Group-Quark matrix and Leech-K3 analogue of del Pezzo surfaces 2/21/23(T) S. Mukai

Abstract: Groups are analyzed by taking normalizer of suitable subgroups. For example, the (quaternionic) Wolf space and Weyl group are attached to each simple Lie group in this way. As a discrete analogy, Harada(2001) considered a 3 by 3 arrangement of certain finite groups, terminated at the Monster, and asked their algebro-geometric meaning. The first row is one of Arnold's trinities and del Pezzo surfaces play the main role there. I will discuss the groups in the second focusing on their (mostly conjectural) connection with symplectic (or "quaternionic") varieties related to supersingular K3 surfaces.

§1 Background/Motivation

X : compact complex manifold, two long standing problems QK&HK

$\boxed{\text{QK}}$ contact & Fano $\Rightarrow$ homogeneous?

$(X^{2n+1}, \underset{\text{1-form}}{\omega})$, $\omega \wedge (d\omega)^n$ nowhere vanishing, $c_1 > 0 \Rightarrow X = G/P_\theta$?

the unique closed orbit of $G \curvearrowright \mathbb{P}(\mathfrak{g})$

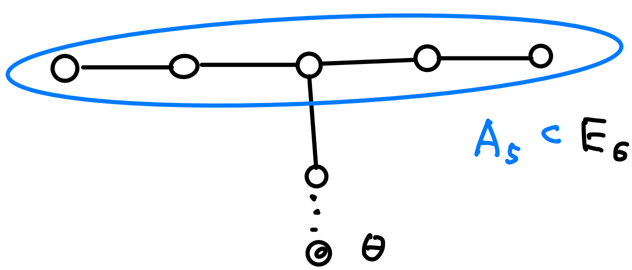
$P_\theta = (\text{Heisenberg}).G_0$

$1 \to \mathbb{C}^* \to H^{2g-3} \to \mathbb{C}^{2g-4} \to 0$, g: dual Coxeter number

Example 1. A_n-type $G=SL(n+1) \Rightarrow$ flag in $\mathbb{P}^n \times \mathbb{P}^{n,*}$

2. E_6 type, g=12

$P_\theta = \mathbb{C}^*.\mathbb{C}^{20}.SL(6)$



$\boxed{\text{HK}}$ Classify irreducible holomorphic simplectic manifolds (K3-like mfds).

(X^{2n}, ω), $\underbrace{\omega\wedge\cdots\wedge\omega}_{n}$ nowhere vanishing (hence $c_1=0$)

dim=2 $\Rightarrow$ K3 surface, e.g., quartic surface in $\mathbb{P}^3$

$K3^{(n)}$ n-th symmetric product

$K3^{(n)} \xleftarrow{\text{min. res. of singularities}} K3^{[n]}$

$\overline{M}_S(2v) \leftarrow \text{OG10}$

torus analogue

def. class	K3	$K3^{[n]}$	OG10	...	...
dim.	2	2n	10	2n	6
B_2	22	23	24	7	8

$(n \geq 2)$

Are these all (deformation types)?

§2 Group-Quark matrix and its suggestion

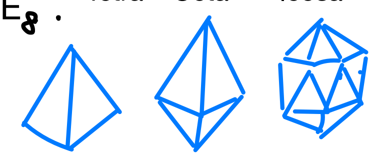
Harada (2001)

	$W(E_6)$	$W(E_7)$	$W(E_8)$	lattice
	$U_4(2).2$ $\boxed{L_3(4)}$	$Sp(6).2$ $\boxed{M_{22}}$	$O_8^+(2)$ $\boxed{M_{24}}$	$I_{1,9}$
$Fi_{21}=$	$U_6(2).2$	Co_2	Co_1	$II_{1,25}$
	${}^2E_6(2)$	$\mathbb{B}$	$\mathbb{M}$	no lattices but VOA's
	Fischer	Baby monster	Monster	

vertical: take normalizer of a suitable 2-extraspecial subgroup 2^{1+2n}, discrete analogy of Heisenberg group.

horizontal: Arnold's trinities, say, E_6, E_7, E_8.

Tetra Octa Icosa



Result:	Tetra	Octa	Icosa
g=Coxeter #	12	18	30
g-8	4	10	22
K3-like (g-8)-fold	$K3^{[2]}$	OG10?	$\exists$?

