$\boxed{\text{Igusa 3-fold and Enriques surfaces}}$

4/4/23(T)

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Abstract: This quartic was first found as the projective dual of Segre's 10-nodal cubic, the moduli of 6 points on the projective line. It was re-discovered as moduli of p.p.a.s's by Igusa(1962). I explain its new interpretation (Contemp. Math., 2012) as the moduli of Enriques surfaces of certain root type.

Two modular interpretations of Igusa (+ Steiner)

2-dim'l analogue of

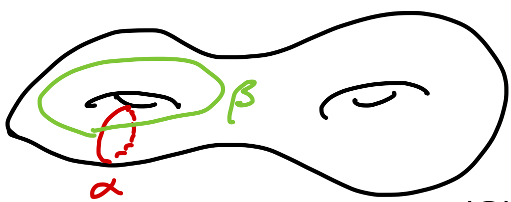
$$X(2) = \mathfrak{h}/\Gamma(2) \cong \mathbb{P}^1 \setminus \{0, 1, \infty\} \quad \boxed{A}$$

$$X_1(2) = \mathfrak{h}/\Gamma_0(2) \cong \mathbb{P}^1 \setminus \{0, \infty\} \quad \boxed{B}$$

$\boxed{A}$ moduli of genus 2 curves with (full) level 2 str.

$\boxed{B}$ Enrique's surface of HG-type

$\boxed{A}$ Igusa as moduli of curves



g(C)=2

1 Period

$C \quad y^2 = f_6(x)$

$\downarrow$

$\mathbb{P}^1$

$$\Omega := \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} dx/\sqrt{f_6} \\ x\,dx/\sqrt{f_6} \end{pmatrix}$$

periods $\displaystyle\int_{\alpha_i} \Omega, \int_{\beta_i} \Omega \in \mathbb{C}^2 \quad (i=1,2)$

$$\mathrm{Jac}\, C := \mathbb{C}^2 \Big/ \left(\mathbb{Z}^4 \text{ generated by periods}\right) \overset{\text{Abel}}{\xleftarrow{\sim}} \mathrm{Pic}^d C$$

2

$$\begin{cases} d=0 & \ni \text{origin} \\ d=1 & \ni \Theta = C \end{cases}$$

period matrix $\sim (1_2 \ Z)$

$$Z \in \mathfrak{H}_2/\Gamma(1) \subset \overline{\mathfrak{H}_2/\Gamma(1)} \quad \text{Satake cptfn}$$

$$\begin{cases} {}^tZ = Z \\ \mathrm{Im}\, Z > 0 \end{cases}$$

$$\uparrow \ /\mathfrak{S}_6$$

$$\Gamma(1) = Sp(4, \mathbb{Z})/\{\pm 1_4\} \qquad \overline{\mathfrak{H}_2/\Gamma(2)}$$

$$\Gamma(2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv 1_4 \mod 2 \right\}$$

$$\Gamma(2)/\Gamma(1) \simeq Sp(4, \mathbb{Z}/2) \cong \mathfrak{S}_6$$

Igusa (1964) embedding by 10 even theta "constants"

$\exists$ 5 linear rel's

$$\begin{array}{ccccc} \overline{\mathfrak{H}_2/\Gamma(2)} & \hookrightarrow & \mathbb{P}^4 & \hookrightarrow & \mathbb{P}^9 \\ \rotatebox{90}{$\in$} & & & & \\ Z & \longmapsto & \multicolumn{3}{l}{(\vartheta_m(Z)^4)_{m:\text{ even}}} \end{array}$$

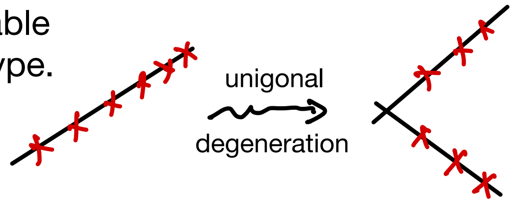
Image is a quartic 3-fold whose singular locus is union of 15 lines.

