Prime Fano 3-folds and Leech-like lattices

12/23/21 (Th)
YMSC
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§1 Introduction & main observation

§2 Computation in Beijing 2019

§3 Two more models

§4 $Q_8(1)$, Umemura 3-fold and conj. K3

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Zoom from Kyoto

Prime Fano 3-folds and Leech-like lattices

12/23/21 (Th)

Fano 3-folds $\longleftrightarrow$ $K3^{ss1}$ supersingular Artin inv. 1

(Leech-like lattices)

§1

① V prime Fano 3-fold $\Leftrightarrow \begin{cases} -K_V \text{ ample} \\ \operatorname{Pic} V = \mathbb{Z} K_V \end{cases}$

$(-K_V)^3 = 2g-2$, $|-K_V|$ very ample w. 2 exceptions

($g=2$ & h.e. $g=3$)

$$V_{2g-2} \underset{|-K_V|}{\hookrightarrow} \mathbb{P}^{g+1} \quad \text{anti-canonical model}$$

(Fano–Iskovskikh) $g = 2, 3, \ldots, 10, \cancel{11}, 12$

Key varieties

g	7	8	9	10
	$OG(5,10)^+ \hookrightarrow \mathbb{P}^{15}$ spinor	$G(2,6) \hookrightarrow \mathbb{P}^{14}$ Plücker	$Sp\text{-}G(3,6) \hookrightarrow \mathbb{P}^{13}$	$G_2\text{-}G(2,7)$ $\mathbb{P}^{13}$
Lie alg.	$\mathfrak{so}(10)$	$\mathfrak{sl}(6)$	$\mathfrak{sp}(6)$	$\mathfrak{g}_2$
Dynkin R	D_5	A_5	C_3	G_2

$g=12$

(M.-) Umemura 3-fold $\quad U_{22} = \overline{PGL(2)/\mathrm{Icosa}} \subset \mathbb{P}^{12}$

All V_{22}'s are deformation of U_{22} (6-dim'l family)

$=$ orbit closure of $[f(x,y) = xy^{11} + 11x^6y^6 - x^{11}y]$

② Most algebraic K3 surfaces ($K_S = 0$, $q = 0$)

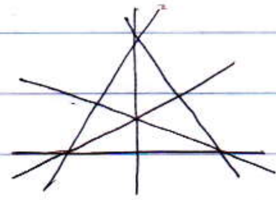
$$\mathrm{Pic}\, S \times \mathrm{Pic}\, S \longrightarrow \mathbb{Z} \qquad \mathrm{sgn} = (1, *)$$

/$\mathbb{C}$ Vinberg (1983) Two most alg. K3's S_3 & S_4

Picard number is maximal ($= 20$), and

$|\mathrm{disc}\ \mathrm{Pic}|$ is smallest ($= 3, 4$)

$S_4 : \tau^2 = xyz(x-y)(y-z)(z-x)$



$\mathrm{Aut}\, S_4 = 2 . C_2^{*5} . \mathfrak{S}_5$

/$\overline{\mathbb{F}}_p$ Picard number is ≤ 20 or 22, called supersingular. In s.s. case

$$|\mathrm{disc}\ \mathrm{Pic}| = p^{2\sigma}, \quad \sigma : \text{Artin inv.}$$

$1 \leq \sigma \leq 10$ $\qquad \sigma = 1$ is unique.

/$\mathbb{F}_p$ Picard number ≤ 21

$\boxed{\text{Theorem - Observation}}$ For each $g = 7, 8, 9, 10$, $\exists$ model $S/\mathbb{F}_q$ of $K3^{ss1}$ such that $\mathrm{Pic}_{\mathbb{F}_q} S$ is the orthogonal complement of the (negative) root lattice of Key variety in the extended Leech-like lattice

$$\mathrm{Pic}_{\mathbb{F}_q} S \cong Q(R)^{\perp} \text{ in } U \oplus \Lambda^{g} \quad \cdots\cdots (*)$$

$$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \mathrm{II}_{1,1}$$

Table

g	7	8	9	10	12 (conj.)
R	D_5	A_5	C_3	G_2	A_1 ?
ord g	1		2	3	5 ?
Λ^g	Leech		Barnes-Wall	Coxeter-Todd	$Q_8(1)$
rk Λ^g	24		16	12	8
$\lvert\mathrm{disc}\,\Lambda^g\rvert$	1		2^8	3^6	5^4
$\mathfrak{g}$	2	3	2	4	?
$K3^{ss1}$	Rd_2	Rd_3	Hermitian model	cubic twist	?
	§2		§3		

$\Lambda = (\mathbb{Z}^{24}, \langle\, ,\, \rangle)$ Leech lattice

(neg.) definite, unimodular, $\nexists$ (-2)-vector

constructed from Steiner system $S(5,8,24)$

$M_{23} \subset M_{24} \curvearrowright \Lambda$ Mathieu groups

Conjecture / Wish $\exists$ S $K3^{ss1}/\mathbb{F}_q$ satisfying $(\star)$

with $R = A_1$ and ord $g = 5$, that is,

$$\mathrm{Pic}_{\mathbb{F}_q} S \cong \langle 2 \rangle + Q_8(1)$$

In particular, $\rho(S/\mathbb{F}_q) = 9$.

