

NEW DEVELOPMENTS
IN THE THEORY OF FANO THREEFOLDS:
VECTOR BUNDLE METHOD AND MODULI PROBLEMS

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A compact complex manifold X is called a Fano variety if its first Chern class $c_1(X)$ is positive. A typical example of a Fano variety is the projective space P^n . In the case of dimension 1, this is the only example of a Fano variety. There are more examples in dimension 2, such as the blow-up of the projective plane and the direct product $P^1 \times P^1$; yet, if we assume that they have second Betti number equal to 1, such Fano varieties are isomorphic to P^2 . Compared with these, if we are in dimension 3, then there exist already 17 kinds of Fano threefolds even in the case $B_2 = 1$. The classification of Fano threefolds has been investigated by using "double projection from lines" introduced by Fano. Recently, however, the relationship with homogeneous spaces ([39]) and the discovery of those arising from quasi-homogeneous spaces (§5), etc., have inspired us to consider a classification in terms of vector bundles. The method is based on an explicit description of a system of defining equations for the projective variety $X_{2g-2} \subset P^{g+1}$, which is called the anti-canonical model of a Fano threefold. Our fundamental tools are projective geometry of homogeneous spaces and higher linear systems on K3 surfaces (§4); they have been refined and improved several times, and by now we can classify Fano threefolds with Gorenstein singularities under the assumption that they are Brill-Noether theoretically¹ general. We expect that the new approach will be useful also in various aspects of the study of Fano varieties, such as their moduli spaces, degeneration, periods, rational points and the Cremona transformation groups in three variables, etc.

In Sections 1–3, we explain fundamental objects and the background of the subject; in Sections 4–7, we describe the important steps toward a new classification, and in Sections 8 and 9, we discuss moduli problems (these two sections are independent of the others).

TERMINOLOGY AND NOTATION

Varieties always mean algebraic varieties over the complex number field C . For an (algebraic) vector bundle E and a vector space V , E^\vee and V^\vee denote their duals. Let $H^i(E)$ be the i -th cohomology group with coefficient in the sheaf E ; we write $h^i(E)$ for the dimension of $H^i(E)$. $\chi(E)$ denotes the alternating sum

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¹The Brill-Noether theory is concerned with the behavior of special divisors on curves (e.g. [1]). Definitions 3.8 and 6.4 have their origin in the theory; but we omit the details.

$\sum (-1)^i h^i(E)$. If V is a vector space, then $P_*V = V - \{0\}/\mathcal{C}^*$ is the space of 1-dimensional subspaces of V and P^*V is the space of 1-dimensional quotient spaces of V .

1. INTRODUCTION TO FANO THREEFOLDS

Very roughly speaking, the family of algebraic varieties X is divided into 3 classes according as $c_1(X) > 0$, $c_1(X) = 0$ or $c_1(X) < 0$. For hypersurfaces X with $F_d(x_0, \dots, x_m) = 0$ in the projective space \mathbf{P}^m , this corresponds to a division by degree: $d < m + 1$, $d = m + 1$ or $d > m + 1$. Of course, there are a large number of algebraic varieties that satisfy none of these conditions. But, a viewpoint of classification theory is that their properties are combinations of those in the 3 classes.

Among the three classes of varieties above, those with $c_1(X) > 0$ are our main object, that is, **Fano varieties** (Definition 2.2). There are far fewer Fano varieties than there are varieties in the other two classes². Nevertheless, mathematicians have obtained a meaningful classification only for those of dimension up to 3. We do not know yet how to deal with Fano varieties of dimension greater than or equal to 4.

For a historical reason, 2-dimensional Fano varieties are called **del Pezzo surfaces**. Interpolating relevant properties of del Pezzo surfaces, we explain a classification of Fano threefolds. One of the most fascinating features of Fano threefolds is that we can describe them explicitly by such a simple condition as $c_1(X) > 0$. If S is a del Pezzo surface, then the self-intersection number $d := c_1(S)^2$ is the only discrete invariant³. For a Fano threefold X , on the other hand, the **degree** $c_1(X)^3$ and the following invariants are important.

(1.1) **The second Betti number:** $B_2(X) := \text{rk} H^2(X, \mathbf{Z})$.

(1.2) **Index:** the largest positive integral divisor of $c_1(X) \in H^2(X, \mathbf{Z})$.

(1.3) **Analytic genus** $p = B_3(X)/2$: the dimension of the intermediate Jacobian variety $\text{Jac}X$ of X . Here $\text{Jac}X$ is the principal polarized abelian variety obtained from the periods of p linearly independent harmonic forms on X of type (2,1).

The degree is always even; we put $g = c_1(X)^3/2 + 1$ and call it the **genus** of X . We use genus rather than degree. As we mentioned in the introduction, there are 17 classes of Fano threefolds even in the case $B_2(X) = 1^4$. We give their "periodic table" with respect to the analytic genus p and index r (see Table 1.4).

Here Q^3 denotes a quadric hypersurface in \mathbf{P}^4 . Fano threefolds B_d , $1 \leq d \leq 5$, of index 2 are also called del Pezzo varieties of degree d ([12], [24]), where d is $1/8$ of the degree (defined earlier in this section). It is known that the 8 classes with $p = 0, 2, 3, 7$ are rational⁵ and the remaining 9 classes are non-rational⁶ ([26]); but we do not pursue it here. Among rational Fano threefolds, those with $p = 0$ are very special. In fact, many of them are compactifications of \mathbf{C}^3 ([13]; [42], §8).

The most typical Fano threefolds are the 10 classes A_{2g-2} of index 1. To study such threefolds, Fano considered double projection from a line on the anti-canonical model (§2) of it. Let us give an example.

²The family of Fano varieties of a fixed dimension is bounded ([31]; [30], Chap. 5).

³Only in the case $d = 8$, the intersection form on $H^2(S, \mathbf{Z})$ is also a discrete invariant.

⁴See [36], [37] for a classification of Fano threefolds with $B_2 \geq 2$.

⁵We prove the rationality of A_{12} in Example 7.4.

⁶To be more precise, A_{10} , B_2 and B_1 are known to be non-rational only for generic ones.