G-Q matrix's suggestion:
Does there exist a K3-like 22- or 20-fold with $B_2 = 25$?

§3 Basic 1st line $\rightleftarrows$ del Pezzo surface of degree d=3, 2, 1

(1, 1) $S_3 \subset \mathbb{P}^3$ smooth cubic surface

Well known: $\exists$ 27 lines on S_3

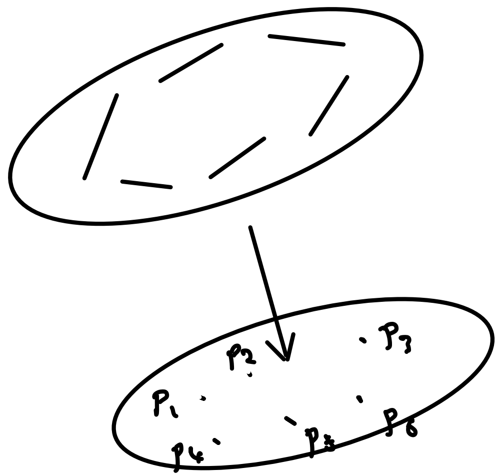
Aut(config. of 27 lines) = $W(E_6)$

G-Q understanding: consider special case $x^3 + y^3 + z^3 + t^3 = 0$ over $\overline{\mathbb{F}}_2$, Fermat/Hermitian. Hence we have $U_4(2) \to W(E_6)$, injective & image of index 2.

Standard way of understanding: 6 disjoint lines $l_1, \dots, l_6$

$S = \mathrm{Bl}_{6\,\text{pts}}\mathbb{P}^2$

$H^2(S,\mathbb{Z})$ w. cup product $= (\mathbb{Z}^7, t^2 - x_1^2 - \dots - x_6^2)$



$c_1(S) \rightleftarrows (3, 111111)$

$c_1(S)^{\perp} = E_6$-lattice (neg. definite)

Aut(config.) $\to O(E_6, \mathbb{Z}) = W(E_6)$ is an isomorphism, and RHS is generated by (-2)-reflections $x \mapsto x + (x.a)a$, with $(a^2) = -2$.

§4 $I_{1,9}$ and $II_{1,25}$

Both are unimodular hyperbolic lattices. I is odd and II is even, that is, (x^2) even for all $x \in II$.

Both have beautiful fundamental domains & str. of orthog. grps

$$\star \quad O(I_{1,9}, \mathbb{Z}) \sim \begin{pmatrix} \text{gp generated} \\ \text{by } (-1) \\ \text{reflections} \end{pmatrix} \cdot \underbrace{\mathbb{Z}^8 \cdot W(E_8)}_{W(\tilde{E}_8)}$$

Coxeter group

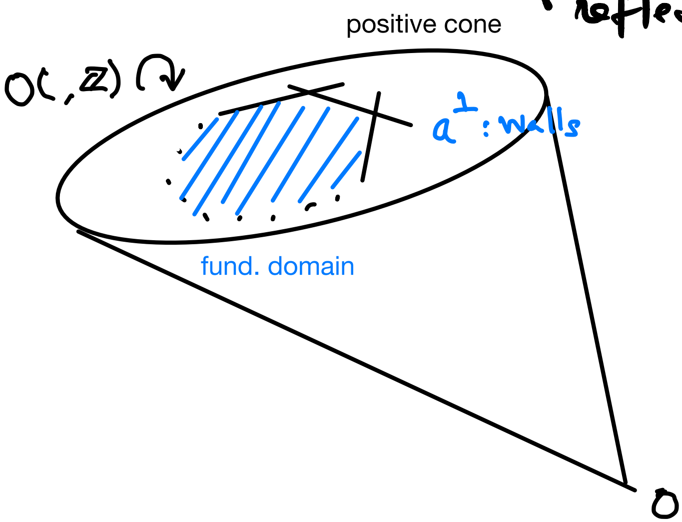
Symmetry of a fund. domain

(Conway '80's)

$$\star\star \quad O(II_{1,25}, \mathbb{Z}) \sim \begin{pmatrix} \text{gp generated} \\ \text{by } (-2) \\ \text{reflections} \end{pmatrix} \cdot \mathbb{Z}^{24} \cdot Co_0$$

Leech lattice

O(Leech)



§5 From del Pezzo to K3-like

Pic $S_d = H(S_d, \mathbb{Z}) = (\langle -1\rangle^{d})^{\perp}$ in $I_{1,9}$

Pic X = Div X/ { (f), f: rat. fn}

Div $X = \bigoplus_{D\subset X} \mathbb{Z}[D]$
codim 1, irred.

