ is effective and C is irreducible, we have $c_1(T) = 0$ (Lemma 3.5 [19]). Hence \tilde{H}/H is of finite length. Since H is locally free, \tilde{H} is equal to H , that is, $T=0$. Hence E satisfies the condition (*)

of Lemma 5.2. By Lemma 5.2, there are at most finite $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\rho_x}(P, E) \neq 0$. Therefore E is of U -type by Proposition 4.1.

q.e.d.

Remark 5.3. Actually we have proved above that E is of U -type if E is μ -semi-stable.

A general sheaf of U -type is locally free but special ones are not. Sheaves of U -type which are not locally free form a subvariety of $M(r, \ell, -1)$ of codimension $r - 1$. In fact, we have:

Proposition 5.4. *Let E be a sheaf of U -type which is not locally free and $r = r(E) \geq 2$. Then E is isomorphic to the kernel of a nonzero homomorphism $f: T_x^* F \otimes P \rightarrow k(y)$ for $x, y \in X$ and $P \in \text{Pic}^0 X$, where F_r is the Picard bundle of rank r on X (cf. § 4 [12]).*

Proof. Let \tilde{E} be the double dual $E^{\vee\vee}$ of E . \tilde{E} contains E canonically and \tilde{E}/E is of finite length. Since E is μ -stable, so is \tilde{E} . By Proposition 3.8, $\chi(\tilde{E}) = (c_1(\tilde{E})^2)/2 - r(\tilde{E})\chi(\tilde{E}) \geq 0$. Since $r(\tilde{E}) \geq 2$ and $\chi(\tilde{E}) = \chi(E) + \text{length } \tilde{E}/E \geq 0$, we have $\chi(\tilde{E}) = 0$ and $\text{length } \tilde{E}/E = 1$. Hence E is isomorphic to the kernel of a nonzero homomorphism from \tilde{E} to $k(y)$ for some $y \in X$. On the other hand \tilde{E} is isomorphic to $T_x^* F_r \otimes P$ for some $x \in X$ and $P \in \text{Pic}^0 X$ by Theorem 5.4 [12].

q.e.d.

By Lemma 4.10 [12], $T_x^* F_r \otimes P \cong T_y^* F_r \otimes Q$ if and only if $x = y$ and $P \cong Q$. On the other hand $P(F_r)$ is isomorphic to the $(r + 1)$ -st symmetric product $S^{r+1} C$ of C . Hence we have:

Corollary 5.5. *The subset of $M(r, \ell, -1)$ consisting of the points corresponding to non-locally-free sheaves is isomorphic to $X \times X \times S^{r+1} C$ and a closed subvariety of codimension $r - 1$.*

Next we consider the moduli space $M(r, -\ell, -1)$. If a sheaf E of U -type is locally free, then the dual vector bundle E^\vee belongs to $M(r, \ell, -1)$ because E is μ -stable and hence so is E^\vee (Theorem 2.14). But E^\vee does not belong to $M(r, \ell, -1)$ if E is not locally free,

Proposition 5.6. *Let E be a member of $M(r, -\ell, -1)$, $r \geq 2$. Then E is isomorphic to the dual of a vector bundle of U -type or to the kernel of a nonzero homomorphism $f: T_x^* F_r^\vee \otimes P \rightarrow k(y)$ for $x, y \in X$ and $P \in \text{Pic}^0 X$, where F_r is the Picard bundle of rank r .*

Proof. Let E be a member of $M(r, -\ell, -1)$. If E is locally free,

then E^\vee is of U -type by Theorem 0.3 and Remark 5.3 because E is μ -semi-stable and hence so is E^\vee . In the case E is not locally free, we have by Proposition 3.11 and the same argument as in the proof of Proposition 5.4 that E is isomorphic to the kernel of a nonzero homomorphism from a vector bundle \tilde{E} to $k(y)$ for some $y \in X$. Since E^\vee is μ -semi-stable, E^\vee is isomorphic to $T_x^*F_r \otimes P^{-1}$ for some $x \in X$ and $P \in \text{Pic}^0 X$ by Theorem 5.4 [12] q.e.d.