$r \backslash p$	0	2	3	5	7	10	14	20	21	30	52
4	P^3										
3	Q^3										
2	B_5	B_4		B_3		B_2				B_1	
1	A_{22}	A_{18}	A_{16}	A_{14}	A_{12}	A_{10}	A_8	A_6		A_4	A_2

Table 1.4. Fano threefolds with $B_2 = 1$.

Example 1.5. Let $C_7 \subset P^3$ be a space curve of genus 3 and degree 7. Consider a homogeneous polynomial $F(x)$ of degree 7 that vanishes twice along C_7 , i.e., all the partial derivatives $\partial F/\partial x_i$ of F are identically zero on C_7 . Choosing a basis $F_1(x_0, x_1, x_2, x_3), \dots, F_{11}(x_0, x_1, x_2, x_3)$ for the vector space of such polynomials, we can define a rational map

$$\begin{array}{ccc} P^3 & \xrightarrow{\psi_{C_7}} & P^{10} \\ \cup & & \cup \\ (x_0 : x_1 : x_2 : x_3) & \longmapsto & (F_1(x) : \dots : F_{11}(x)). \end{array}$$

For a Fano threefold A_{16} of index 1 and genus 9, there exists a space curve $C_7 \subset P^3$ such that A_{16} is isomorphic to the image of ψ_{C_7} and is birational to P^3 via ψ_{C_7} .

An advantage of this result is that if A_{16} exists, then we can conclude immediately that it is rational. But, when we want to construct A_{16} from $C_7 \subset P^3$, we come up with a lot of cumbersome problems, such as when the vector space of polynomials of degree 7 has dimension 11, when ψ_C becomes birational, or when the image of ψ_C is non-singular, etc. Contrary to this, an example of A_{14} of genus 8 is easy to construct by a different method that Fano already knew.

Example 1.6. Let $G(2, 6) \subset P^{14}$ be the Grassmannian variety⁷ consisting of 2-dimensional subspaces of a 6-dimensional vector space C^6 . Non-singular threefolds obtained as 5-times hyperplane sections of $G(2, 6)$ are Fano threefolds A_{14} of index 1 and genus 8.

It is generally recognized that Grassmannian varieties are representatives of homogeneous projective varieties; so, it is natural to look into them for more examples similar to the above. In fact, we find three more examples as follows:

(1.7) $g = 7$: 10-dimensional orthogonal Grassmannian variety

$$\Sigma_{12}^{10} = SO(10)/U(5) \subset P^{15}.$$

(1.8) $g = 9$: 6-dimensional symplectic Grassmannian variety

$$\Sigma_{16}^6 = Sp(6)/U(3) \subset P^{13}$$

(cf. Example 5.1).

(1.9) $g = 10$: G_2 -variety⁸ $\Sigma_{18}^5 \subset P^{13}$ (cf. Example 5.2).

⁷We regard it as a projective variety with Plücker coordinates (cf. [17]).

⁸ G_2 -varieties are introduced originally by Borcea [5] to give an example of a K3 surface of genus 10.

A Fano threefold A_{16} in Example 1.5 can be obtained as a 3-times hyperplane section of $\Sigma_{10}^6 \subset \mathbf{P}^{13}$ in (1.8). In a new classification, we will discuss the following properties by making use of the four homogeneous spaces Σ_{2g-2} with $g = 7, 8, 9, 10$.

(A) All non-singular Fano threefolds with $B_2 = 1$ and $g = 7, 8, 9, 10$ can be obtained as linear sections of $\Sigma_{2g-2} \subset \mathbf{P}$.

(B) There does not exist (unfortunately?) any homogeneous space which realizes Fano threefolds of genus 12 as its linear sections; but, there is a way of describing them in a 12-dimensional Grassmannian variety.

(C) There exist no non-singular Fano threefolds with $B_2 = 1$ and genus $g = 11$ or $g \geq 13$.

(D) The three properties above can be generalized to Fano threefolds with canonical Gorenstein singularities.

These statements are not independent of each other. In fact, they are related closely in the following way: (B) will be shown in the course of proving (A) (§§4,5); (D) leads to a simple proof of (C) (thanks to singularities); the proof of (C) is based on (A) and (B) (§§6,7). In particular, (D) contains a problem of how to deal with the condition $B_2 = 1$ when singularities are involved. Here we summarize the result for non-singular Fano threefolds. The anti-canonical line bundle of a Fano threefold X with $B_2 = r = 1$ has no base point. Let $\Phi_{-K} : X \rightarrow \mathbf{P}^{g+1}$ denote the anti-canonical morphism (§2).

Theorem 1.10. *The genus g of X is $g \leq 10$ or $g = 12$.*

(1) *If $g = 2$, then $\Phi_{-K} : X \rightarrow \mathbf{P}^3$ is a double covering ramified at a sextic surface.*

(2) *If $g = 3$, then $\Phi_{-K} : X \rightarrow \mathbf{P}^4$ is either*

(2.a) *an embedding whose image is a quartic hypersurface, or*

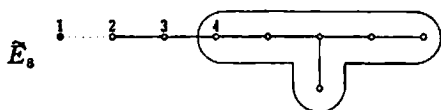
(2.b) *a double covering of a quadric hypersurface $Q^3 \subset \mathbf{P}^4$.*