Each $\vartheta_m^4(Z) = 0$ cuts a (double) quadric surface Q_m for even m.

② $$\begin{array}{ccccc} \widetilde{\mathcal{M}}_2 & \xrightarrow[\text{map}]{\text{period}} & \overline{\mathfrak{H}_2/\Gamma(2)} & \underset{\text{Igusa}}{\hookrightarrow} & \mathbb{P}^4 \\ \mathfrak{S}_6\text{-covering} \downarrow & & & & \\ \mathcal{M}_2 & & & & \end{array}$$

Fact: $\widetilde{\mathcal{M}}_2$ is the complement of $\bigcup\limits_{m:\text{even}} Q_m$.

<u>Rmk</u> $Q_m \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong \overline{X(2)} \times \overline{X(2)}$ parametrizes product abelian surfaces $E_1 \times E_2 = \mathrm{Jac}\, E_1 \vee E_2$, the Jacobean of stable curve of compact type.



③ Bielliptic curves and Steiner surfaces

C is bielliptic $\overset{\text{def.}}{\iff}$ $\exists$ involution σ such that C/σ is elliptic

$\left(\Rightarrow \sigma \curvearrowright \{w_1, \dots, w_6\} \text{ is permutation type } (2)^3 \right)$
Weierstrass points $\quad (2)^3 \subset \mathfrak{S}_6$

$\iff \{\overline{w}_1, \dots, \overline{w}_6\} \subset \mathbb{P}^1$ is a line section of a complete quadrangle $\bigcup\limits_{1 \le i < j \le 4} \overline{p_i p_j}$ in $\mathbb{P}^2$.

This classical theorem gives a morphism

$$\mathbb{P}^{2,*} \twoheadrightarrow \overline{\begin{pmatrix}\text{moduli of}\\ \text{bielliptic}\\ \text{curves}\end{pmatrix}}$$

onto one of 15 components in Igusa.

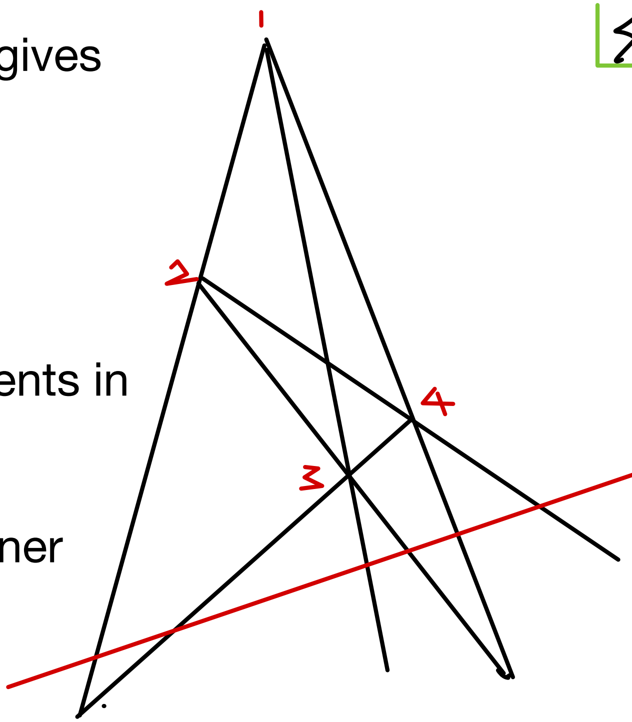
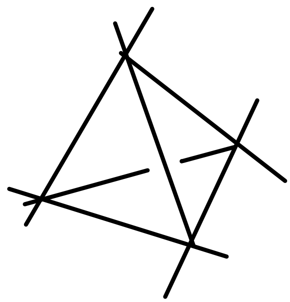


Image of $\mathbb{P}^{2,*}$ is a Steiner (Roman) surface.

$$\begin{pmatrix}\text{tetrahedron}\\ xyzt = 0\end{pmatrix} \Big/ \begin{pmatrix}\text{Klein's}\\ \text{4-group}\end{pmatrix}$$

Non-normal quartic surface singular along (line)$\vee$(line)$\vee$(line).