Corollary 5.7. *The moduli space $M(r, -\ell, -1)$ is irreducible and birationally equivalent to $X \times \text{Hilb}^{r+1} X$.*

Proof. In the case $r=1$ or 2 , the map $E \rightarrow E \otimes \mathcal{O}_X(-(2/r)C)$ gives an isomorphism between $M(r, \ell, -1)$ and $M(r, -\ell, -1)$. Hence the assertion is clear in this case. By Proposition 5.6, in the case $r \geq 3$ the subset of $M(r, -\ell, -1)$ consisting of the points corresponding to non-locally-free sheaves is isomorphic to $X \times X \times P(F_r^\vee)$ whose dimension is equal to $r+5 < 2r+4$. Hence the rational map $E \rightarrow E^\vee$ from $M(r, \ell, -1) \cong X \times \text{Hilb}^{r+1} X$ to $M(r, -\ell, -1)$ (Proposition 5.4 and Corollary 5.5) is birational by Proposition 3.6. q.e.d.

§ 6. $M(r, \ell, -1)$ in the case C is reducible

In this section we prove Theorem 0.2 in the case C is reducible. In this case, $C=C_1+C_2$ for two elliptic curves C_1 and C_2 which intersect transversally at one point and X is isomorphic to $C_1 \times C_2$.

By Corollary 3.6, every component of $M(r, \ell, -1)$ is smooth and has dimension $2r+4$. On the other hand, a general sheaf of U -type is stable ([20] Lemma 9) and $\text{Hilb}^{r+1} X$ is irreducible. Hence for the proof of Theorem 0.2 it suffices to show the following:

Theorem 6.1. *The μ -semi-stable sheaves E on X with $r(E)=r$, $c_1(E) \approx C$ and $\chi(E)=-1$ and which are not of U -type are parametrized by a union of algebraic varieties of dimension $< 2r+4$.*

Lemma 6.2. *Let N be a line bundle algebraically equivalent to $\mathcal{O}_X(C_k)$ and \mathcal{I} an ideal of \mathcal{O}_X of finite colength, where $k=1$ or 2 . Then we have:*

- 1) *W.I.T. holds for $N \otimes \mathcal{I}$ and its index is equal to 1, that is, $\hat{S}(N \otimes \mathcal{I}) = R^2 \hat{S}(N \otimes \mathcal{I}) = 0$.*
- 2) *\hat{N} is a line bundle on a translate C'_{3-k} of C_{3-k} .*
- 3) *$(N \otimes \mathcal{I})^\wedge$ is torsion free if $\mathcal{I} \neq \mathcal{O}_X$.*

Proof. 1) By the Serre duality, $h^2(X, N \otimes \mathcal{I} \otimes P) = \dim \text{Hom}_{\mathcal{O}_X}(N \otimes \mathcal{I}, P^{-1}) = 0$ for every $P \in \text{Pic}^0 X$. Hence by the base change theorem,

$R^2\hat{S}(N\otimes\mathcal{S})=0$. Since $N\otimes\mathcal{S}$ is torsion free, so is $\hat{S}(N\otimes\mathcal{S})=\pi_{x,*}(\mathcal{P}\otimes\pi_x^*(N\otimes\mathcal{S}))$. On the other hand there is a $P\in\text{Pic}^0X$ such that $H^0(X, N\otimes\mathcal{S}\otimes P)=0$. Hence by the base change theorem $\text{Supp } \hat{S}(E)\neq\hat{X}$. Therefore we have $\hat{S}(E)=0$.

2) It is easy to check that $c_1(\hat{N})$ is algebraically equivalent to C_{3-k} by the identification $\phi_c: X\rightsquigarrow\hat{X}$. On the other hand, $\text{Hom}_{\mathcal{O}_x}(k(\hat{x}), \hat{N})=0$ for every $\hat{x}\in\hat{X}$ since $S(\hat{N})=0$. Hence we have 2).