(3) *If $g \geq 4$, then Φ_{-K} is an embedding and the following holds, where X_{2g-2} denotes its image.*

g	Anti-canonical model
4	$X_6 = (2) \cap (3) \subset \mathbf{P}^9$ complete intersection
5	$X_8 = (2) \cap (2) \cap (2) \subset \mathbf{P}^6$ complete intersection
6	$X_{10} \subset \mathbf{P}^7$ is a quadric hypersurface section of a quintic del Pezzo 4-fold $V_5 \subset \mathbf{P}^7$. $V_5 \subset \mathbf{P}^7$ may be chosen as the cone of a quintic del Pezzo 3-fold.
7	$X_{12} \subset \mathbf{P}^8$ is a linear section of a 10-dimensional orthogonal Grassmannian variety. $[\Sigma_{12}^{10} \subset \mathbf{P}^{15}] \cap H_1 \cap \dots \cap H_7$
8	$X_{14} \subset \mathbf{P}^9$ is a linear section of an 8-dimensional Grassmannian variety. $[G(2, 6) \subset \mathbf{P}^{14}] \cap H_1 \cap \dots \cap H_5$
9	$X_{16} \subset \mathbf{P}^{10}$ is a linear section of a 6-dimensional symplectic Grassmannian variety. $[\Sigma_{16}^6 \subset \mathbf{P}^{13}] \cap H_1 \cap H_2 \cap H_3$
10	$X_{18} \subset \mathbf{P}^{11}$ is a linear section of a G_2 -variety. $[\Sigma_{18}^5 \subset \mathbf{P}^{13}] \cap H_1 \cap H_2$
12	$X_{22} \subset \mathbf{P}^{13}$ is isomorphic to a non-singular threefold $G(3, 7, N) \subset \mathbf{P}^{13}$ (cf. §5) obtained from a non-degenerate 3-dimensional subspace $N \subset \mathbb{A}^2 C^7$.

We remark on a relation between the four homogeneous spaces and Dynkin diagrams, although this has no direct connection with our classification. It is well

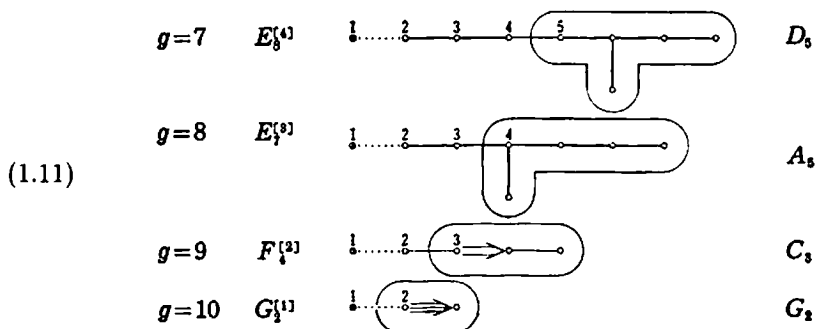
known that del Pezzo surfaces are related with Dynkin diagrams ([33]). For instance, there are exactly 27 lines on a cubic surface, and the configuration of these lines has symmetry isomorphic to a Weyl group $W(E_6)$. More precisely, it is related with $E_8^{[3]}$, where $E_8^{[3]}$ is defined in the following extended Dynkin diagram of E_8



as a pair consisting of the diagram E_6 obtained from \widetilde{E}_8 by removing 3 consecutive points from the maximal root \odot and (the fundamental weight of E_6 corresponding to) the 4-th root from \odot .

In particular, it determines an irreducible representation V of an exceptional Lie group of type E_6 , and the 27 lines can be regarded $W(E_6)$ -equivariantly as a basis for V (an exceptional Jordan algebra). A similar property holds⁹ for the configuration of lines on a del Pezzo surface of degree d ($1 \leq d \leq 6$) and a representation of $E_8^{[d]}$.

For a given irreducible representation V of a semisimple algebraic group G , the orbits of the elements $v \in V$ of the maximal weight in the projective space $P \cdot V$ form a homogeneous projective variety. The four homogeneous spaces Σ_{2g-2} that yield Fano threefolds can be obtained in this manner from the representations corresponding to the following diagrams:



The situation differs considerably from the case of del Pezzo surfaces, and we do not know how to construct a diagram corresponding to genus 12. But the phenomenon seems very interesting.

2. DEL PEZZO SURFACES

Before we start a discussion on a classification of Fano threefolds, we explain some terminology and recall a 2-dimensional classification.

A vector bundle E on a variety X is defined by a system of transition functions $\{\varphi_{\alpha\beta}\}$, $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{C})$, which take value in matrices. Here $\{U_\alpha\}$ is a suitable open covering of X , and r is called the rank of E . The system of transition functions defined by the determinants $\{\det \varphi_{\alpha\beta}\}$ gives a line bundle on X ; this is

⁹The results may be found in [29], together with its relation with the intermediate Jacobian varieties of threefolds having del Pezzo surface bundles. The configuration of planes on the anti-canonical model of the blow-up of P^3 at $(8-d)$ points also has symmetry isomorphic to $E_7^{[d]}$.

nothing but the exterior power $\wedge^r E$ over E of highest degree. In particular, if E is the tangent vector bundle T_X , then $\wedge^n T_X$ is called the **anti-canonical line bundle** on X . If (z_1, z_2, \dots, z_n) is a local coordinate of X with $n = \dim_{\mathbb{C}} X$, then a local section of $\wedge^n T_X$ is expressed in terms of a (regular) function f as

$$(2.1) \quad f(z_1, z_2, \dots, z_n) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \cdots \wedge \frac{\partial}{\partial z_n}.$$

We call it a (local) **anti-canonical form** on X .

Definition 2.2. A projective algebraic variety X is called a **Fano variety** if $\wedge^n T_X$ is ample. It is called a **Fano variety of the principal series** if $\wedge^n T_X$ is very ample.

A line bundle, L , is said to be **very ample** if the vector space $H^0(X, L)$ of the global sections has a basis $\alpha_1, \alpha_2, \dots, \alpha_N$ satisfying the following two conditions:

(2.3) L has **no base point**: that is, $\alpha_1, \alpha_2, \dots, \alpha_N$ have no common zeros.

(2.4) The map

$$\begin{array}{ccc} \Phi_L : X & \longrightarrow & \mathbb{P}^{N-1} \\ \downarrow & & \downarrow \\ p & \longmapsto & (\alpha_1(p) : \alpha_2(p) : \cdots : \alpha_N(p)) \end{array}$$

is an embedding.

If $L = \wedge^n T_X$ and (2.3) is satisfied, then the map of (2.4) is denoted by Φ_{-K} . The elements $\alpha_1, \alpha_2, \dots, \alpha_N$ themselves are not functions, but their ratios α_i/α_j are (holomorphic) functions on X by (2.1). Hence Φ_{-K} is defined globally (the same is true for a more general Φ_L). The morphism is called the **anti-canonical morphism**. If (2.4) holds, then the image of Φ_{-K} (as a projective variety) is called the **anti-canonical model** of a Fano variety of the principal series.

