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Moduli of abelian surfaces, and regular polyhedral groups

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Abstract : Let A_d be the moduli space of polarized abelian surfaces of type (1, d). For $d = 2$, 3 and 4, the Satake compactification of A_d is isomorphic to the quotient of $\mathbb{P}^3$ by an action of $PSL(2, Z/d) \times PSL(2, Z/d)$. Let $G_5 \subset PGL(2)$ be the icosahedral group and $PGL(2) \hookrightarrow \mathbb{P}^3$ the natural embedding. A small resolution of the Satake compactification of A_5 at the point cusp is isomorphic to the quotient of the blow-up $\tilde{\mathbb{P}}^3$ of $\mathbb{P}^3$ at the 60 points G_5 by an action of $G_5 \times G_5$.

Let $X(d)$ be the moduli of elliptic curves E with a full level d structure, i.e., a symplectic isomorphism between the standard symplectic module

$$2[Z/d] := \left(Z/d \oplus Z/d, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

and the group E_d of d-torsion points with the Weil pairing. The modulr curve $X(d)$ is rational if and only if $d \leqslant 5$. In particular, the finite group $PSL(2, Z/d)$ is a regular poly-

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hedral group $G_d \subset PGL(2)$ for $d = 2, 3, 4$ and 5. The compactified modular curve $\overline{X(d)}$ is identified with the circumscribed Riemann sphere of the regular polyhedron P_d with t vertices, where t is the number of cusps. The order of G_d is equal to dt.

d	2	3	4	5
P_d	triangle	tetrahedron	octahedron	icosahedron
t	3	4	6	12
G_d	S_3	A_4	S_4	A_5

Regular polyhedral groups are also closely related with Hilbert modular surfaces of small discriminant:

Example Let $O_{\sqrt{5}}$ be the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{5})$ and put

$$\Gamma = SL(2, O_{\sqrt{5}} : \sqrt{5})$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathbb{1}_2 \mod \sqrt{5}, \quad a, b, c, d \in O_{\sqrt{5}}, \quad ad - bc = 1 \right\}$$

Then Γ acts on the product $H \times H$ of two copies of the upper half planes. Let $\tilde{\Gamma}$ be the group generated by Γ and the switch involution of $H \times H$. Then the Hilbert modular surface $Y_\Gamma = \overline{\tilde{\Gamma} \backslash H \times H}$ added with 6 point cusps is isomorphic to the projective plane $\mathbb{P}^2$. This Y_Γ has an action of the icosahedral group G_5.

Moreover, $X_\Gamma = \overline{\Gamma \backslash H \times H}$ is the double cover of Y_Γ with branch a G_5-invariant plane curve of degree 10.

In the 3-fold case the wreath product $2 \wr G_d$ plays the role of G_d. Let $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be the Segre embedding. The ambient space is the projectivization of the space of 2 by 2 matrices and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ paramatrizes the rank one matrices. Hence the complement is naturally identified with PGL(2). This $\mathbb{P}^3$ is an equivariant compactification of the algebraic group PGL(2) and the polyhedral group G_d acts on it from both sides. Let τ be the involution of this $\mathbb{P}^3$ interchanging $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its cofactor matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The fixed locus is the union of $\mathbb{1}_2$ and the traceless matrices. τ interchanges the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$. So the bipolyhedral group $2 \wr G_d$ acts on $\mathbb{P}^3$.

For a polarized abelian surface (X, L) of type $(1, d)$, a symplectic isomorphism between the standard module $2[Z/d]$ and the group

$$K(L) = \{x \in X \mid T_x^* L \simeq L\}$$

with the Weil pairing is called a <u>canonical</u> <u>level</u> <u>structure</u>. By a canonical <u>colevel</u> structure, we mean a symplectic isomorphism between $2[Z/d]$ and the quotient $X_d/K(L)$. We denote the moduli space of polarized abelian surfaces of type $(1, d)$ with canonical level and colevel structure by $A(1, d)$ and $A(d, 1)$, respectively. The forgetful morphisms

$$A(1, d) \longrightarrow A_d \quad \text{and} \quad A(d, 1) \longrightarrow A_d$$

are both Galois covering with Galois group $PSL(2, Z/d)$. The fibre product

$$A_d^{bl} := A(1, d) \times_A A(d, 1)$$

is called the moduli of abelian surfaces with a (weak) <u>bilevel</u> structure.

