Igusa quartic and Steiner surfaces

Dedicated to Prof. Tetsuji Shioda on his 70th birthday

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Abstract. The Igusa quartic has a morphism of degree 8 onto itself. Via this self-morphism, the Satake compactification $\mathcal{A}_1^s(2)$ of the moduli of principally polarized abelian surfaces with Göpel triples (as well as usual p.p.a.s.'s with full level-2 structures) is isomorphic to the Igusa quartic. We also determine the action of Fricke involution on the moduli.

In the workshop in the University of Georgia in October 2011, I gave a talk on Enriques surfaces of type E_7, which is a continuation of [**9**] and will appear elsewhere. In this article, instead I report on a new interpretation of the Igusa quartic as a moduli, which was found in my study of such Enriques surfaces (*cf.* Remark 7).

The Satake compactification $\mathcal{A}^s(2)$ of the moduli space $H_2/\Gamma(2)$ of principally polarized abelian surfaces is a quartic hypersurface in $\mathbb{P}^4$, called the *Igusa quartic*, where H_2 is the Siegel upper half space of degree 2 and $\Gamma(2)$ is the principal congruence subgroup of level 2 in $Sp(4,\mathbb{Z})$. We characterize the Igusa quartic using Steiner quartic surfaces, or Steiner's Roman surfaces. As a corollary, we show that the Satake compactification $\mathcal{A}_1^s(2)$ of the moduli of principally polarized abelian surfaces with Göpel triples is also isomorphic to the Igusa quartic.

A *Steiner surface* is an irreducible quartic surface in $\mathbb{P}^3$ whose singular locus is the union of three lines meeting at a point ([**10**]). A Steiner surface has seven planes which cut out double conics, or tropes, from it. Three are the unions of two double lines. The other four are linearly independent and cut out irreducible double conics. Taking these four planes as the reference tetrahedron $x_0x_1x_2x_3 = 0$ of homogeneous coordinates, a Steiner surface is normalized in the form

$$(s_1^2 - 4s_2)^2 = 64s_4, \tag{1}$$

where s_i is the elementary symmetric polynomial of degree i in the coordinates x_0, x_1, x_2, x_3. (See (10) for another equation.) In particular, all Steiner surfaces are isomorphic to each other.

Let X be a hypersurface in $\mathbb{P}^4$ and σ a linear and *reflective* involution of $X \subset \mathbb{P}^4$, that is, a lift of σ to $GL(5,\mathbb{C})$ has four 1's and (only) one -1 as its eigenvalues.

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The fixed point set of the action of σ on $\mathbb{P}^4$ consists of an isolated point and a hyperplane $\mathbb{P}^3$. The projection $X \cdots \to \mathbb{P}^3$ from the isolated fixed point factors through the quotient X/σ.

The following is called the *Steiner property* of such a pair (X, σ).

$(*)$ The fixed point locus of σ is a Steiner surface R and the map $X/\sigma \cdots \to \mathbb{P}^3$ is a double covering with branch the union of four planes which cut out irreducible double conics from R.

A hyperquartic X is said to satisfy the Steiner property if there exists an involution σ such that (X, σ) satisfies it. Such a hyperquartic is isomorphic to the standard one

$$(x_4^2 - s_1^2 + 4s_2)^2 = 64s_4 \tag{2}$$

in $\mathbb{P}^4_{(x_0:\ldots:x_4)}$.

The following observation is the starting point of our consideration.

Proposition 1. *The Igusa quartic satisfies the Steiner property and has a morphism of degree* 8 *onto itself.*

(See Remark 4 for the geometric meaning of the involution σ in this case.)

We denote the congruence subgroup of $Sp(4, \mathbb{Z})$ consisting of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $C \equiv 0$ (2) by $\Gamma_0(2)$, and with $A - I_2 \equiv C \equiv 0$ (2) by $\Gamma_1(2)$. The quotient $H_2/\Gamma_0(2)$ is the moduli space of pairs (A, G) of principally polarized abelian surfaces A and Göpel subgroups $G \subset A_{(2)}$. (G is *Göpel* if it is maximally totally isotropic with respect to the Weil pairing.) The quotient $H_2/\Gamma_1(2)$ is the the moduli space of pairs (A, ψ)'s, where $\psi : (\mathbb{Z}/2)^{\oplus 2} \to A_{(2)}$ is an isomorphism onto a Göpel subgroup.

The element $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I_2 \\ -2I_2 & 0 \end{pmatrix} \in Sp(4, \mathbb{R})$ belongs to the normalizer of $\Gamma_1(2)$, and induces involutions of the quotient $H_2/\Gamma_0(2)$ and $H_2/\Gamma_1(2)$, which are called the *Fricke involutions*. More explicitly, the Fricke involution maps a pair (A, G) to $(A/G, A_{(2)}/G)$. Since $A_{(2)}/G$ is isomorphic to G via Weil pairing, the Fricke involution of $H_2/\Gamma_1(2)$ is also well defined. Two pairs (A, G) and $(A/G, A_{(2)}/G)$ are geometrically related by Richelot's theorem. See Remark 7.