3) Since $\mathcal{S}\neq\mathcal{O}_x$, there are at most finitely many $P\in\text{Pic}^0X$ such that $H^0(X, N\otimes\mathcal{S}\otimes P)\neq 0$. Hence we have 3) in a way similar to the proof of 2) \Rightarrow 3) in Proposition 4.1. q.e.d.

Proposition 6.3. *Let E be a torsion free sheaf with $c_1(E)\approx C$ and $\chi(E)=-1$. Assume that E has infinitely many $P\in\text{Pic}^0X$ such that $\text{Hom}_{\mathcal{O}_x}(P, E)\neq 0$ and that $(c_1(F), C)\leq 1$ for every proper subbundle F of E . Then $\hat{S}(E)=0$ and the torsion part of $R^1\hat{S}(E)$ is zero or a line bundle on a translate of C_1 or C_2 .*

Proof. We prove the lemma by induction of $r(E)$. In the case $r(E)=1$, E is isomorphic to $\det E\otimes\mathcal{S}$, where \mathcal{S} is an ideal of \mathcal{O}_x of colength 2. Since $\det E$ is algebraically equivalent to $\mathcal{O}_x(C_1+C_2)$ and since $\text{Hom}_{\mathcal{O}_x}(P, E)\neq 0$ for infinitely many $P\in\text{Pic}^0X$, $\text{Spec } \mathcal{O}_x/\mathcal{S}$ is a subscheme of a translate C'_k of C_k , where $k=1$ or 2 . Hence E contains a line bundle M algebraically equivalent to $\mathcal{O}_x(C_{3-k})$ and E/M is isomorphic to a line bundle on C'_k of degree -1 . By the base change theorem, $\hat{S}(E/M)=0$ and $R^1\hat{S}(E/M)$ is a line bundle on \hat{X} . Since \hat{S} and $R^1\hat{S}$ are semi-exact functors, $\hat{S}(E)=0$ and the torsion part of $R^1\hat{S}(E)$ is a line bundle on a translate of C_{3-k} by Lemma 6.2. Assume that $r(E)\geq 2$. Let $f: P\rightarrow E$ be a nonzero homomorphism from $P\in\text{Pic}^0X$ to E .

When $\text{Coke } f$ is torsion free, put $E'=\text{Coke } f$. Then E' has infinitely many $P\in\text{Pic}^0X$ such that $\text{Hom}_{\mathcal{O}_x}(P, E')\neq 0$ and $(c_1(F'), C)\leq 1$ for every proper subbundle F' of E' . Hence by induction hypothesis, $\hat{S}(E')=0$ and the torsion part of $R^1\hat{S}(E')$ is zero or a line bundle on a translate of C_1 or C_2 . On the other hand, operating the Fourier functor to the exact sequence $0\rightarrow P\rightarrow E\rightarrow E'\rightarrow 0$, we have $\hat{S}(E)\rightsquigarrow\hat{S}(E')=0$ and the injection $0\rightarrow R^1\hat{S}(E)\rightarrow R^1\hat{S}(E')$, since $\hat{S}(P)=R^1\hat{S}(P)=0$. Hence the torsion part of $R^1\hat{S}(E)$ is zero or a line bundle on a translate of C_1 or C_2 .

When $\text{Coke } f$ is not torsion free, let E' be the quotient of $\text{Coke } f$ by its torsion part. Let E'' be the kernel of the composite $E\rightarrow\text{Coke } f\rightarrow E'$. Since $r(E'')=1$, E'' is isomorphic to $\det E''\otimes\mathcal{S}$ for an ideal \mathcal{S} of \mathcal{O}_x of finite colength. Since $\det E$ contains P and $\det E''\neq P$, $c_1(E'')$ is algebraically equivalent to a nonzero effective divisor. Since $(c_1(E''), C)\leq 1$ by assumption, $c_1(E'')$ is algebraically equivalent to C_1 or C_2 . On the other