A line bundle L is said to be **ample** if $L^{\otimes m}$ becomes very ample for some $m > 0$. About this, the Nakai-Kleiman criterion is used so often that it is sometimes confused with the definition. We recall it in the case of surfaces.

Theorem 2.5 (Nakai [54]). *A line bundle L on a projective algebraic surface S is ample if and only if $c_1(L)^2 > 0$ and $(c_1(L), C) > 0$ for all (irreducible) curves C on S .*

Let $S = S(p_1, p_2, \dots, p_n)$ be the surface obtained by blowing up the projective plane \mathbb{P}^2 at n distinct points p_1, p_2, \dots, p_n . Write h for the pull-back of a line and E_i for (the cohomology class of) the exceptional divisor over p_i . Then the first Chern class of S is

$$(2.6) \quad c_1(S) = 3h - E_1 - E_2 - \cdots - E_n$$

and $c_1(S)^2 = 9 - n$. Furthermore, if S is a del Pezzo surface, then $n \leq 8$.

Theorem 2.7 ([27]). *$S(p_1, \dots, p_n)$ is a del Pezzo surface if and only if $n \leq 8$ and the points p_1, \dots, p_n satisfy the following three conditions:*

(i) *no three of them are on a line;*

(ii) *no six of them are on a conic;*

(iii) *if $n = 8$, then there exists no (rational) cubic curve passing through all these points and being singular at a point p_i .*

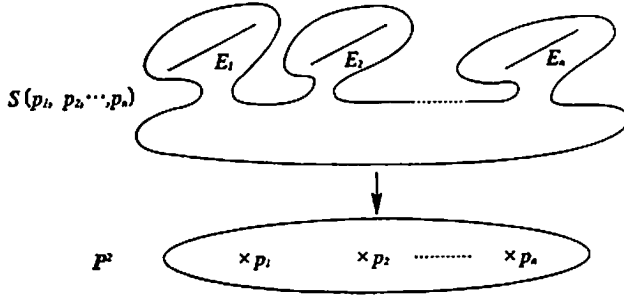


FIGURE 1

Under these conditions, the set of all homogeneous cubic polynomials $F(x, y, z)$ that vanish at p_1, \dots, p_n has dimension $10 - n$. If F_0, F_1, \dots, F_{9-n} denote a basis for it, then the following rational map induces the anti-canonical morphism of $S(p_1, \dots, p_n)$:

$$\begin{array}{ccc} \mathbf{P}^2 & \longrightarrow & \mathbf{P}^{9-n} \\ \cup & & \cup \\ (a_0 : a_1 : a_2) & \longmapsto & (F_0(a) : F_1(a) : \dots : F_{9-n}(a)). \end{array}$$

If $n \leq 6$ in the above, then the anti-canonical bundle is very ample and the anti-canonical model is a surface S_d of degree d in \mathbf{P}^d with $d = 9 - n$. Conversely, such a projective surface is the anti-canonical model of a rational surface if it is projectively normal (del Pezzo [8]). It is well-known that a cubic surface in \mathbf{P}^3 is the anti-canonical model of a blow-up $S(p_1, \dots, p_6)$ of \mathbf{P}^2 at 6 points.

Example 2.8. On the Fermat cubic surface

$$X^3 + Y^3 + Z^3 + T^3 = 0$$

the 6 points are $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : 1 : 1)$, $(1 : \omega : \omega^2)$, $(1 : \omega^2 : \omega)$, where ω is an imaginary cube root of unity.

The next proposition is another important result in the theory of del Pezzo surfaces.

Proposition 2.9. *Del Pezzo surfaces with $B_2 = 1$ are isomorphic to \mathbf{P}^2 .*

Based essentially on these properties, we obtain the following (e.g. [37], §1).

Theorem 2.10. *A del Pezzo surface is isomorphic either to a blow-up $S(p_1, \dots, p_n)$ of \mathbf{P}^2 which satisfies the conditions of Proposition 2.7, or to the product $\mathbf{P}^1 \times \mathbf{P}^1$.*

Although the correspondence is not so simple, the classification of Fano threefolds with $B_2 \geq 2$ in [36] and Theorem 1.10 are, in some sense, 3-dimensional versions of Theorem 2.7 and Proposition 2.9, respectively.

3. CANONICAL CURVES AND FANO VARIETIES

A hyperplane section $S \subset \mathbf{P}^g$ of the anti-canonical model $X \subset \mathbf{P}^{g+1}$ of a Fano threefold of the principal series is a projective model of a polarized K3 surface. Its hyperplane section $C \subset \mathbf{P}^{g-1}$ is then a canonical curve. We are able to classify Fano threefolds, since we have a good understanding of K3 surfaces and of (canonical) curves. We begin with the latter.

In what follows, a curve means a compact Riemann surface or a complete non-singular algebraic curve. They are controlled by the Riemann-Roch theorem:

$$(3.1) \quad \begin{cases} \dim H^0(L) - \dim H^1(L) = \deg L + 1 - g, \\ H^1(L) \cong H^0(\omega_C L^{-1})^\vee, \end{cases}$$

where g is the genus of C , ω_C denotes the canonical line bundle on C and L is an arbitrary line bundle. The theorem implies that the vector space $H^0(\omega_C)$ of differential forms of the first kind has dimension g . Choosing a basis $\omega_1, \dots, \omega_g$ for it, we define a map Φ_K by

$$(3.2) \quad \begin{array}{ccc} C & \longrightarrow & \mathbf{P}^{g-1} \\ \cup & & \cup \\ p & \longmapsto & (\omega_1(p) : \dots : \omega_g(p)). \end{array}$$

A straightforward application of (3.1) gives the following.

Proposition 3.3. ω_C is very ample (§2) if and only if C is not hyperelliptic.

In this situation, the image of Φ_K is a projective curve $C_{2g-2} \subset \mathbf{P}^{g-1}$. We call it the **canonical model** of C . When C is not specified, it is called simply a canonical curve of genus g .

