On the moduli space of K3 surfaces

12/22/89 (Fri)

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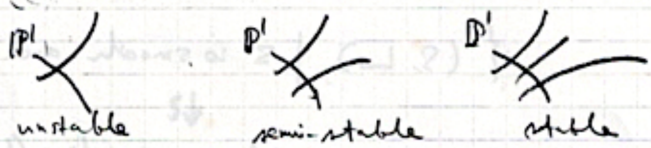
<u>Problem</u>. Compactify K_{2d}

§1 Background — Review of basic facts on degeneration

- Mumford's semi-stable reduction theorem.
- DM stable curve. The moduli space $\overline{M_g}$ is projective.

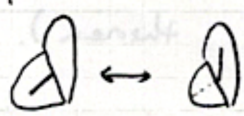


§2 Degeneration of K3 surfaces

Kulikov model (1977)

K_{2d}^{naive} or $K_{2d}^{\text{modif.}}$ is not Hausdorf.

↑ reason (F) ↑ reason (T) flop



<u>Example</u> (plane) $\cup$ (cubic) $\subset \mathbb{P}^3$

$\updownarrow$ twist

(quartic with $\widetilde{E}_8$) $\subset \mathbb{P}^3$

§3 Stable K3 surface

Definition & Main Theorem

§4 Moduli of stable K3 surfaces

$$(\Gamma_{2d}\backslash D)_0^{\text{toroidal}} \dashrightarrow K_{2d}^0 \xrightarrow[\text{map}]{\text{period}} (\Gamma_{2d}\backslash D)_0^{SBB}$$

Hausdorf proper

§5 Open Problems

§6 K_{2d}^0 for small d

Stable K3 surfaces

12/20/89 (Wed.)

<u>Definition</u> A polarized K3 surface (S, L) is a pair of a K3 surface S, i.e., $K_S \equiv 0$ and $q=0$, and a nef & big line bundle L on it. $(L^2) = 2d > 0$ is called the degree of (S, L).

Let K_{2d} be the moduli space of polarized K3 surface of degree $2d$.

$$\{(S, L) \mid S \text{ is smooth and } (L^2)=2d\}/\text{isom.}$$
$$\downarrow\wr$$
$$K_{2d} = \left\{ (S, L) \;\middle|\; \begin{array}{l} L \text{ is ample}, (L^2)=2d \\ \text{and Sing } S \text{ are RDP's} \end{array} \right\} \Big/ \text{isom}$$

K_{2d} is a quasi-projective variety (by the Torelli type theorem). <u>Example</u> $K_4 = \left\{\begin{array}{c}\text{RDP} \\ \text{quartic} \\ \text{surface} \\ \subset \mathbb{P}^3\end{array}\right\} \cup \left\{\begin{array}{c}\text{RDP} \\ \text{bigonal}\end{array}\right\} \cup \left\{\begin{array}{c}\text{RDP} \\ \text{monogonal}\end{array}\right\} \Big/ \text{isom.}$

19-dim 18-dim 18-dim.

<u>Problem</u> Construct a <u>geometric</u> compactification of K_{2d}. Find a class of (suitable) degeneration of K3 surfaces whose moduli is compact and Hausdorf.

§1 <u>Background</u> (Review of basic facts on degeneration)

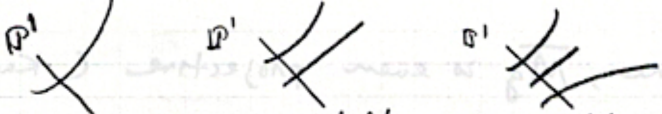
Let $\pi : X \longrightarrow \Delta$ be a degeneration of smooth (complete) varieties.

<u>Theorem</u> (Mumford-Knudsen-Waterman) There exists a proper $\tilde{\pi} : \tilde{X} \longrightarrow \Delta$ such that

i) $\tilde{X}^*/\Delta^*$ is an n-th root fibration of X^*/Δ^*, and

ii) $\tilde{\pi}^{-1}(0)$ is reduced and normal crossings.

$\dim \pi = 1$



D-M unstable D-M semi-stable D-M stable

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<u>Definition</u> (Deligne–Mumford) A connected reduced curve C is D-M stable (resp. semi-stable) if

i) Sing C are nodes, and

ii) each nonsingular rational component R has ≥ 3 (resp. ≥ 2) points in common with $\overline{C-R}$.

Let $\pi : X \longrightarrow \Delta$ be a degeneration of smooth curves with $\pi^{-1}(0)$ reduced normal crossings.

