Inose quartic surfaces & type II degeneration

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Arithmetic

A1 Even unimodular lattice of signature (1, 17)

$$II_{1,17} = U + E_8 + E_8 \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ E_8 = II_{0,8}$$

A2 $II_{1,17} + \langle -2 \rangle$

Geometry

G1 Aut (Inose) & type II degeneration of K3 surfaces

G2 Bir-Aut (Inose$^{[2]}$) & type II degeneration of hol. symp. 4-folds

§1 Lattice $II_{1,17}$ as Pic (Inose)

$$H^2(K3, Z) = II_{3,19} \hookleftarrow II_{1,17}$$

$\exists$ 2-parameter family of K3 surfaces whose Picard lattice contains $II_{1,17}$

Explicitly given by Inose(1978) as quartic surface

$$S_{a,b} : y^2 z + \cdot\cdot\cdot = 0 \subset \mathbb{P}^3$$

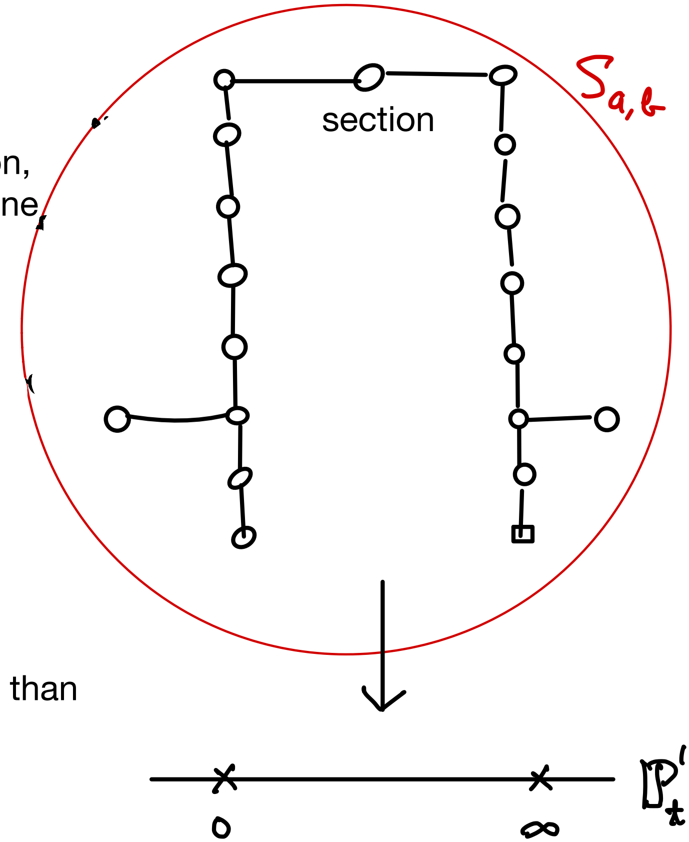
Simpler realization is

$$y^2 = x^3 - at^4 x + t^5(t^2 + 1 - 2bt),$$

elliptic K3 surface with 2 $\tilde{E}_8$-fibers at $t = 0, \infty$ and a section

We assume a,b are very general and Picard number $\rho = 18$.

By vinberg's computation, these 19 (-2) classes define a fundamental domain in the positive cone.

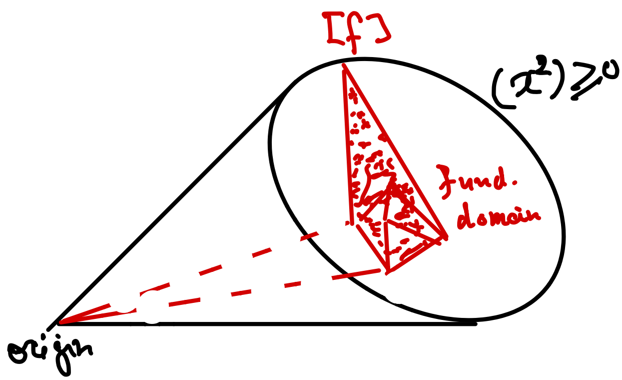


Geometric implication

(1) No other (-2)P 's other than what we see.