§2 Computation in Beijing 2019 ; Reid sextic

$Rd : \sum_1^6 x_i = \sum_{i<j} x_i x_j = \sum_{i<j<k} x_i x_j x_k = 0 \subset \mathbb{P}^5$

K3 surface of degree 6 with $\mathfrak{S}_6$-action

$/\mathbb{C}$ Picard number $= 20$, $|\mathrm{disc\ Pic}| = 120$

mod p reduction is smooth $\Leftrightarrow p \nmid 120$

p	2	3	5
Sing (Rd mod p)	$15A_1$	$10A_2$	$6A_1$

Minimal resolution Rd_p of Rd mod p is a K3 surface $/\mathbb{F}_p$ and Picard number $\begin{cases} 22 & /\mathbb{F}_{p^e} \text{ s.s.} \\ 21 & /\mathbb{F}_p \end{cases}$

Moreover Artin inv. is 1, "the most algebraic K3."

$\boxed{\text{Theorem}}$ (1) $\mathrm{Aut}_{\mathbb{F}_2} Rd_2 \cong \mathfrak{S}_6 \underset{3^2C_4}{*} 3^2 D_8$

(2) $\mathrm{Aut}_{\mathbb{F}_3} Rd_3 \cong \langle \mathfrak{S}_7, \sigma_4, \sigma_7 \rangle$

Proof is Borcherd's – Kondo method. Use (✱), i.e.,

$\exists\ \mathrm{Pic}_{\mathbb{F}_p} Rd_p \xrightarrow[\text{embedding}]{\text{primitive}} U \oplus \Lambda$ extended Leech ($g = 7, 8$)

and use Leech root system (of Conway). D_5, A_5

§3 Two more models

$\boxed{g=9}$ Hermitian model of $K3^{ssl}$ $(p=2)$

$$\boxed{S_{14} \subset \mathbb{P}^2 \times \mathbb{P}^2 \quad \sum_1^3 x_i^2 y_i = \sum_1^3 x_i y_i^2 = 0}$$

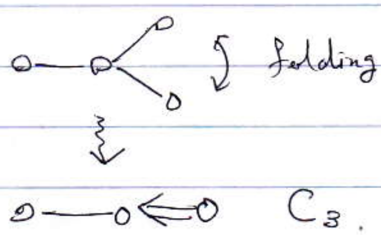
attains Picard number maximal $(=22)$ $/\mathbb{F}_4$

$$\exists \quad \mathrm{Pic}_{\mathbb{F}_4} S_{14} \xhookrightarrow[\text{emb.}]{\text{prim.}} U + \Lambda \quad \text{w. complement } D_4$$

$\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ – invariant part

$$\mathrm{Pic}_{\mathbb{F}_2} S_{14} \hookrightarrow U + \Lambda^g \qquad \text{ord } g = 2$$

with Λ^g = Barnes-Wall and complement



$\boxed{g=10}$ $\exists$ cyclic covering of degree 3 $/\mathbb{F}_4$

$$S_{14} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \qquad \tau^3 = (x-y^2)(y-x^2)$$

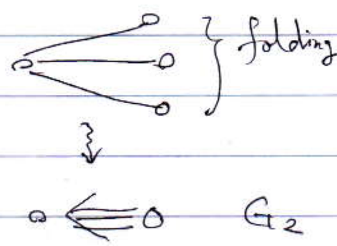
Take cubic twist

$$\boxed{S_{14}^{(3)} \qquad \tau^3 = \omega(x-y^2)(y-x^2)}$$

to get $\mathrm{Gal}(\mathbb{F}_{64}/\mathbb{F}_4)$ – inv. part $\qquad \omega = \sqrt[3]{1}$

$$\mathrm{Pic}\, S_{14}^{(3)} \hookrightarrow U + \Lambda^g \qquad \text{ord } g = 3$$

with Λ^g = Coxeter-Todd. and complement



§4 $Q_8(1)$, Umemura 3-fold and conjectural K3

$Q_8(1) = \Lambda^g$, $g \in M_{23}$ of order 5, rk 8, disc $= 5^4$

① Underlying $\mathbb{Z}$-lattice of rk 4 $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$-lattice

② $O(Q_8(1)) = 2 . \mathfrak{A}_{5,5} . 2$ $\qquad \mathfrak{A}_{5,5} = (\mathfrak{S}_5 \times \mathfrak{S}_5) \cap \mathfrak{A}_{10}$

order 2880 $\qquad$ in $\mathfrak{S}_{10}$

③ vectors of maximal norm $(= -4)$ $\qquad \# = 120$

$\Updownarrow$

vertices of regular 4-polytope $\{3,3,5\}$ w. 600-cells

$\Updownarrow$

{unit icosians} $= \Gamma_{120}$ $\qquad$ binary icosahedral gp

$\xrightarrow{2:1}$ Γ_{60} icosahedral group

Projectification

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{P}^3 \supset PGL(2) \supset \Gamma_{60}$$

$\downarrow$ $\qquad$ $\downarrow$ Aluffi-Faber map

$[f(ax+by, cx+dy)] \in \mathbb{P}^{12}$

$f = f(x,y) = x^{11}y + 11x^6y^6 - xy^{11}$

invariant of Γ_{120}

U_{22} = Image of AF map

Conjectural K3 / $\mathbb{F}_5$? (Or K3-like objects, such as HK mfds or CY_2 categories, etc. ?)

$$\mathrm{Pic}_{\mathbb{F}_5} S \cong \mathbb{Z}h \oplus Q_8(1) \qquad (h^2) = 2$$

$\qquad \cup \qquad\qquad \cup$

$[C_i] \longleftrightarrow h + x_i \qquad (x_i^2) = -4$

$i = 1, \ldots, 120$

120 $\mathbb{P}^1$'s on S

with 600-cell configuration ?!