Theorem 2. *The Satake compactification $\mathcal{A}_1^s(2)$ of $H_2/\Gamma_1(2)$ is a hyperquartic in $\mathbb{P}^4$ and the Fricke involution φ acts linearly on $\mathcal{A}_1^s(2) \subset \mathbb{P}^4$. Moreover, the pair $(\mathcal{A}_1^s(2), \varphi)$ satisfies the Steiner property. In particular, $\mathcal{A}_1^s(2)$ is isomorphic to the Igusa quartic and its quotient $\mathcal{A}_1^{*,s}(2)$ by the Fricke involution is the double cover of $\mathbb{P}^3$ with branch the union of four planes.*

As Terasoma [**11**] observes, the Fricke involution fixes the moduli of abelian surfaces with real multiplications by $\sqrt{2}$. The fact that the fixed point locus is a Steiner surface also follows from Hirzebruch [**5**]. It is interesting to compare our description with the computation of Siegel modular forms in [**7**] but we do not pursuit it here.

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1. Self-morphism of degree 8

We first construct a self-morphism (of degree 8) of the quartic hypersurface (2).

Let Y be the double $\mathbb{P}^3$ with branch the union of four linearly independent planes. The symmetric group $\mathfrak{S}_4$ of degree four acts on Y permuting the four planes.

LEMMA 3. *The quotient of the above threefold Y by the action of the Klein's 4-group $K_4 \subset \mathfrak{S}_4$ is isomorphic to* (2).

PROOF. Y is expressed as $z^2 = y_0y_1y_2y_3$ for a homogeneous coordinate $(y_0 : y_1 : y_2 : y_3)$ of $\mathbb{P}^3$. To compute the quotient we make the following coordinate transformation:

$$x_0 = (y_0 + y_1 + y_2 + y_3)/2, \quad x_1 = (y_0 + y_1 - y_2 - y_3)/2, \tag{3}$$

$$x_2 = (y_0 - y_1 + y_2 - y_3)/2, \quad x_3 = (y_0 - y_1 - y_2 + y_3)/2.$$

Then Y is expressed as

$$16z^2 = (x_0+x_1+x_2+x_3)(x_0+x_1-x_2-x_3)(x_0-x_1+x_2-x_3)(x_0-x_1-x_2+x_3)$$

and as $16z^2 = S_1^2 - 4S_2 + 8x_0x_1x_2x_3$, where S_i is the elementary symmetric polynomial of degree i in the new variables $X_0 := x_0^2, \dots, X_3 := x_3^2$. Since K_4 interchanges even number of signs of x_1, x_2 and x_3, the quotient Y/K_4 is $(S_1^2 - 4S_2 - 16z^2)^2 = 64X_0X_1X_2X_3$. Hence the quotient Y/K_4 is isomorphic to (2). □

When (X, σ) has the Steiner property, the quotient X/σ is isomorphic to the threefold Y in the lemma. Therefore, (2) has a self-morphism of degree 8. Its explicit form is give by

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto \tag{4}$$

$$((x_0 + x_1 + x_2 + x_3)^2 : \cdots : (x_0 - x_1 - x_2 + x_3)^2 : 2(S_1^2 - 4S_2 - x_4^2)).$$

2. Proof of Proposition 1

We give three proofs.

Proof 1. To use the equation

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4) = 0 \tag{5}$$

in Igusa [**8**, p. 397] is the simplest. The interchange of y_3 and y_4 is an involution of this hyperquartic. Its fixed point locus

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3^2)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + 2y_3) = 0$$

is isomorphic to the Steiner surface (1) by regarding $y_0 + y_1 + y_2 + 2y_3$ as a new coordinate. Therefore, (5) is isomorphic to (2) and satisfies the Steiner property.

Proof 2. As is well-known, the Igusa quartic is isomorphic to the hyperquartic

$$\sigma_1 = \sigma_2^2 - 4\sigma_4 = 0 \tag{6}$$

which is invariant under the natural action of the symmetric group of degree six ($\simeq Sp(4,\mathbb{Z})/\Gamma(2)$), ([**4**, Sections 4, 5]), where σ_i is the elementary symmetric polynomial of degree i in the six coordinates $x_1, \dots, x_6$. It is easy to see from this equation that the Igusa qaurtc has 15 double lines. The complement of these 15 lines is isomorphic to $H_2/\Gamma(2)$.