(2) Nef cone is polyhedral with 19 facets.
The Π-diagram is the Coxeter diagram.

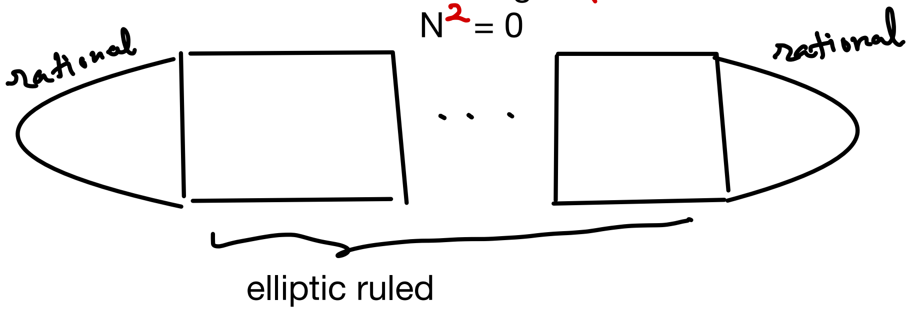
(3) Aut($S_{a,b}$) is finite.



§2 $II_{1,17}$ as type II K3 lattice Λ_{II}

Kulikov model of type II degeneration of K3

$N = \log P \neq 0$
$N^2 = 0$

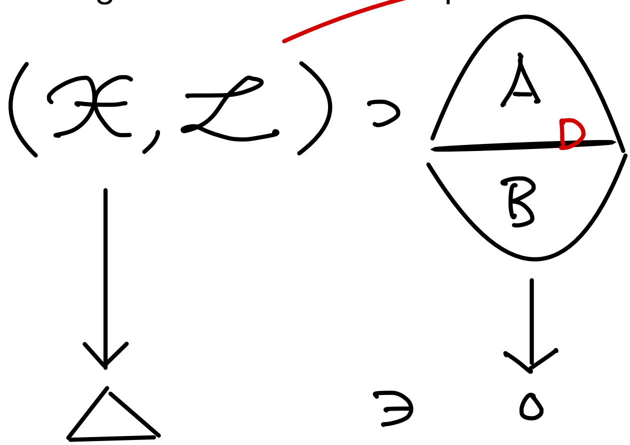


One difficulty of (compact) moduli construction of polarized K3 (X, L) is that naive

$\{RDP\} \cup \{\text{type II}\} / \text{isom}$

is not Hausdorff by the following reason. (Another reason caused by flop is discussed in other lectures and omitted here.)

(T) Assume that the central fiber is reducible, say, A B. Then tensoring line bundle changes the distribution of polarizations.



Restriction

(L_A, L_B)

$\rightsquigarrow \otimes \mathcal{O}_{\mathcal{X}}(\pm A)$

$(L_A(\mp D), L_B(\pm D))$

$D = A \cap B$

Example

$$\begin{array}{ccc} & & \text{cubic} \quad \text{plane} \\ \mathbb{P}^3 \supset X_4 & \xrightarrow{\text{degeneration}} & T \cup P \subset \mathbb{P}^3 \end{array}$$

$$\begin{array}{ccc} \mathcal{X} & \supset & T \cup P' \\ \downarrow & & \downarrow \\ \Delta & \ni & 0 \end{array}$$

$T \cup P' \to T \cup P$: 12-pt blow-up to make family d-semi-stable

	deg			$\otimes \mathcal{O}_{\mathcal{X}}(T)$	deg
P'	1	ℓ	$\longmapsto$	$\ell + D = 4\ell - \sum_1^{12} p_i$	4
T	3	$\mathcal{O}_T(1)$	$\longmapsto$	$\mathcal{O}_T(1) - D$	0

Which one among two (or more) polarized type II K3 surfaces should we take when (partially) compactifing moduli?