First contract all $E \cong \mathbb{P}^1$ with $(E^2) = -1$ in $\pi^{-1}(0)$, We obtain: $\pi_{ss} : X_{ss} \longrightarrow \Delta$. Then contracting all $E \cong \mathbb{P}^1$ with $(E^2) = -2$, we obtain $\pi_s : X_s \longrightarrow \Delta$.

	π_{ss}	π_s
central fibre	DM semi-stable	DM stable
total space	smooth	RDP
K_X	(relatively) nef	relatively ample

<u>Corollary</u> The moduli space $\overline{M}_g$ of DM stable curve is compact and Hausdorf.

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<u>Remark</u> A 1-dim. scheme C of arithmetic genus $g \geq 2$ is DM stable iff its m-canonical model is GIT stable, where m is any integer ≥ 5 (Mumford, Gieseker).

[Therefore, $\overline{M}_g$ is even projective (Knudsen).

§2 Degeneration of K3 surfaces

Let $\pi' : X' \longrightarrow \Delta$ be a degeneration of K3 surfaces with $\pi'^{-1}(0)$ reduced normal crossing. Assume that all components of $\pi'^{-1}(0)$ are algebraic.

Kulikov (1977) There exists a birational modification

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \Delta & \end{array}$$

in the central fibre such that

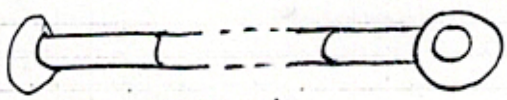
i) $\pi^{-1}(0)$ is still reduced normal crossing, and

ii) X is smooth and $K_X \equiv 0$

Moreover, $X_0 = \pi^{-1}(0)$ is one of the following:

Type I $\quad X_0$ is a smooth K3 surface.

Type II $\quad X_0 = V_0 \cup V_1 \cup \cdots \cup V_{n-1} \cup V_n$



V_0, V_n are rational.

V_i, $1 \leq i \leq n-1$, is elliptic ruled.

$V_{i-1} \cap V_i$ and $V_i \cap V_{i+1}$ are sections of the ruling $V_i \longrightarrow E$.

Type III $\quad$ All components are rational.

<u>Remark</u>

Types I, II and III are distinguished by the Picard-Lefschetz transformation T on $H^2(X_t)$.

Type	I	II	III
$N = \log T$	$N = 0$	$N^2 = 0$	$N^3 = 0$

K_{2d}^{naive} = (the moduli space of polarized Type I & II & III K3 surfaces)

$$= \left\{ (S, L) \;\middle|\; \begin{array}{l} L \text{ is nef and } (L^2) = 2d \\ \cancel{S \text{ has no } E \cong \mathbb{P}^1 \text{ with}} \end{array} \right\} \Big/ \text{isom.}$$

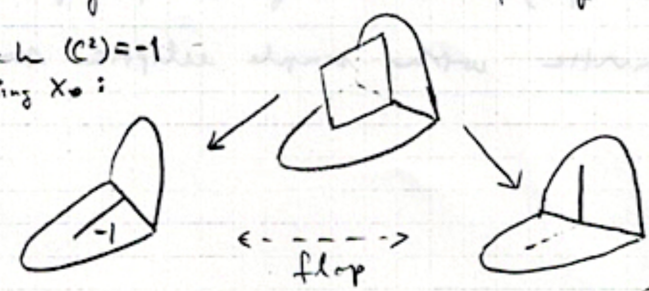
Compact ... missing [illegible]

(Shepherd-Barron 1982) K_{2d}^{naive} is compact.

K_{2d}^{naive} is not Hausdorff by the following two reasons:

flop ; false } (F) assume that the central fibre X_0 contains $C \cong \mathbb{P}^1$ with $(C^2) = -1, -2$ and $(L \cdot C) = 0$. Then the flop with center C gives rise to a new family.

Case in which $(C^2) = -1$ and $C \not\subset \operatorname{Sing} X_0$:



Every C with $(L.C) = 0$ can move

$$K_{2d}^{\text{modif.}} = \left\{ (S, L) \;\middle|\; \begin{array}{l} L \text{ is nef and } (L^2) = 2d \\ \cancel{S \text{ has no } C \cong \mathbb{P}^1 \text{ with}} \\ \cancel{(C^2) = -1, -2 \text{ and } (L.C) = 0} \\ \operatorname{Sing} S \text{ are RDP's} \end{array} \right\} \Big/ \text{isom.}$$

This is still non-Hausdorff.

twisting ; true } (T) assume that central fibre X_0 is reducible: $X_0 = A \cup B$.