$\subset \mathbb{P}^3$

Fact: Igusa quartic has 15 linear involutions σ with Fix σ Steiner surfaces

Key for passing from $\boxed{A}$ to $\boxed{B}$

$$\Gamma_1(2) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1_2 & * \\ 0_2 & 1_2 \end{pmatrix} \bmod 2 \right\}$$

$\cup$ index 8

$\Gamma(2)$

$$\overline{\mathfrak{H}_2/\Gamma(2)} \xrightarrow{\text{quotient by } C_2^3} \overline{\mathfrak{H}_2/\Gamma_1(2)}$$

<u>Fact</u>: $\overline{X_1(2)}$ is again Igusa quartic.

moduli of (C, G)

$G < (\mathrm{Jac}\, C)_{(2)}$

Göpel, i.e., $\# G = 4$

Weil pairing$|_G \equiv 0$

$\boxed{\text{Igusa quartic has a self-morphism of degree 8.}}$

$\boxed{B}$ Igusa as moduli of Enrique's surfaces

Motivation

3-dim'l $\qquad$ 19 dim'l

Abelian surface $\mathrm{Jac}\, C$ or $E_1 \times E_2$ $\longmapsto$ Kummer surface $\sim \mathrm{Jac}\, C/\pm 1 \subset$ {alg. K3 surfaces}

$\vdots$

Abelian surface $\overset{?}{\longmapsto}$ $* \subset$ {Enriques surfaces}

10-dim'l family

Q. Find $\ast$.
Answer by M.-Ohashi(2013):
$\ast$ should be Hutchinson-Gopel (HG) type.

mini-history

Kummer(1864) Found 3-dim'l family of quartic surfaces with 16 nodes (16 is maximal possible)

Borchardt(1877) Uniformized them by abelian surfaces, or h.e. functions.
Kummer's equation is equivalent to Gopel's one found in 1847.

Hutchinson(1901) Found a new equation of Kummer quartic $\overline{Km}(C) \subset \mathbb{P}^3$, with

reference to Gopel subgroup $G \subset (\mathrm{Jac}\, C)_{(2)}$,

which is invariant under standard Cremona transformation $(x:y:z:t) \longleftrightarrow \left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{t}\right)$

of $\mathbb{P}^3$.

More precisely, we have

7

(C, σ) bielliptic & $G = \{a \in (\mathrm{Jac}\, C)_{(2)} \mid \sigma(a) = a\}$

$\Rightarrow \langle G \rangle =$ plane $\subsetneqq \mathbb{P}^3$ (degenerate case)

Otherwise standard Cremona transformation induces an involution

$$\varepsilon_G \in \mathrm{Aut}\,(\mathrm{Km}\, C).$$

Quotient $\mathrm{Km}\, C / \varepsilon_G$ is called Enriques of HG type if ε_G is free.

Enriques surface $S = X/\varepsilon$ K3 surf./free inv.

1 [period of S] $\in \mathcal{D}^{10}$ bdd symmetric domain of type IV

well-defined modulo $O_{\mathbb{Z}}(2, 10)$,

the orthogonal group of diag$[1, 1, \underbrace{-1, \ldots, -1}_{10}]$

<Torelli type thm> $S \simeq S' \Leftrightarrow$ periods are the same (modulo $O_{\mathbb{Z}}(2,10)$)

<surj. thm> {periods of S} $\overset{\text{open}}{\hookrightarrow} \mathcal{D}^{10}/O_{\mathbb{Z}}(2, 10)$

is the complement of a divisor $\mathcal{C}$

8

Rmk (1) $\mathcal{C}$ is the zero locus of Borcherds' Φ. ($\Rightarrow$ quasi-projectivity of moduli)

(2) $\mathcal{C}$ parametrizes Coble surfaces X/ε, s.t. $\mathrm{Fix}\ \varepsilon = \{m \text{ nodes}\}$, $1 \leq m \leq 10$, X/ε has m singular points of type (1, 1)/4.