A surface S (a 2-dimensional compact complex manifold) is called a K3 surface if $\overset{2}{\wedge} T_S \cong \mathcal{O}_S$ and $H^1(\mathcal{O}_S) = 0$ ([48], §1). They play an intermediate role between the theory of Fano threefolds and the theory of curves.

Theorem 3.4 ([65]). *If X is a Fano threefold, then there exists a non-zero anti-canonical form $\alpha \in H^0(\overset{3}{\wedge} T_X)$ on X such that the zero locus $S = (\alpha)_0$ of α is non-singular.*

It follows from the adjunction formula, the Kodaira vanishing theorem, the Lefschetz theorem, etc. (cf. [17]), that S is a K3 surface. Further, if L denotes the restriction of $\overset{3}{\wedge} T_X$ on S , then the following sequence is exact:

$$(3.5) \quad 0 \longrightarrow \mathcal{C}\alpha \longrightarrow H^0(\overset{3}{\wedge} T_X) \longrightarrow H^0(S, L) \longrightarrow 0.$$

Since $\overset{3}{\wedge} T_X$ is ample, L is also ample.

In general, a **polarized K3 surface** is defined to be a pair (S, L) consisting of a K3 surface S and an ample line bundle L on S ; the integer $g = c_1(L)^2/2 + 1$ is called its **genus**. As in Definition 2.2, we say that (S, L) is of the principal series if L is very ample. Polarized K3 surfaces outside the principal series can be classified completely ([61]). Based on this result and (3.5), we may classify Fano threefolds outside the principal series ([24]). From a Riemann-Roch type formula

$$(3.6) \quad \dim H^0(\overset{3}{\wedge} T_X) = g + 2, \quad \dim H^0(S, L) = g + 1$$

we find that the anti-canonical model of X is a subvariety $X_{2g-2} \subset \mathbf{P}^{g+1}$ of degree $2g - 2$ and its hyperplane section $S_{2g-2} \subset \mathbf{P}^g$ is the image of a K3 surface S via Φ_L . The latter is called the projective model of (S, L) .

Proposition 3.7. *For a projective surface $T \subset \mathbf{P}^g$ of degree $2g - 2$, the following three conditions are equivalent:*

(i) $T \subset \mathbf{P}^g$ is isomorphic to the projective model of a polarized K3 surface (S, L) of the principal series of genus g ;

- (ii) non-singular hyperplane sections of $T \subset \mathbf{P}^g$ are canonical curves of genus g ;
- (iii) T is not a ruled surface.

Similarly, a projective threefold whose hyperplane section is the projective model of some K3 surface is an anti-canonical model of dimension 3. Hence, as we find from the title of [10], Fano threefolds of the principal series are nothing but “threefolds with canonical curve sections.”

The following terminology is useful for a classification of Fano threefolds with certain singularities (Theorem 6.5).

Definition 3.8. A polarized K3 surface (S, L) is said to be **BN-general** if

$$h^0(M)h^0(N) < h^0(L)$$

for every pair of non-trivial line bundles M and N such that $M \otimes N \cong L$.

In the moduli space of polarized K3 surfaces of genus g , the family of BN-general surfaces forms a non-empty Zariski open subset. For low genera, their projective models are given as follows:

genus	projective model of a BN-general K3 surface
2	$S_2 \rightarrow \mathbf{P}^2$ double covering ramified at a sextic curve
3	$S_4 \subset \mathbf{P}^3$ quartic surface
4	$S_6 = (2) \cap (3) \subset \mathbf{P}^4$ complete intersection
5	$S_8 = (2) \cap (2) \cap (2) \subset \mathbf{P}^5$ complete intersection
6	$S_{10} = V_5 \cap (2) \subset \mathbf{P}^6$

The case $g \leq 5$ of Theorem 1.10 follows immediately from this¹⁰. When $g = 7, 8, 9, 10$, write P for a g -dimensional subspace of \mathbf{P} which intersects the homogeneous space $\Sigma_{2g-2} \subset \mathbf{P}$ of §1 transversally. Then $S = P \cap \Sigma_{2g-2} \subset \mathbf{P}^g \cong \mathbf{P}$ is the projective model of a polarized K3 surface. We furthermore have the following.

Theorem 3.10. *The polarized K3 surfaces $S = P \cap \Sigma_{2g-2} \subset \mathbf{P}^g$ ($g = 7, 8, 9, 10$) are BN-general.*

Conversely, every BN-general polarized K3 surface of genus 7,8,9,10 can be obtained in this way. This is the first step toward a proof of Theorem 1.10, as we will explain in the next section.

4. VECTOR BUNDLES ON K3 SURFACES

In [48], we discussed the structure of the moduli space of vector bundles on K3 surfaces together with their applications to the study of (holomorphic) symplectic varieties. As representations are important for the study of groups and as the central object of ring theory has shifted naturally from ideals to modules, I believe that vector bundles can still play many more significant roles in the study of algebraic varieties. Here we make use of the Grassmannian morphisms associated to vector bundles in order to investigate Fano varieties. We prepare some terminology about higher linear systems before going into a proof of Theorem 1.10 (and its generalization).

Classically, a linear system is defined to be a pair consisting of an (algebraic) line bundle L on an algebraic variety X and a space $V \subset H^0(L)$ of global sections;

¹⁰A classification of those of genus 6 can also be obtained by using vector bundles. See [4] for a different approach.

the case $V = H^0(L)$ is used most frequently. By Nakayama's lemma, L has no base point if and only if the following canonical homomorphism of sheaves is surjective:

$$\begin{array}{ccc}
 ev_L : H^0(L) \otimes \mathcal{O}_X & \longrightarrow & L \\
 \cup & & \cup \\
 \sum_i s_i \otimes f_i & \longmapsto & \sum_i f_i s_i
 \end{array}$$

We call ev_L the evaluation homomorphism of L . Let E be an (algebraic) vector bundle on X of rank r . Then, as in the case of line bundles, the pair consisting of E and a subspace $V \subset H^0(E)$ is called a linear system of rank r . Again, the case $V = H^0(E)$ is important.