Now we consider the involution of (6) interchanging x_5 and x_6. The hyperplane $x_5 = x_6$ contains three of 15 double planes and cut out a Steiner surface. Let us see more throughly. The hyperquartic (6) is defined by

$$(x_5x_6 - s_1^2 + s_2)^2 = 4(x_5x_6s_2 - s_1s_3 + s_4),$$

where s_i is the elementary symmetric polynomial of degree i in the four coordinates $x_1, \dots, x_4$. Putting $x_0 = x_5 - x_6$, (6) is expressed as a hyperquartic

$$(x_0^2 + 3s_1^2 + 4s_2)^2 = 64(x_2+x_3+x_4)(x_1+x_3+x_4)(x_1+x_2+x_4)(x_1+x_2+x_3) \tag{7}$$

in $\mathbb{P}_{x_0:\dots:x_4}$. The fixed point locus

$$(3s_1^2 + 4s_2)^2 = 64(x_2+x_3+x_4)(x_1+x_3+x_4)(x_1+x_2+x_4)(x_1+x_2+x_3) \tag{8}$$

is a Steiner surface and (7) satisfies the Steiner property.

Proof 3. A principally polarized abelian surface which is not of product type is mapped onto a Kummer quartic surface in $\mathbb{P}^3$ by the linear system of twice the theta divisor. Its equation

$$a(x^4+y^4+z^4+t^4)+b(x^2y^2+z^2t^2)+c(x^2z^2+y^2t^2)+d(x^2t^2+y^2z^2)+16exyzt = 0 \tag{9}$$

(with coefficients $a, \dots, e \in \mathbb{C}$) is classically known ([**6**]) and is invariant under the action of the Heisenberg group. The Satake compactification $\mathcal{A}^s(2)$ of $H_2/\Gamma(2)$ is the quotient of the ambient $\mathbb{P}^3$ by the Heisenberg (projective) action of $B \simeq (C_2)^4$. More precisely, the ambient $\mathbb{P}^3$ is the Satake compactification $\mathcal{A}^s(2,4)$ of $H_2/\Gamma(2,4)$ ([**3**, Proposition 1.7]). The group B has an exact sequence $0 \to B_1 \to B \to B_2 \to 0$ such that $B_1 \simeq B_2 \simeq C_2^2$, that B_1 changes even number of signs of the coordinates x, y, z, t, and that B_2 permutes them like Klein's 4-group modulo sign. The quotient Y of $\mathbb{P}^3$ by B_1 is the double $\mathbb{P}^3$ with branch the union of four coordinate planes. Hence the quotient $\mathbb{P}^3/B$ is isomorphic to (2) by Lemma 3 and satisfies the Steiner property.

3. Proof of Theorem 2

First we prove the following part of the theorem:

Claim: the Satake compactification $\mathcal{A}_1^s(2)$ is isomorphic to the Igusa quartic.

Proof. We restart from the expression (7) of $H_2/\Gamma(2)$ and take its quotient by the group $\Gamma_1(2)/\Gamma(2) \simeq (C_2)^3$. When a principally polarized abelian surface A is the Jacobian of a curve C of genus two, a Göpel subgroup G corresponds to a partition of the six Weierstrass points into three pairs. For example, $K_C - w_1 - w_2, K_C - w_3 - w_4, K_C - w_5 - w_6$ and 0 form a Göpel subgroup G_0. The group $\Gamma_1(2)/\Gamma(2)$, which preserves G_0, is generated by three transpositions $(12), (34)$ and (56). The action of the symmetric group of degree 6 on the coordinates of (6) is twisted by a nontrivial outer automorphism. Hence $\Gamma_1(2)/\Gamma(2)$ acts on $x_1, \dots, x_6$

by the permutation $C_2 \times K_4$, where C_2 is the symmetric group of two coordinates, say x_5 and x_6, and K_4 is the Klein's 4-group acting on the rest. The quotient Y of (7) by C_2, generated by σ_{56}, is the double $\mathbb{P}^3$ with branch the union of the four planes $x_2+x_3+x_4=0, x_1+x_3+x_4=0, x_1+x_2+x_4=0$ and $x_1+x_2+x_3=0$. Since K_4 permutes these four planes like Klein's 4-group, the quotient Y/K_4 is isomorphic to the Igusa quartic by Lemma 3. □

REMARK 4. The fixed point locus of σ_{56} contains the Jacobians of curves C of genus two with bi-elliptic involutions α ([**9**]) such that the action of α on the cohomology group $H^1(C, \mathbb{Z}/2)$ is the same as the element of $Sp(4, \mathbb{Z}/2)$ corresponding to σ_{56}.

Now we determine the action of the Fricke involution.

