Geometric realization of root systems and the Jacobians of del Pezzo surfaces

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In the conference talk, I reviewed the geometric part of my article [2] and announced the following result on the Jacobian of a del Pezzo surface, whose details will be published elsewhere.

A smooth complete algebraic surface S (over an algebraically closed field) is *del Pezzo* if the anti-canonical class $-K_S$ is ample. The self-intersection number $(-K_S^2) =: d$ is called the degree, which ranges from 1 to 9. A del Pezzo surface S is isomorphic to the projective plane $\mathbb{P}^2$ if $d = 9$, and to either a smooth quadric surface Q or the blow-up of $\mathbb{P}^2$ at a point if $d = 8$. A del Pezzo surface S of degree $d \leq 7$ is isomorphic to the blow-up of $\mathbb{P}^2$ at $(9 - d)$ points in a general position.

The anti-canonical system $|-K_S|$ is of dimension d and its general member is a smooth elliptic curve. Let $\mathcal{C} \subset S \times \mathbb{P}^d$ be the universal family of anti-canonical members $C \in |-K_S| = \mathbb{P}^d$ and C_η be the generic fiber of $\mathcal{C} \longrightarrow \mathbb{P}^d$.

Definition A morphism $\varphi : \tilde{J} \longrightarrow \mathbb{P}^d$ is *a Jacobian fibration of* S if all fibers are of dimension one and its generic fiber is the Jacobian of the generic anti-canonical member C_η. A $(d + 1)$-dimensional variety J with a smooth point p is *a reduced Jacobian of* S if the blow-up of J at p have a Jacobian fibration φ such that the exceptional divisor over p is the 0-section of φ.

In the case of degree $d = 1$, the anti-canonical system $| - K_S|$ is a pencil with a unique base point. Hence a del Pezzo surface S itself is its reduced Jacobian.

Theorem *For a del Pezzo surface S, there exists a reduced Jacobian $(J(S), p)$ which satisfies the following properties:*

(1) *$J(S)$ is a $(d + 1)$-dimensional weak del Pezzo variety of degree one, that is, $-K_S = dH$, H being a nef and big divisor with $(H^{d+1}) = 1$,*

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(2) *p is the unique base point of the d-dimensional linear system $|H|$,*

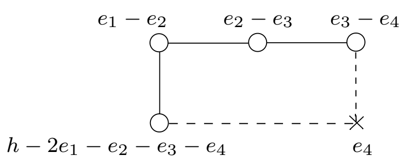
(3) *$J(S)$ is the blow-up of the projective space $\mathbb{P}^3$ at seven points in a general position if $d = 2$,*

(4) *$J(S)$ is the blow-up of the product $\mathbb{P}^2 \times \mathbb{P}^2$ at five points in a general position if $d = 3$,*

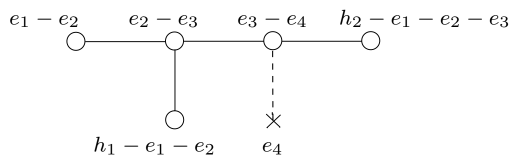
(5) *$J(S)$ is the blow-up of the 6-dimensional Grassmannian $G(2, 5)$ at four points $p_1, \ldots, p_4$ in a general position if $d = 5$, and*

(6) *$J(S)$ is the blow-up of a singular hyperplane section $G(2, 5)'$ of $G(2, 5) \subset \mathbb{P}^9$ at four points $p_1, \ldots, p_4$ in a general position if $d = 4$.*

In the case $d = 1, 2, 3$, the reduced Jacobian $J(S)$ belongs to the class of rational varieties studied in [2]. The augmented root system $N(E_{9-d}, \text{adjoint})$ (cf. [3, §4] and [4, §4]) is realized in the second cohomology group $H^2(J(S), \mathbb{Z})$. The Weyl group $W(E_{9-d})$ birationally acts on the universal family of $J(S)$ over the configuration space of $(9 - d)$ points on $\mathbb{P}^2$. Similar properties hold true for a del Pezzo surface S_d of degree $d = 5, 4$ with the following augmented root system.



$d = 5$, $(A_4, \text{adjoint})$



$d = 4$, $(D_5, \text{adjoint})$

In these diagrams in $H^2(J(S_d), \mathbb{Z})$, h denotes the pull back of a hyperplane section of the Plücker embedding $G(2, 5) \subset \mathbb{P}^9$, and h_i denotes that of a divisor class of $G(2, 5)'$ with $\dim |h_i| = i$. $e_1, \ldots, e_4$ are the classes of exceptional divisors over $p_1, \ldots, p_4$. The reflection with respect to the (-2)-class $h - 2e_1 - e_2 - e_3 - e_4 \in H^2(J(S_5), \mathbb{Z})$ is realized by the composite of two birational involutions of $J(S_5) = Bl_{p_1,\ldots,p_4} G(2, 5)$. One is *the Geiser involution* the blow-up of $G(2, 5)$ at p_2, p_3 and p_4, that is, the covering involution of the morphism

$$\Phi_{|H - e_2 - e_3 - e_4|} : Bl_{p_2,p_3,p_4} G(2, 5) \longrightarrow \mathbb{P}^6$$

of degree 2. The other is *the Bertini involution*, that is, the involution of $J(S_5)$ induced from the $(-1)_{\tilde{J}}$ of the elliptic fibration $\varphi : \tilde{J}(S_5) \longrightarrow \mathbb{P}^5$. (See [1, §7] for the Geiser and Bertini involutions of del Pezzo surfaces.)

References

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