$\otimes \mathcal{O}(\pm A)$ changes the distribution of polarization.

$$\begin{cases} L_A = \mathcal{L}|_A \rightsquigarrow \mathcal{L}(\pm A)|_A = L_A(\mp D) \\ L_B = \mathcal{L}|_B \rightsquigarrow \mathcal{L}(\pm A)|_B = L_A(\pm D) \end{cases}$$

$$\begin{array}{c} X \swarrow \mathcal{L} \\ \downarrow \\ \Delta \end{array}$$

where $D = A \cap B$.

Example $X_0 = P \cup T \subset \mathbb{P}^3$
plane cubic

Blow up P at 12 points on $P \cap T$.

$X_0' = P' \cup T$

The pull back L of $\mathcal{O}_{\mathbb{P}}(1)$ is the pull back l of a line over P', and $\mathcal{O}_T(1)$ over T.

	degree		twist by $\mathcal{O}(T)$	new distribution
P'	1	$l \longmapsto$	$l + D$	
			$= 4l - p_1 - \cdots - p_{12}$	4
T	3	$\mathcal{O}_T(1) \longmapsto$	$\mathcal{O}_T(1) - D = 0$	0

The projective model of new polarized surface is a quartic with simple elliptic singularity of type $\widetilde{E_6}$.

§3 Stable K3 surface

Definition A pair (S, L) of a type II K3 surface S $(= V_0 \cup \cdots \cup V_n)$ and nef l.b L is *semi-stable* if

i) each V_i, $1 \le i \le n-1$, is minimal and flat, and $(L.\ \text{fibre}) = 0, 1$,

Rem. The universal covering of V_i^0, $1 \le i \le n-1$, is $\mathbb{C}^* \times \mathbb{C}$

$1 \to \mathbb{Z} \to \pi_1 \to \mathbb{Z} \times \mathbb{Z} \to 1$ 中村[illegible].

The restriction $L|V \not\equiv 0$ for any component of X_0. Components V with $(L^2)_V = 0$ are contracted.

ii) ~~both ends $V = V_0, V_n$ are~~ ~~Del Pezzo surfaces~~ ~~blow up of $\bar{V}$ which are~~ ~~a $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, and $L|V$ comes from $\bar{V}$, and~~

$-K_{V_0}$ and $-K_{V_n}$ are nef, and

iii) $N_{D_i/V_i} \otimes N_{D_i/V_{i+1}} \cong \mathcal{O}_{D_i}$ for $0 \le i \le n-1$, where $D_i = V_i \cap V_{i+1}$

a curve $C \subset S$ with $(C.L) = 0$ is isomorphic to $\mathbb{P}^1$ and $(C^2) = 0, -1, -2$.

Both ends V_0 and V_n are weak Del Pezzo.

Definition A stable (polarized) type II K3 surface is a pair (S', L') obtained from semi-stable (S, L) by contracting all C with $(L.C) = 0$. Components V with $(L')_V = 0$ disappear.

S' is $V_0' \cup V_2' \cup \cdots \cup V_s'$ $(0 \le s \le n)$. (S', L') satisfies

i) V_i', $1 \le i \le s-1$, is minimal and flat, and $(L'.\ \text{fibre}) = 1$,

ii) Let V be an end surface V_0', V_s'. When $s \ge 1$, we have

(a) V is RDP Del Pezzo surface, or

(b) V is non-normal along $C \simeq \mathbb{P}^1$ and normalization is a flat $\mathbb{P}^1$-bundle over an elliptic curve.

$$\begin{array}{ccc} \tilde{V} & \supset & E \ \text{elliptic curve} \\ \downarrow & & \downarrow 2:1 \\ V & \supset & C \simeq \mathbb{P}^1 \end{array}$$

V has 4 pinch points $y^2 = xz^2$ on C. Other points are nodes

iii) L' is ample.

Theorem Let $\pi : X \longrightarrow \Delta$ be an algebraic type II degeneration of K3 surfaces i.e. $N^2=0$ and $\mathcal{L}$ a nef line bundle on X. Then there exists $\pi_{\Delta\Delta} : X_{\Delta\Delta} \longrightarrow \Delta$ and $\mathcal{L}_{\Delta\Delta}$ which satisfy

i) $(X^*_{\Delta\Delta}, \mathcal{L}^*_{\Delta\Delta})/\Delta^*$ is an m-th root fibration of $(X^*, \mathcal{L}^*)/\Delta^*$

ii) Put $X_0 = \pi_{\Delta\Delta}^{-1}(0)$ and $L_0 = \mathcal{L}|_{X_0}$. Then (X_0, L_0) is semi-stable polarized type II K3 surface

iii) $X_{\Delta\Delta}$ is smooth.