LEMMA 5. *The automorphism group of the Igusa quartic is the symmetric group* $\mathfrak{S}_6$ *of degree six.*

PROOF. First, we note that the automorphism group $\mathrm{Aut}\,(X)$ as an abstract variety coincides with that $\mathrm{Aut}\,(X \subset \mathbb{P}^4)$ as a projective variety, since $X \subset \mathbb{P}^4$ is an anti-canonical embedding of X.

The singular locus of the Igusa quartic $X \subset \mathbb{P}^4$ is the union of 15 lines. We construct a homomorphism $\mathrm{Aut}\,(X) \to \mathfrak{S}_6$ using an incidence relation of these lines and show its injectivity. Note that there are exactly six sets $D_1, \ldots, D_6$ of five disjoint double lines. Moreover, each intersection $D_i \cap D_j$, $i \neq j$, consists of one line, and every line is contained exactly two of $D_1, \ldots, D_6$. Hence we have an homomorphism $\mathrm{Aut}\,(X) \to \mathfrak{S}_6$, and if an automorphism belongs to the kernel it preserves each of 15 double lines. Since the intersection points of all pairs of distinct lines span the ambient project space $\mathbb{P}^4$, such an automorphism is the identity. □

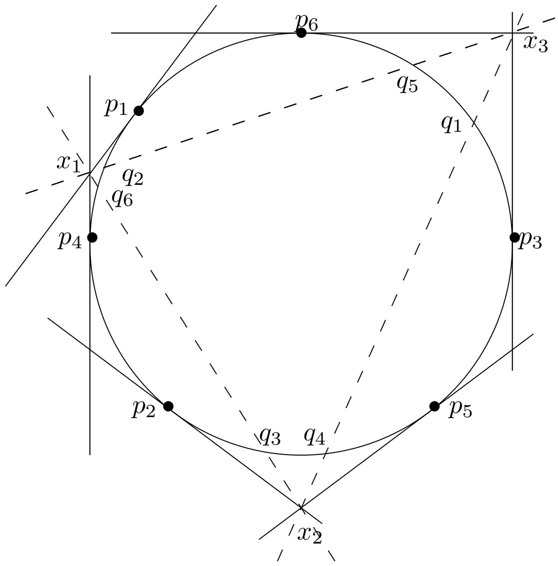
By the claim and the lemma, the automorphism group of the Satake compactification $\mathcal{A}_1^s(2)$ is $\mathfrak{S}_6$. Hence there are three types of involutions, that is, permutation type $(2), (2)^2$ and $(2)^3$. Since the Fricke involution fixes the moduli points of abelian surfaces with real multiplication by $\sqrt{2}$ and such abelian surfaces forms a 2-dimensional family, the permutation type of the Fricke involution is (2). Hence the pair of $\mathcal{A}_1^s(2)$ and the Fricke involution satisfies the Steiner property. Thus the proof of Theorem 2 is completed.

REMARK 6. When we regard (2) as the Stake compactification $\mathcal{A}_1^s(2)$, the hyperplane section $\tau = 0$ is an Humbert surface of discriminant 8 as we already saw above. We find two kinds of other Humbert surfaces in $\mathcal{A}_1^s(2)$. They are the hyperplane sections $\tau = \pm(-x_0+x_1+x_2+x_3)$. As surfaces, they are defined by

$$x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = 4x_0x_1x_2x_3 \tag{10}$$

in $\mathbb{P}^3$. This is again a Steiner surface and singular along three lines $x_1 = x_2 = 0$, $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$. One of them, say $\tau = -x_0+x_1+x_2+x_3$ parametrizes abelian surfaces of product type and the other parametrizes bi-elliptic ones. The Fricke involution $\tau \mapsto -\tau$ interchanges these two Humbert surfaces.

REMARK 7. Let A be the Jacobian of a (smooth) curve C of genus 2 and $p_1, \ldots, p_6$ be the images of the Weierstrass points $P_1, \ldots, P_6$ of C by the bi-canonical morphism $\Phi_{2K} : C \to \mathbb{P}^2$. Assume that a Göpel sugbroup G of C is not bi-elliptic ([**9**]). Then the quotient abelian surface A/G is again the Jacobian of a curve C' of genus 2. Moreover, the bi-canonical images of the Weierstrass points of C' is projectively equivalent to the the $q_1, \ldots, q_6 \in \mathbb{P}^2$ in the figure below by Richelot's theorem (cf. [**1**], [**2**, §4]). Here G consists of the divisor classes $[P_i - P_{i+3}]$, $i = 1, 2, 3$, and 0, and x_i is the intersection of two tangent lines of the conic $\Phi_{2K}(C)$ at p_i and p_{i+3}. This is the geometric interpretation of the Fricke involution $(A, G) \mapsto (A/G, A_{(2)}/G)$ of $\mathcal{A}_1(2)$, and plays an essential role in our sturdy of Enriques surfaces of type E_7.



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