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Abstracts

Enriques surfaces with many (semi-)symplectic automorphisms

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An automorphism of a K3 surface S is *sympletic* if it acts on $H^0(\mathcal{O}_S(K_S))$ trivially. All finite groups which have symplectic actions on K3 surfaces are classified in terms of the Mathieu group M_{24} by Mukai [4] and Kondo [2]. An automorphism of an Enriques surface S is *semi-symplectic* if it acts on $H^0(\mathcal{O}_S(2K_S))$ trivially. A smart classification similar to K3 surfaces is desirable for semi-symplectic actions of Enriques surfaces but still far from complete investigation. Here I propose a restricted class of semi-symplectic actions.

Definition An effective semi-symplectic action of a finite group G on an Enriques surface is *M-sympletic* if the Lefschetz number of g equals 4 for every automorphism $g \in G$ of order 2 and 4.

Here the Lefschetz number of an automorphism σ is the Euler number of the fixed point locus Fix σ, and equal to the trace of the cohomology action of σ on $H^*(S, \mathbb{Q})$.

M-semi-symplectic actions are closely related to the symmetric group S_6 of degree 6 via the Mathieu group M_{12} though S_6 itself has no semi-symplectic actions. It is known that S_6 has six maximal subgroups upto conjugacy, and four modulo automorphisms. The four subgroups are

(1) the alternating group A_6,
(2) the symmetric group S_5 of degree 5,
(3) $(C_3)^2.D_8$, the normalizer of a 3-Sylow subgroup, and
(4) the direct product $S_4 \times C_2$,

where C_n and D_n denote a cyclic and a dihedral group of order n, respectively.

Theorem *The three maximal subgroups $A_6, S_5, (C_3)^2.D_8$ and the abelian group $(C_2)^3$ have M-semi-symplectic actions on Enriques surfaces.*

Remark By Kondo [1], there are two Enriques surfaces whose automorphism groups are isomorphic to S_5. One is called type VII and the other is the quotient of the Hessian of a special cubic surface (type VI). The action of S_5 is M-semi-symplectic for the former and not for the latter.

The action of the three maximal subgroups are constructed refining the method of [5]. We use

(1) embeddings of S_6 into the Mathieu group M_{12},
(2) the action of $M_{12} \times C_2$ on the Leech lattice, and
(3) Torelli type theorem for Enriques surfaces.

An Enriques surface $S = Km(E_1 \times E_2)/\varepsilon$ of Lieberman type has a semi-symplectic action of $(C_2)^4$ by translation by 2-torsion points. One involution $\sigma \in (C_2)^4$ is *numerically trivial* in the sense of [3], that is, its Lefschetz number is the maximal $(= 12)$. Moreover, the action of $(C_2)^4$ is M-semi-symplectic except for σ. Hence S has an M-semi-symplectic action of $(C_2)^3$

Question Is a finite group isomorphic to a proper subgroup of the symmetric group S_6, if it has an (effective) M-semi-symplectic action on an Enriques surface?

The definition of M-semi-symplectic action is modeled on the permutation group M_{12} of degree 12. The permutation type of $g \in M_{12}$ depends only on its order n if it has a fixed point (on the operator domain of cardinality 12). The type and the number of fixed points $\mu_+(n)$ are as follows.

n	1	2	3	4	5	6	8	11
permutation type	(1)	$(2)^4$	$(3)^3$	$(4)^2$	$(5)^2$	(6)(3)(2)	(8)(2)	(11)
$\mu_+(n)$	12	4	3	4	2	1	2	1

It is well known that a symplectic involution of a K3 surface have exactly 8 fixed points. But for an involution σ of an Enriques surface, the fixed point set $\mathrm{Fix}\,\sigma$ is not necessarily finite and the Lefschetz number varies from -4 to 12. (Note that every involution of an Enriques surface is semi-symplectic.) The required number 4 in our definition is one half of 8, the mean of -4 and 12 and equal to $\mu_+(2)$. A semi-symplectic action of G on an Enriques surface is M-semi-symplectic if and only if the Lefschetz number and μ_+ are the same on G since the order of semi-symplectic automorphism is either ≤ 6 or ∞ by H. Ohashi.

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