Definition 4.1. We say that E is **base point free** if the evaluation homomorphism $ev_E : H^0(E) \otimes \mathcal{O}_X \rightarrow E$ is surjective.

When E is base point free, the fiber E_x of E at each point $x \in X$ is an r -dimensional quotient space of $H^0(E)$. The dual space E_x^\vee is an r -dimensional subspace of $H^0(E)^\vee$. Hence by the correspondence $x \mapsto [E_x^\vee]$, one can define a map

$$(4.2) \quad \Phi_E : X \rightarrow G(r, H^0(E)^\vee).$$

Since E is algebraic, Φ_E is regular.

The Grassmannian variety $G(r, H^0(E)^\vee)$ is a projective variety with Plücker coordinates. In our case, it is embedded into $\mathbf{P}^* \overset{r}{\wedge} H^0(E)$. Let $X \rightarrow \mathbf{P}^* \overset{r}{\wedge} H^0(E)$ be the composite of this embedding with Φ_E . To explain more about it, we consider the r th exterior product of the evaluation map ev_E :

$$\overset{r}{\wedge} ev_E : \overset{r}{\wedge} H^0(E) \otimes \mathcal{O}_X \rightarrow \overset{r}{\wedge} E.$$

This induces a linear map

$$(4.3) \quad \lambda_r : \overset{r}{\wedge} H^0(E) \rightarrow H^0(\overset{r}{\wedge} E)$$

and coincides with the composite of $\lambda_r \otimes 1$ with the evaluation map of $\overset{r}{\wedge} E$. Thinking of the meaning of the Plücker embedding, we obtain the following proposition easily.

Proposition 4.4. *Let E be of rank r and base point free. Assume that $\lambda_r : \overset{r}{\wedge} H^0(E) \rightarrow H^0(\overset{r}{\wedge} E)$ is surjective. Then the following diagram is commutative:*

$$\begin{array}{ccc}
 X & \xrightarrow{\Phi_E} & G(r, H^0(E)^\vee) \\
 \Phi_L \downarrow & & \cap \text{Plücker} \\
 \mathbf{P}^* H^0(L) & \xrightarrow[\alpha]{} & \mathbf{P}^* \overset{r}{\wedge} H^0(E)
 \end{array}$$

where L denotes $\overset{r}{\wedge} E$ and α is a linear embedding induced from a surjective map λ_r .

We now come back to K3 surfaces. A polarized K3 surface has a stable vector bundle which may be characterized by topological conditions¹¹. We apply it for a classification of Fano threefolds.

¹¹ Note that by the Riemann-Roch-Hirzebruch Theorem, the Euler-Poincaré characteristic $\chi(E)$ of a vector bundle E is a topological invariant.

Theorem 4.5. *Let (S, L) be a polarized K3 surface of genus g . Assume that g is equal to the product of two positive integers, rs . Then*

- (1) *there exists a semi-stable vector bundle E_r of rank r satisfying $\overset{r}{\wedge} E_r \cong L$ and $\chi(E_r) = r + s$;*
- (2) *if there exists a stable vector bundle among such E_r 's, then all the vector bundles in (1) are isomorphic to it.*

In general, we have $g - r(\chi(E) - r) \geq 0$ for every stable vector bundle E with $\overset{r}{\wedge} E \cong L$. Twice the left-hand side of this inequality is equal to the moduli number of E ([40], Corollary 0.2). In this sense, E_r of the theorem is sometimes called a rigid bundle. The values for g, r, s that we actually use in our classification are as follows:

(4.6)

g	6	8	9	10	12
rs	$2 \cdot 3$	$2 \cdot 4$	$3 \cdot 3$	$2 \cdot 5 = 5 \cdot 2$	$3 \cdot 4 = 2 \cdot 6$

Theorem 4.7. *The projective model of a BN-general polarized K3 surface (S, L) of genus $g = 7, 8, 9, 10$ is isomorphic to a linear section $\mathbf{P}^9 \cap \Sigma_{2g-2}$ of the homogeneous space $\Sigma_{2g-2} \subset \mathbf{P}$ in (1.6)–(1.9).*

If (S, L) is BN-general, then Theorem 4.5 (2) holds and $H^1(E_r) = H^2(E_r) = 0$. In particular, $\dim H^0(E_r) = r + s$.

First, in the case of genus 8, one has $\dim H^0(E_2) = 6$ for the rigid bundle E_2 of rank 2. The assumption of Proposition 4.4 is satisfied, and the following diagram is commutative:

$$\begin{array}{ccc}
 S & \xrightarrow{\Phi_E} & G(2, 6) \\
 \Phi_L \downarrow & & \cap \text{Plücker} \\
 \mathbf{P}^8 & \hookrightarrow & \mathbf{P}^{14}.
 \end{array}$$

The theorem amounts to saying that the diagram is cartesian, i.e. $S = \mathbf{P}^8 \cap G(2, 6)$. Although we omit the details here, the essential part of proof is a thorough consideration of why $\mathbf{P}^8 \cap G(2, 6)$ is of dimension 2 (see [45]). The classification of singular hyperplane sections of $\Sigma_{2g-2} \subset \mathbf{P}$, i.e. the structure of projective dual, is a key to the proof; this applies also to the other cases.

In the case of genus 9, we use the rigid bundle E_3 of rank 3. It is not sufficient to consider λ_3 of (4.3). We also need a linear map

(4.8)
$$\lambda_2 : \overset{2}{\wedge} H^0(E_3) \longrightarrow H^0(\overset{2}{\wedge} E_3),$$

which can be obtained similarly. Since λ_2 is not injective, the image of Φ_E can be embedded into a symplectic Grassmannian variety Σ_{16} (cf. Example 5.1) and the following diagram is commutative:

$$\begin{array}{ccc}
 S & \xrightarrow{\Phi_E} & \Sigma_{16} \subset G(3, 6) \\
 \Phi_L \downarrow & & \cap \quad \cap \text{Plücker} \\
 \mathbf{P}^9 & \hookrightarrow & \mathbf{P}^{13} \subset \mathbf{P}^{19}.
 \end{array}$$

Theorem 4.7 claims that the left half of the diagram is cartesian.