<u>Remark</u> The moduli space A_d is the quotient of the Siegel upper half space of degree 2 by the full paramodular group

$$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & d\mathbb{Z} \\ d\mathbb{Z} & \mathbb{Z} & d\mathbb{Z} & d\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & d\mathbb{Z} \\ \mathbb{Z} & \frac{1}{d}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp_4(\mathbb{Q})$$

and A_d^{bl} is the quotient by the subgroup generated by

$$\mathbb{1}_4 + \begin{pmatrix} d\mathbb{Z} & & d\mathbb{Z} & d\mathbb{Z} \\ d\mathbb{Z} & d\mathbb{Z} & d\mathbb{Z} & d^2\mathbb{Z} \\ d\mathbb{Z} & & d\mathbb{Z} & d\mathbb{Z} \\ & & & d\mathbb{Z} \end{pmatrix} \cap Sp_4(\mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

For a polarized abelian surface (X, L) of type $(1, d)$, its dual $\hat{X}$ has a natural polarization $\hat{L}$ of the same type such that $\phi_{\hat{L}} \circ \phi_L = d_X$. The colevel structure of (X, L) is equivalent to the level structure of its dual $(\hat{X}, \hat{L})$ and vice versa. Therefore, the moduli space A_d^{bl} has an action of the wreath product $2 \wr PSL(2, Z/d)$.

<u>Theorem</u> (1) For $d = 2$, 3 and 4, the Satake compactification of A_d^{bl} is $(2 \wr G_d)$-equivariantly isomorphic to the projective 3-space $\mathbb{P}(M_{2\times 2}\ \mathbb{C})$.

(2) There exists a $(2 \wr G_5)$-eqivariant morphism

$$\psi : \tilde{\mathbb{P}}^3 \longrightarrow \bar{A}_5^{bl, \vartheta}$$

onto the Satake compactification and ψ contracts the strict transform of the 72 special lines (see below) to the 72 point cusps. ψ is an isomorphism elsewhere. Moreover, the exceptional divisors over the 60 points G_5 are the Hilbert modular surface Y_Γ in Example and parametrize the Comesatti surfaces, i.e., abelian surfaces with real multiplication by $O_{\sqrt{5}}$. ($\tilde{\mathbb{P}}^3$ is the blow-up of $\mathbb{P}^3$ at the 60 points G_5.)

(3) In both cases $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}(M_{2\times 2}\ \mathbb{C})$ parametrizes the products of two elliptic curves (of degree 1 and d).

Let $p_1, \ldots, p_t$ be the cusps of $\overline{X(d)}$. Then, by the theorem, the 2t lines $p_i \times \mathbb{P}^1$ and $\mathbb{P}^1 \times p_i$, $1 \leq i \leq t$, on $\mathbb{P}^1 \times \mathbb{P}^1$ are 1-dimensional boundaries of A_d^{bl}. A_2^{bl} and A_3^{bl} are the complement of these 2t lines. In order to describe A_4^{bl} and

A_5^{bl}, we need the following:

Definition A line in $\mathbb{P}^3$ joining two points $[g_1]$ and $[g_2]$ of $G_d \subset PGL(2)$ is special if $g_1 g_2^{-1} \in G_d$ is of order d.

The number of special lines is equal to 9, 16, 18 and 72 for d = 2, 3, 4 and 5. In the case d = 2, 3, the special lines parametrize the polarized abelian surfaces (X, L) which have symplectic automorphisms of order d.

Proposition (1) The moduli space A_4^{bl} is the complement of 12 lines $p_i \times \mathbb{P}^1$ and $\mathbb{P}^1 \times p_i$, and the 18 special lines in $\mathbb{P}^3$.

(2) The moduli A_5^{bl} is the complement of the strict transform of the 24 lines $p_i \times \mathbb{P}^1$, $\mathbb{P}^1 \times p_i$ and the 72 special lines in the blow-up $\widetilde{\mathbb{P}}^3$.

Let K_4 be the Klein's subgroup of the octahedral group G_4. The action of $K_4 \times K_4$ on $\mathbb{P}^3$ is the projectivization of the Schrödinger representation of the Heisenberg group. There are 15 involutions in $K_4 \times K_4$ and each has the union of two skew lines as its fixed locus. The 18 and 12 lines in (1) of the proposition coincide with these 30 fixed lines. Hence $\bar{A}_4^{bl,\dagger}$ is identified with the common ambient space of all Kummer's quartic surfaces. The quotient of $\bar{A}_4^{bl,\dagger} \simeq \mathbb{P}^3$ by the action of $K_4 \times K_4$ is the moduli space $\overline{A_1(2)}^{\dagger}$ of principally polarized abelian surfaces with a full level 2 structure.

I close this note with a remark on the Voronoi (troidal) compactification $\bar{A}_d^{bl,Vor}$ of A_d^{bl}. There exists a natural morphism

$$\bar{A}_d^{bl,Vor} \longrightarrow \bar{A}_d^{bl,\dagger}.$$

In the case $d = 2, 3$, this morphism is the composition of the blowing up along t lines $p_i \times \mathbb{P}^1$ and the blowing up along the strict transform of the remaining t lines $\mathbb{P}^1 \times p_i$. The universal family of abelian surfaces over A_d^{bl}, which exists in the sense of stack, extends to the family of semi-abelian surfaces over $\bar{A}_d^{bl,Vor}$. The blow-up in the reverse order is not $\bar{A}_d^{bl,Vor}$ but the family of dual abelian surfaces extends to semi-abelian surfaces over it.

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