Make analogy.

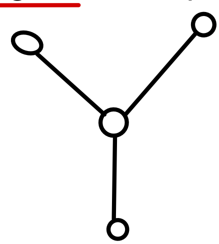
K3-like variety X is Leech-K3-analogue of del Pezzo if
(Pic X, BBF-form) $= (\langle -2\rangle^{d})^{\perp}$ in $II_{1,25}$ in narrow sense
$= R^{\perp}$ for root sublattice $R \subset II_{1,25}$ in broad sense

§6 Example (broad sense, Dolgachev-Kondo, supersingular, char. p=2)

$S = S_{DK}: \sum_1^3 x_i y_i^2 = \sum_1^3 x_i^2 y_i = 0$ in $\mathbb{P}^2 \times \mathbb{P}^{2*}/\overline{\mathbb{F}}_2 \rightleftarrows R = D_4$

$\rho = B_2 = 22$

def.



§7 Example (narrow sense)

Tetra: $X = S_{DK}^{[2]}$, Pic $X = (\langle -2\rangle + \langle -2\rangle + \langle -2\rangle)^{\perp}$ in $II_{1,25}$, ρ=23
birational action of $U_6(2)$

Octa: $X = \widetilde{M}_S(2v)$ for suitable Mukai vector v with $(v^2)=2$ on $S = S_{DK}$
(conjecture, Co_2 action not found yet, $\exists$ Co_2-config. of (-2)-divisors)

Icosa : X might be 22- or 20dimensional?! (A reason given below.)

§8 Maximal subgroup and Lagrangian fibration

Lagrangian fibration = h. dim'l generalization of elliptic fibrtaion of K3 surface

$$X^{2n} \xrightarrow{\text{(quasi-) abelian finration}} \text{“}\mathbb{P}^n\text{”}$$

Tetra case: $2^9.L_3(4)$ is contained in $U_6(2)$.

X has a Lagrangian fibration over $\mathbb{P}^2$.

$L_3(4)$ is $\mathrm{Aut}(\mathbb{P}^2)$ over $\mathbb{F}_4$ and 2^9 acts in fiber direction

(Rem: $L_3(4) = M_{21}$)

Octa case: $2^{10}.M_{22}$ is contained in Co_2.

X has a Lagrangian fibration over $\mathbb{P}^5$, on which M_{22} acts.

(projectivization of irrep. of central extension $3.M_{22}$)

Icosa case: $2^{11}.M_{24}$ is contained in Co_1.

Minimal projective space with action of M_{24} is 10-dimensional. So minimal possibility of dim X is it's twice, 20.

Space of Golay codes is a natural 12-dim'l rep. of M_{24}. In this another possibility, X is 22 dim'l.

Postscript (3/3/23(F)) The possibility of dim X = 20 above was overlooked in my talk.

References

§1

Beauville, A., Holomorphic symplectic geometry: a problem list, in "Complex and differential geometry", pp. 49-63, Springer-Verlag, 2011. (See Sect. 3 for the conjecture "contact+Fano implies homogeneous".)

§2

Harada, K., Group-quark matrix?, RIMS Kokyuroku, 1228(2001), 153-160.

Arnold, V.I., Polymathematics: is mathematics a single art or a set of arts?

§3

Manin, Yu.I., Cubic forms: Algebra, Geometry, Arithmetic.

§4

Conway, J.H., The automorphism group of the 26-dimensional even unimodular Lorentzian lattice, J. Algebra, 1983. Chap. 27 of [SPLG].

§6

Dolgachev, I. and Kondo, S., A supersingular K3surface in characteristic 2 and the Leech lattice, Int'l. Math. Res. Notices, 2003, pp. 1-23.