For K3 surfaces of genus 10, the assertion follows from the non-injectivity of either of the following two linear maps:

$$S^2 H^0(E_2) \longrightarrow H^0(S^2 E_2), \quad \wedge^4 H^0(E_5) \longrightarrow H^0(\wedge^4 E_5).$$

In the latter case, we use the expression of G_2 -manifolds by trivectors (Example 5.2).

The proof of the theorem in the case of genus 7 is different from the other three cases; it employs a vector bundle E of rank 5 with $\wedge^5 E \cong L^{\otimes 2}$. Specifically, E is chosen to be a twist $N_{S/P}^\vee \otimes L^{\otimes 2}$ of the conormal bundle $N_{S/P}^\vee$ of the projective model $S_{12} \subset P^7$, whereas, in other cases, E_r is constructed from special divisors on curves. The set V of homogeneous quadratic polynomials in 8 variables vanishing on S_{12} has dimension 10, and the canonical homomorphism $V \otimes \mathcal{O}_S \longrightarrow E$ is surjective. The linear system (E, V) of rank 5 defines an embedding of S into a 25-dimensional Grassmannian variety. The image is in Σ_{12} . Moreover, it is a complete linear section of $\Sigma_{12} \subset P^{15}$. These follow from analogous properties of curves of genus 7 ([46]).

5. FANO THREEFOLDS OF GENUS 12

Fano threefolds of genus 12 were overlooked by Fano himself ([60], p.91) and discovered by Iskovskih [25]. Subsequently, a special type of Fano threefolds U_{22} which carry an action of $PGL(2)$ were constructed in [50] from a different point of view (cf. Remark 5.6). The normal bundle of a line on U_{22} has a very special property. It gave a motivation to reconsider the classification by the double projection method ([28]). On the other hand, it has been recognized that the study of lines on a generic Fano threefold of genus 12 is related with the theory of plane quartic curves. A consequence of this is the discovery of an expression of such Fano threefolds in terms of varieties of sums of powers ([41], Theorem 5; [42], §6). Like this, Fano threefolds of genus 12 have a particular history; in addition, they are not linear sections of homogeneous spaces like others. But, if we look at them from the viewpoint of vector bundles we explained in the previous section, then they are not exceptional at all. Moreover, in this framework, we can describe them naturally in Grassmannian varieties. Here we proceed by choosing the case associated with the rigid bundle E_3 of Theorem 4.5; but, if we choose E_2 instead, then we will obtain another interesting expression ([41], §3).

Let

$$m : \underbrace{V \times \cdots \times V}_k \longrightarrow C$$

be a skew-symmetric k -linear form on a vector space V . Write $G(r, V, m)$ for the set of r -dimensional subspaces U of V such that the restriction of m on $U \times \cdots \times U$ becomes zero. If \mathcal{E} denotes the dual of the universal vector bundle on the Grassmannian variety $G(r, V)$, then

$$H^0(\mathcal{E}) \cong V^\vee, \quad H^0(\wedge^k \mathcal{E}) \cong \wedge^k V^\vee$$

and $G(r, V, m)$ is a closed subscheme of $G(r, V)$ defined as the zero locus of the global section s of $\wedge^k \mathcal{E}$ corresponding to m . By the higher Bertini's theorem ([43], §1), if $m \in \wedge^k V^\vee$ is generic, then $G(r, V, m)$ is non-singular and has codimension

equal to the rank of $\wedge^k \mathcal{E}$. Here m can be regarded as a (partial) differential operator of the exterior algebra $\wedge V$. Note that Plücker coordinates on elements of $G(r, V, m)$ are annihilated by m .

Example 5.1. If $k = 2$, $\dim V = 2r$ and m is non-degenerate, then $G(r, V, m)$ is non-singular and of dimension $r(r + 1)/2$. It parametrizes the Lagrange subspaces of the symplectic space (V, m) and can be embedded into the projective space of dimension $\binom{2r}{r} - \binom{2r}{r-2} - 1$ by Plücker coordinates. When $r = 3$, $G(r, V, m)$ coincides with $\Sigma_{16} \subset \mathbf{P}^{13}$ of (1.8).

Example 5.2. If $k = 4$, $\dim V = 7$ and m is generic¹², then $G(5, V, m) \subset \mathbf{P}^{13}$ is a G_2 -variety in (1.9).

We now come back to Theorem 4.5. Let (S, L) be a BN-general polarized K3 surface of genus 12 and let E_3 be the rigid bundle of rank 3 in Theorem 4.5. As in the case of genus 9, we consider a map

$$\lambda_2 : \wedge^2 H^0(E_3) \longrightarrow H^0(\wedge^2 E_3).$$

Its kernel, N , is of dimension 3, and the image of $\Phi_E : S \longrightarrow G(3, H^0(E_3)^\vee)$ is contained in the intersection $\bigcap G(3, H^0(E_3)^\vee, m)$, where m ranges over N .

Given a 7-dimensional vector space V and a 3-dimensional space $N \subset \wedge^2 V^\vee$ of skew-symmetric bilinear forms, we denote by $G(3, V, N)$ the subscheme $\bigcap G(3, V, m)$ of the 12-dimensional Grassmannian variety. Let $\{m_1, m_2, m_3\}$ be a basis for N and let s_1, s_2, s_3 be the corresponding global sections of $\wedge^2 \mathcal{E}$. Then $G(3, V, N)$ is the zero locus of the global section (s_1, s_2, s_3) of the vector bundle $(\wedge^2 \mathcal{E})^{\oplus 3}$ of rank 9. Hence, if N is generic, then $G(3, V, N)$ is non-singular and of dimension 3. It is embedded into \mathbf{P}^{13} by Plücker coordinates and has degree 22 by the Schubert calculus ([17], Chap. 1). The adjunction formula implies that $G(3, V, N) \subset \mathbf{P}^{13}$ is an anti-canonical model. More precisely, we have the following.

Definition 5.3. $N \subset \wedge^2 V^\vee$ is said to be **non-degenerate** if the intersection of the subspace $P_*(N \wedge V^\vee)$ with the Grassmannian variety $G(3, V^\vee) \subset P_* \wedge^3 V^\vee$ is empty.