My proposal in 1989: Choose it following Vinberg's fundamental domain (or diagram) since $\mathrm{II}_{1,17}$ is the Picard lattice of the standard type II K3 surface.

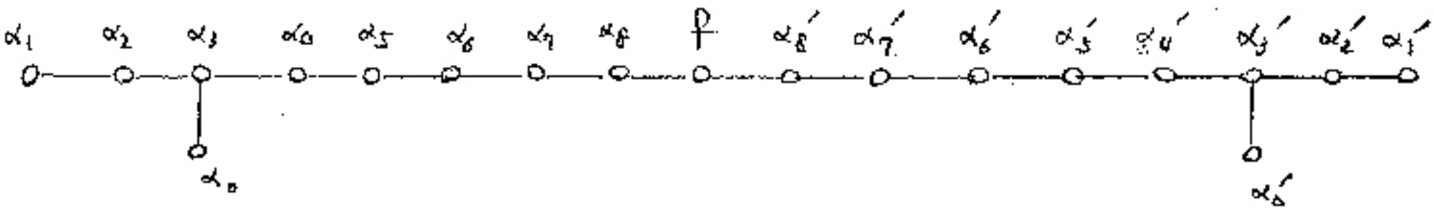
$$K3^{std}_{\mathrm{II}} := Bl_9\, \mathbb{P}^2 \underset{B=B'}{\cup} Bl_9\, \mathbb{P}^2$$

and the fundamental domain coincides with the new cone.

Standardization of type II K3's to make moduli Hausdorff.
The next page shows Π-diagram in Λ_{II}, the Picard lattice of $K3^{std}_{\mathrm{II}}$.

Thus for Example above, our choice is the union $T \cup P$, not a rational surface with an elliptic singularity of type $\widetilde{E}_6$.

Π - Diagram of the lattice Λ_{II}



$$\alpha_0 = h - e_1 - e_2 - e_3$$
$$\alpha_0' = h' - e_1' - e_2' - e_3'$$
$$\alpha_i = e_i - e_{i+1} \qquad 1 \le i \le 8$$
$$\alpha_i' = e_i' - e_{i+1}' \qquad 1 \le i \le 8$$
$$f = 3h - e_1 - e_2 - \cdots - e_9 = 3h' - e_1' - \cdots - e_9'$$

lattice

$$\Lambda_{II} = \delta^{\perp}/\mathbb{Z}\delta \quad \text{in} \quad \langle h, h', e_1, \ldots, e_9, e_1', \ldots, e_9' \rangle_L$$
$$\delta = (3h - e_1 - \cdots - e_9) - (3h' - e_1' - \cdots - e_9')$$

Vector in the fundamental domain

$$w = w_+ + af + w_- \qquad w_+ = \sum_{i=0}^{8} b_i w_i \qquad w_- = \sum_{j=0}^{8} c_j w_j'$$

$$w \in \Lambda_{II} \Leftrightarrow a,\ b_i,\ c_j \ \text{integer}$$
$$(w_+ . f) = (w_- . f) \left(\equiv_{\text{mod}} 2\right)$$