hand, $(c_1(F'), C) \leq 0$ for every proper subbundle F' of E' by our assumption. Hence F'/H is torsion free for every homogeneous vector bundle H with $r(H) < r(F')$ and contained in F' . Since E has infinitely many $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(P, E) \neq 0$, E' or E'' has this property. If E' has this property then there is a homogeneous vector bundle H such that $H \subset E'$, $r(H) = r(E') - 1$ and E'/H is torsion free. Hence E'/H is isomorphic to $\det(E'/H) \otimes \mathcal{I}$ for an ideal \mathcal{I} of finite colength. Since $c_1(E'/H) \approx c_1(E')$ is algebraically equivalent to C_2 or C_1 and since E'/H has infinitely many $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(P, E'/H) \neq 0$, we have by Lemma 6.2 that $\mathcal{I} = \mathcal{O}_X$, $\hat{S}(E'/H) = 0$ and $R^1\hat{S}(E'/H)$ is a line bundle on a translate of C_1 or C_2 . Hence $\hat{S}(E') = 0$ and $R^1\hat{S}(E')$ is a line bundle on a translate of C_1 or C_2 . Since $\text{colength}(\mathcal{I}) + \text{colength}(\mathcal{I}) = -\chi(E) = 1$, $\mathcal{I} \neq \mathcal{O}_X$. Hence by Lemma 6.2, $\hat{S}(E'') = 0$ and $R^1\hat{S}(E'')$ is torsion free. Since \hat{S} and $R^1\hat{S}$ are semi-exact, $\hat{S}(E) = 0$ and the torsion part of $R^1\hat{S}(E)$ is zero or a line bundle on a translate of C_1 or C_2 . If there are at most finite $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(P, E') \neq 0$, then we have in a way similar to the proof 2) \Rightarrow 3) in Proposition 4.1 that $\hat{S}(E') = 0$ and $R^1\hat{S}(E')$ is torsion free. Since $\hat{S}(E'') = 0$ and the torsion part of $R^1\hat{S}(E'')$ is zero or a line bundle on a translate of C_1 or C_2 by Lemma 6.2, $\hat{S}(E) = 0$ and the torsion part of $R^1\hat{S}(E)$ is zero or a line bundle on a translate of C_1 or C_2 . q.e.d.

Proof of Theorem 6.1. Let E be a μ -semi-stable sheaf with $r(E) = r$, $c_1(E) \approx C$ and $\chi(E) = -1$ and which is not of U -type. By Proposition 4.1, there are infinitely many $P \in \text{Pic}^0 X$ such that $\text{Hom}_{\mathcal{O}_X}(P, E) \neq 0$. By Proposition 6.3, $\hat{S}(E) = 0$ and the torsion part of $R^1\hat{S}(E)$ is zero or a line bundle on a translate of C_1 or C_2 . Since $h^2(E \otimes P) = \dim \text{Hom}_{\mathcal{O}_X}(E, P^{-1}) = 0$ for every $P \in \text{Pic}^0 X$, $R^2\hat{S}(E)$ is zero. Hence W.I.T. holds for E . By Proposition 4.1, the Fourier transform \hat{E} is not torsion free. Let T be the torsion part of \hat{E} and put $M = \hat{E}/T$. By Proposition 1.23, we have $c_1(\hat{E}) \approx C$. Since $\text{index}(E) = 1$, we have $r(\hat{E}) = -\chi(\hat{E}) = 1$ and $\chi(\hat{E}) = -r(E)$ by Corollary 2.8 [12]. Since M is torsion free and $c_1(M) \approx c_1(\hat{E}) - c_1(T) \approx C_1$ or C_2 , $\hat{S}(M) = 0$ by Lemma 4.2. Hence $R^1S(T)$ is a subsheaf of $R^1S(\hat{E}) = \hat{E} \cong (-1_X)^*E$. On the other hand, $S(T) = 0$ because it is a subsheaf of $S(\hat{E}) = 0$. Since $\dim \text{Supp } T = 1$, $H^2(X, T \otimes P) = 0$ for every $P \in \text{Pic}^0 X$. Hence $R^2S(T) = 0$. It follows that W.I.T. holds for T . Since the rank of $\hat{T} = R^1S(T)$ is positive, $\chi(T) = -r(\hat{T})$ is negative. Since $\hat{E} \cong (-1_X)^*E$, it suffices to show that the simple sheaves \hat{E} such that $r(\hat{E}) = 1$, $c_1(\hat{E}) \approx C$, $\chi(\hat{E}) = -r$ and the torsion part T of \hat{E} is a negative line bundle on a translate of C_1 or C_2 are parametrized by an algebraic variety of dimension $2r + 4$. The number of moduli of deformations of T as a sheaf on X is equal to 2. Since $M \cong \det M \otimes \mathcal{I}$ for an ideal \mathcal{I} of colength $-\chi(M)$, the number of moduli of deformations of M is equal to $2 +$