2 & 3

$\exists$ natural embeddings

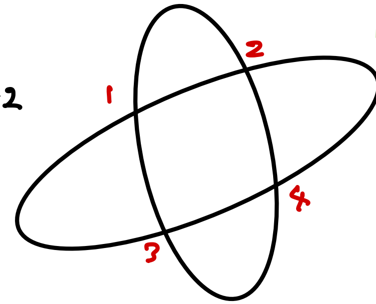
$$\begin{array}{ccc} \mathfrak{H}_2 & \hookrightarrow & \mathcal{D}^{10} \\ \downarrow & & | \\ \overline{X_1(2)} = \overline{\mathfrak{H}_2/\Gamma_1(2)} & \hookrightarrow & \overline{\mathcal{D}^{10}/O_{\mathbb{Z}}(\ )} \end{array}$$

which is geometrically interpreted as follows:

$\boxed{\text{Theorem}}$ **2** $\overline{X_1(2)} \cap \mathcal{C}$ is the union of 2 Steiner surfaces H_4 & H_8. The complement of $H_4 \cup H_8$ is moduli of Enriques surfaces of HG-type. (Root type $D_6 + A_1$)

3 $(C, G) \in H_4 \Rightarrow \exists$ bielliptic involution σ s.t. $G \subset \overline{Km(C)} \subset \mathbb{P}^3$, $\langle G \rangle \cong \mathbb{P}^2$, $\langle G \rangle \cap \overline{Km(C)}$

is the union of 2 conics.
Their strict transforms R_1 and R_2 are disjoint on Km(C).



G = {1,2,3,4}

Contract R's to 2 nodes and take quotient by σ. Then one obtains a Coble surface with two (1,1)/4 singular points (m=2).

3 (cont'd) $(C, G) \in H_8 \iff$ Jac C has real multiplication by $\sqrt{2}$, i.e., End( ) $\supset \mathbb{Z}[\sqrt{2}]$.
Km C has an extra $(-2)\ \mathbb{P}^1$, which is fixed by ε_G
(Coble surfaces with m=1).

References

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(The next page was used at the beginning of my 3rd talk on 5(W) to explain type II & III boundaries.)

self-morphism of Igusa_4

$$\widetilde{\mathcal{M}}_{0,6} \overset{\text{open}}{\hookrightarrow} \overline{\mathfrak{h}_2/\Gamma(2)} \xrightarrow{/C_2\times C_2\times C_2} \overline{\mathfrak{h}_2/\Gamma_1(2)}$$

$$\widetilde{\mathcal{M}}_{0,6} = \widetilde{\mathcal{M}}_2$$

$$\left\{\begin{array}{c}\text{Enriques}\\ \text{of HG}\\ \text{type}\end{array}\right\} \hookrightarrow \overline{\mathfrak{h}_2/\Gamma_1(2)}$$

complement divisor

$$\bigcup_{10} \overline{X_0(2)} \times \overline{X_0(2)}$$

2 Steiner surfaces

These are nterior divisor when regarded as moduli

of ppAS's

of covering K3's

True bdry = $\bigcup_{1\le i<j\le 4} l_{ij}$

True bdry = $\bigcup$ 6 lines

= Sing Igusa_4

Cayley-Richmond configuration $(15_3\text{-}15_3)$

(15 lines)

15 intersection points

complement

type II

type III

Remark Remaining 9 lines parametrize Enrique's surfaces with extra automorphism.

	N	N^2
type II	$\neq 0$	0
III		$\neq 0$

N = log P

quasi-unipotent monodromy P