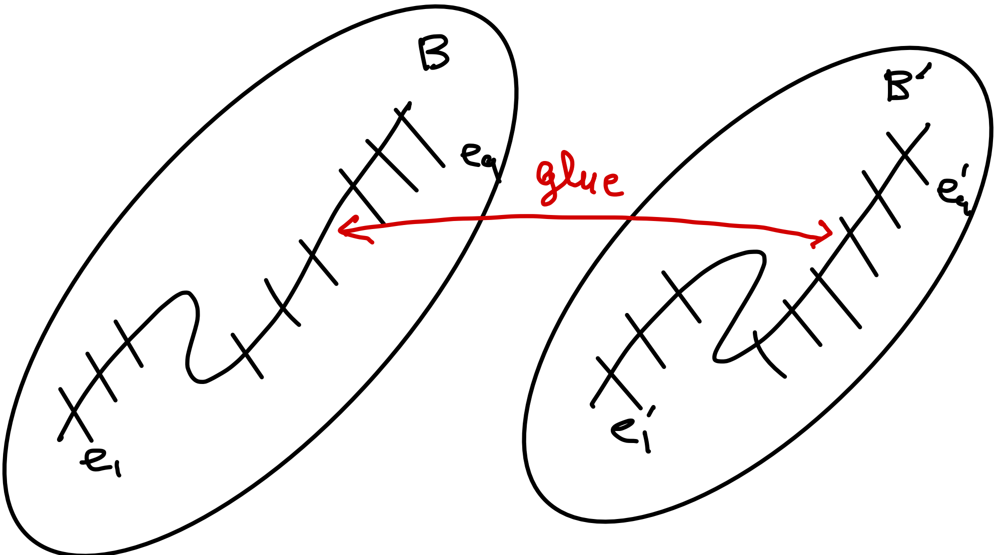
$w \in \Lambda_{II}$ lies in the fundamental domain $\Leftrightarrow$ (w, α_i), (w, α_j'), $(w . f)$ non negative

$$\Leftrightarrow a,\ b_i,\ c_j \ge 0$$

$w_0 = h$	$w_0' = h'$
$w_1 = h - e_1$	$w_1' = h' - e_1'$
$w_2 = 2h - e_1 - e_2$	$w_2' = 2h' - e_1' - e_2'$
$w_i = 3h - e_1 - e_2 - \cdots - e_i$	$w_i' = 3h' - e_1' - e_2' - \cdots - e_i'$

$$3 \le i \le 8$$

$B \in |3h - \sum_1^9 e_i|$ $\qquad$ $B' \in |3h' - \sum_1^9 e_i'|$



$$\mathrm{Pic}(K3^{\mathrm{std.}}_{\mathrm{II}}) = \delta^{\perp} / \mathbb{Z}\delta \cong \mathrm{II}_{1,17}.$$

where $\delta = B - B' \in \mathrm{Pic}(\mathrm{Bl}_9\mathbb{P}^2) \oplus \mathrm{Pic}(\mathrm{Bl}_9\mathbb{P}^2)$

$(\delta^2 = 0)$.

Before moving to §3, recall some
basics of holomorphic symplectic manifold of K3-type

K3	$K3^{[2]}$
S	$X = S^{[2]} \xrightarrow[\text{res.}]{\text{min.}} S^{(2)}$

$$\mathrm{Pic}\, S \hookrightarrow \mathrm{Pic}\, X = \mathrm{Pic}\, S + \mathbb{Z}\delta$$

as lattice

with Beauville form

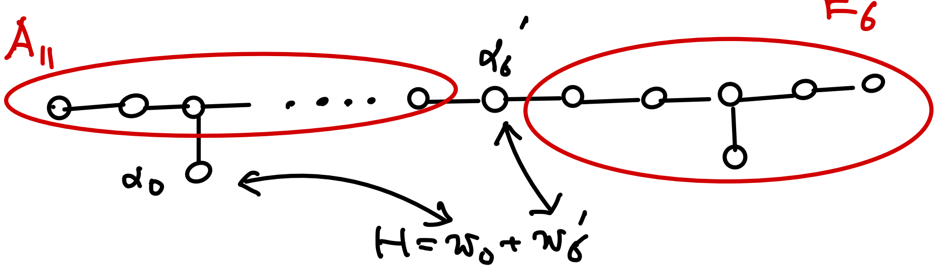
$\delta = \frac{1}{2}$ of exc. div. of

$(\delta^2) = -2$

§3 Bir-Aut (Inose[2])