$\dim \text{Hilb}^{-\chi(M)} X = 2 - 2\chi(M)$. Since \hat{E} is simple, $\text{Hom}_{\mathcal{O}_X}(M, T) = 0$. Since M is torsion free, $\text{Hom}_{\mathcal{O}_X}(T, M) = 0$. Hence $\dim \text{Ext}_{\mathcal{O}_X}^1(M, T) = -\chi(M, T) = -\chi(T) + 1$ by (3.1) and (3.2). Hence \hat{E} 's are parametrized by an algebraic variety of dimension $2 + \{2 - 2\chi(M)\} + \{-\chi(T)\} = 2r + 4 + \chi(T) < 2r + 4$.
q.e.d.

References

- EGA A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Publ. Math. I.H.E.S.
- FGA A. Grothendieck, *Fondaments de la géométrie algébrique*, collected Bourbaki talks, Paris 1962.
- [1] A. Altman and S. Kleiman, Compactifying the Picard scheme, *Adv. Math.*, **35** (1980), 50–112.
- [2] M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.*, **7** (1957), 414–452.
- [3] I. F. Donin, On analytic Banach space of modules of holomorphic fiberings, *Soviet Math. Dokl.*, **11** (1970), 1591–1594.
- [4] O. Forster and K. Knorr, Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen, *Math. Ann.*, **209** (1974), 291–346.
- [5] J. Fogarty, Algebraic families on an algebraic surface, *Amer. J. Math.*, **90** (1968), 511–521.
- [6] D. Gieseker, On the moduli of vector bundles on an algebraic surface, *Ann. of Math.*, **106** (1977), 45–60.
- [7] A. Iarrobino, Reducibility of the families of 0-dimensional schemes on a variety, *Invent. Math.*, **15** (1972), 72–77.
- [8] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures, II, *Ann. of Math.*, **67** (1958), 403–459.
- [9] M. Maruyama, Moduli of stable sheaves, II, *J. Math. Kyoto Univ.*, **18** (1978), 557–614.
- [10] H. Matsumura, *Commutative algebra*, Benjamin, New York, 1970.
- [11] S. Mukai, Semi-homogeneous vector bundles on an abelian variety, *J. Math. Kyoto Univ.*, **18** (1978), 239–272.
- [12] —, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.*, **81** (1981), 153–175.
- [13] —, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. Math.*, **77** (1984), 101–116.
- [14] D. Mumford, *Abelian varieties*, Oxford University Press, 1974.
- [15] M. Namba, *Families of Meromorphic Functions on Compact Riemann Surfaces*, Lecture Notes in Math. n° 767, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
- [16] T. Oda, Vector bundles over an elliptic curve, *Nagoya Math. J.*, **43** (1971), 41–71.
- [17] M. Schlessinger, Functors on Artin rings, *Trans. Amer. Math. Soc.*, **130** (1968), 205–222.
- [18] Y. T. Siu and G. Trautmann, Deformations of coherent analytic sheaves with compact supports, *Memoirs of Amer. Math. Soc.*, **29** (1981), n° 238.
- [19] H. Umemura, On a property of symmetric products of a curve of genus 2, *Proc. Int. Symp. on Algebraic Geometry*, Kyoto 1977, Kinokuniya, Tokyo 1978.
- [20] —, Moduli space of the stable vector bundles over abelian surfaces,

Nagoya Math. J., 77 (1980), 47-60.

*Department of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya, 464 Japan*