Contracting all $C \subset X_0$ with $(\mathcal{L}.C) = 0$, we obtain $\pi_\Delta : \overset{\mathcal{L}_\Delta}{\downarrow} X_\Delta \longrightarrow \Delta$ whose central fibre is stable.

$\boxed{\text{12/22 ここまで と§6を話した}}$

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Remark Non-isolated singularities of X_Δ are nodes along smooth rational curves or A_n along elliptic curves. Isolated singularities are rational singularities with small resolutions.

§4 Moduli of stable K3 surfaces

$\hat{K}_{2d}$ = (the moduli of stable polarized type I and II K3 surfaces (S, L) of degree $2d$)

The period mapping $K_{2d} \longrightarrow D/\Gamma_{2d}$ extends

$$\hat{K}_{2d} \longrightarrow (\Gamma_{2d}\backslash D)^{SBB} = (\Gamma\backslash D) \amalg \underbrace{\begin{pmatrix}\text{1-dim}\\ \text{compact}\end{pmatrix}^{\text{bdry}} \amalg \begin{pmatrix}\text{0-dim}\\ \text{compact}\end{pmatrix}^{\text{bdry}}}$$

Satake-Baily-Borel compactification

put $(\Gamma_{2d}\backslash D)^{SBB}_0$

By the definition and theorem $\hat{K}_{2d}$ is Hausdorff and the period map is proper over $(\Gamma_{2d}\backslash D)^{SBB}_0$.

$D = SO(2,19)/SO(2)\times SO(19)$

bdd symmetric domain of type IV

Let $(\Gamma_{2d}\backslash D)_0^{\text{toroidal}}$ be the inverse image of $(\Gamma_{2d}\backslash D)_0^{SBB}$ in any Mumford's toroidal compactification. This is independent of the choice of rational polyhedral decomposition and dominates $K_{2d}^{\wedge}$.

$$(\Gamma_{2d}\backslash D)_0^{\text{toroidal}} \cong \left\{ \begin{array}{l} \text{semi-stable polarized type I and II} \\ \text{K3 surfaces of degree } 2d \end{array} \right\} \Big/ \text{equivalence}$$

$$\begin{array}{c} (\Gamma_{2d}\backslash D)_0^{\text{toroidal}} \\ \downarrow \\ K_{2d}^{\wedge} \\ \downarrow \\ (\Gamma_{2d}\backslash D)_0^{SBB} \end{array}$$

[Remark: a class of compactification between $(\Gamma\backslash D)^{\text{toroidal}}$ and $(\Gamma\backslash D)^{SBB}$ is studied by Looijenga]

§5 Open problems

(S, L) stable $\Rightarrow$ $L^{\otimes n}$ $n \geq 3$ is very ample

$$\Phi_{|L^{\otimes n}|} : S \hookrightarrow \mathbb{P}^N \qquad N = n^2 d + 1$$

The image is called the n-projective model of (S, L)

There exist an open set $\mathcal{U}$ of $\text{Hilb}\, \mathbb{P}^N$ such that

$$K_{2d}^{\wedge} \cong \mathcal{U} \big/ \text{Aut}\, \mathbb{P}^N$$

Conjecture: The n-projective model of (S, L) is GIT stable iff (S, L) is stable.

Problem: Define a stable type III polarized K3 surfaces.

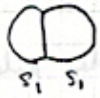
§6 $\hat{K}_{2d}$ for small d

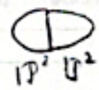
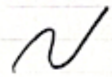
2d=2 (# of 1-dim bdry component) = 4

Shah's compactification K_2^{Shah}

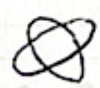
$$(K_2)_0^{\mathrm{toroidal}} \longrightarrow \hat{K}_2 \longrightarrow (K_2)_0^{\mathrm{SBB}}, \qquad \hat{K}_2 \nearrow (K_2)_0^{\mathrm{Shah}} \searrow (K_2)_0^{\mathrm{SBB}}$$

Branch sextic curve of 1-projective "model"

divisor $\xrightarrow{1}$  S_1 S_1 # of moduli = 17    1

divisor $\xrightarrow{17}$  $\mathbb{P}^2$ $\mathbb{P}^2$ 1    1

divisor $\xrightarrow{9}$  $\mathbb{P}^1$ with 4 pinch pts, S_2 8    1

divisor $\xrightarrow{15}$  $\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^1$ with 4 pinch pts.   2    2

↑ # of moduli

$\hat{K}_2$ has no bdry divisors

Theorem

2d	2	4	6	8	10	12
# of 1-dim. bdry components	4	9	10	16	20	to be computed
# of bdry div's of $\hat{K}_{2d}$	0	1	1	1	1	2