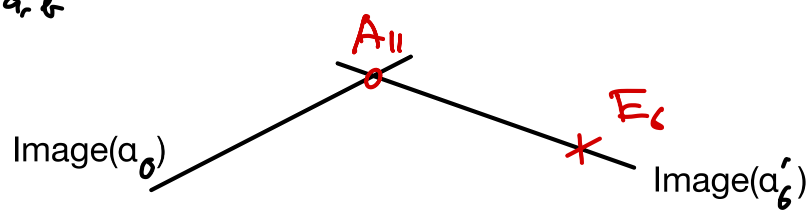
Inose quartic $S_{a,b} = y^2zt + \cdot\cdot\cdot = 0 \subset \mathbb{P}^3$

is $(U+2E_8)$-K3 surface embedded into $\mathbb{P}^3$ by $H = w_0 + w_6'$:



(Rmk cubic-plane, i.e., T^UP-degeneration in Example)

$S_{a,b} \ni$ A_{11}-RDP & E_6-RDP & 2 lines

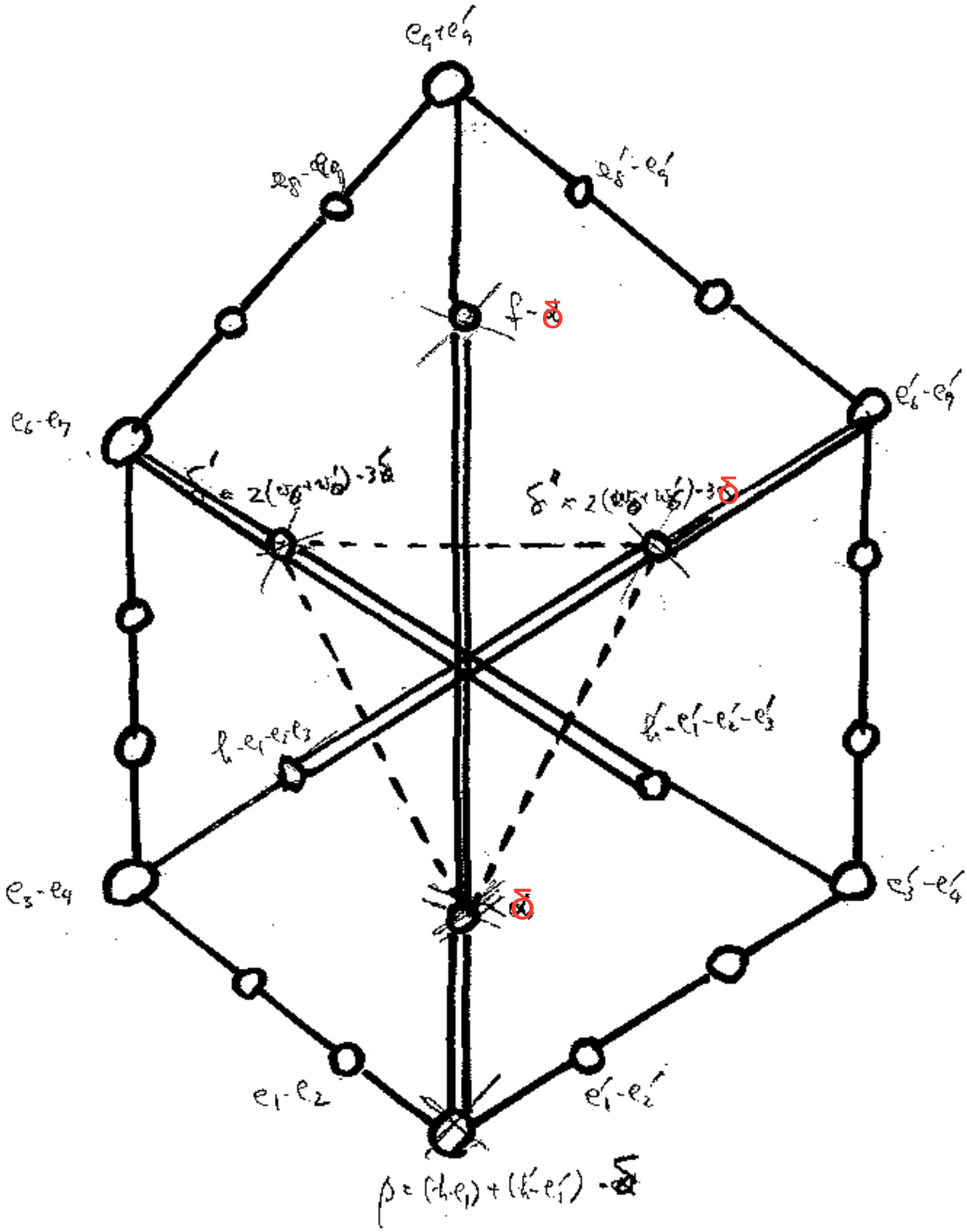


This quartic polarization plays a key role in the proof of the following:

Theorem Bir-Aut (Inose[2]) is finite.

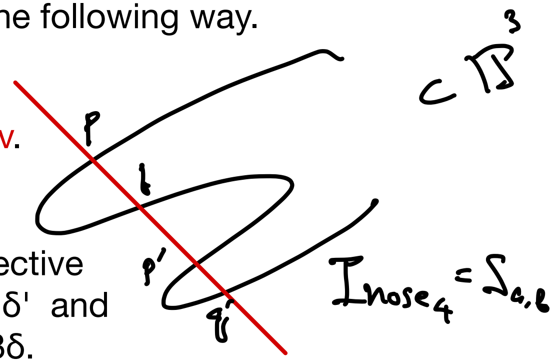
Proof: Similar to the Inose surface case after replacing nef cone, (-2) P', Vinberg's diagram by movable cone, Q-effective (-2) effective divisor, KLS-diagram, respectively.

KLS diagram = Π-diagram + 5 vertices 24=19+5
Vertices are (-2) classes in the Picard lattice of Inose[2].
Basis is written, using double $Bl_9\mathbb{P}^2$ model, on the next page.


e9+e9'
e8-e9
e8'-e9'
f-δ
e6-e7
e6'-e9'
δ' = 2(w8+w8')-3δ
δ'' = 2(w8+w8')-3δ
h-e1-e2-e3
h'-e1'-e2'-e3'
e3-e4
e3'-e4'
δ
e1-e2
e1'-e2'
ρ = (h-e1)+(h'-e1') -δ

<u>Remark</u> KLS stands for Kondo-Looijenga-Scattone. Kondo uses the diagram to compute the automorphism group of a K3 surface. Others, including the recent sophisticated construction by Alexeev-Engel-Thompson, use for type II degeneration of K3's of degree 2.

Proof (cont'd) (*) Show all classes are $\mathbb{Q}$-effective.
1. 19 (-2) classes in sub-Π-diagram comes from (-2)P^1 on Inose surface. Easy to prove their effectiveness.
2. Two comes from the Lagrangian fibration of Inose_4, induced from the elliptic fibration of Inose . Also easy.
3. δ' and δ". Here Inose quartic is crucial. From a pair of points p,q, one obtain new pair p',q' in the following way.



Thus one gets an involution of $\text{Inose}^{[2]}$, which is called Beauville inv.

δ' and δ" are the image of δ by Beauville inv's. Since δ is $\mathbb{Q}$- effective so are δ' and δ". Note that both δ' and δ" are of the form 2(Inose pol) - 3δ.

By (*), cone of movable divisors $\text{Mov}(\text{Inose}^{[2]})$ is finite polyhedral. Hence Bir-Aut$(\text{Inose}^{[2]})$ is finite. QED

§4 <u>Type II degeneration of holomorphic symplectic 4-fold of $K3^{[2]}$-type</u>

We still do'nt have a good theory like Kulikov's about this. I just pose

Problem: Find a standardization of , similar to K3, replacing Vinberg's Π by KLS-diagram.

Seems difficult to answer since geometric meaning of S^3- symmetry of KLS-diagram is still unclear.

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