Theorem 5.4. *If $N \subset \wedge^2 V^\vee$ is non-degenerate, then $G(3, V, N)$ is a non-singular Fano threefold with $B_2 = 1$ and genus 12. The set M of elements of $\wedge^3 V$ annihilated by the elements of N has dimension 14. The Plücker embedding $G(3, V, N) \subset P_* M$ gives the anti-canonical model of M .*

Using the embedding associated to the vector bundle E_3 , we obtain the following.

Theorem 5.5. *The projective model of a BN-general polarized K3 surface of genus 12 is isomorphic to a hyperplane section of $G(3, 7, N) \subset \mathbf{P}^{13}$ for some non-degenerate $N \subset \wedge^2 \mathcal{C}^7$. The converse is also true.*

¹²To be more precise, this means that m belongs to an open orbit of $\wedge^3 \mathcal{C}^7$ with respect to the action of $GL(7)$ (cf. [19], §35).

Remark 5.6. Let V_6 be the irreducible representation of $SL(2)$ of dimension 7. Then the irreducible component V_2 of dimension 3 of the exterior power $\wedge^2 V_6 \cong V_{10} \oplus V_6 \oplus V_2$ is non-degenerate in the sense of Definition 5.3. If W denotes the irreducible component of dimension 3 of the representation V_6 restricted to the icosahedral group Icosa, then a non-singular compactification of the $SL(2)$ -orbit of a point $[W]$ is isomorphic to the Fano threefold $G(3, V_6, V_2)$ of genus 12. Although our method of construction is different, the result is the same as the equivariant compactification U_{22} of $SL(2)/\text{Icosa}$ discovered in [50].

6. GORENSTEIN FANO THREEFOLDS

It turns out to be more natural to classify Fano threefolds with terminal singularities. Furthermore, those with canonical singularities are indispensable to the study of degeneration of Fano threefolds and a compactification of their moduli space (§9). We begin by generalizing Theorem 1.10 to Fano threefolds with such singularities.

First, we give a quick example of a singular Fano threefold by blowing up a projective space \mathbf{P}^3 . In order to emphasize the three-dimensionality, we look at a blow-up $\beta : Y \rightarrow \mathbf{P}^3$ with center a curve C ; the situation is similar when we blow up at a point. Let h be the pull-back of a plane and let E be the exceptional divisor over C . Then we have $c_1(Y) = 4h - E$, as in (2.6). Since there are ∞^4 number of lines in \mathbf{P}^3 , usually there exists a line that intersects with C at 4 points (quadriseccant). In fact, the “number” of such lines is equal to

$$(6.1) \quad \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{p(d^2-7d+13-p)}{2},$$

where d and p are the degree and genus of C , respectively ([17] Chap. 2; [18]). If $l' \subset Y$ denotes the proper transform of a quadriseccant $l \subset \mathbf{P}^3$ by β , then $(c_1(Y), l') = 0$. Hence, for Y to be Fano, we ought to assume that (6.1) is not positive. Indeed, using several inequalities about d and p , we see that there are exactly 12 space curves with no quadriseccants, namely

- (i) rational curves of degree ≤ 4 and elliptic curves of degree 3, 4, 5;
- (ii) $(d, p) = (5, 2), (6, 2), (6, 3), (7, 5), (9, 10)$.

The condition $c_1(Y)^3 > 0$ is weaker than this. If it holds, then $\wedge^3 T_Y$ is base point free in many cases, and the anti-canonical morphism Φ_{-K} becomes “almost embedding”. This means that there are only a finite number of curves C_1, C_2, \dots, C_f on Y having intersection number zero with $c_1(Y)$, and Φ_{-K} is an embedding outside these curves; compare with l' above. In this situation, the image of Φ_{-K} is singular at the images of curves C_i , which are a finite number of points, and the line bundle corresponding to $\wedge^3 T$ is ample. This is a typical example of a Fano threefold with Gorenstein terminal singularities.

Example 6.2. A generic sextic rational curve $R_6 \subset \mathbf{P}^3$ has exactly six quadriseccants l_1, \dots, l_6 . We see that $c_1(Y)^3 = 14$, and the anti-canonical morphism $\Phi_{-K} : Y \rightarrow \mathbf{P}^9$ is an embedding outside the proper transforms of l_1, \dots, l_6 . The image of Φ_{-K} is a Fano threefold of genus 8 with six ordinary double points, and non-singular elsewhere.

A large number of singular Fano threefolds obtained in this manner are a natural 3-dimensional version of the non-singular del Pezzo surfaces, which are all constructed as blow-ups at n points in general position satisfying $c_1(S)^2 = 8 - n > 0$. (Theorem 2.7).

Definition 6.3 (Reid [57], [59]). We say that a normal algebraic variety X has at most **terminal singularities** if X is \mathbb{Q} -Gorenstein and there exists a resolution of singularities $f : Y \rightarrow X$ satisfying the condition:

(*) for all exceptional divisors E_i of f , the coefficients a_i in $K_Y = f^*K_X + \sum_i a_i E_i$ are positive.

If this holds, then the singular locus of X has codimension greater than or equal to 3. In particular, X is non-singular in dimension 2 and it has only isolated singularities in dimension 3. The origin of the hypersurface $x^2 + y^2 + z^2 + t^n = 0$ in \mathbb{C}^4 is a typical example of a Gorenstein terminal singularity. (When $n = 2$, it is called an ordinary double point.) If X is an algebraic variety with only Gorenstein singularities, then the inverse $\mathcal{O}_X(-K)$ of the dualizing sheaf associated with the divisor K_X in Definition 6.3 has the same meaning as $\overset{n}{\wedge} T$ in the non-singular case. Thus, we may call $\mathcal{O}_X(-K)$ the anti-canonical line bundle and define X to be Fano if it is ample.

From now on, we assume that X is a Fano threefold with at most Gorenstein terminal singularities. As in Theorem 3.4, there exists a global section α of $\mathcal{O}_X(-K)$ such that the zero locus S is non-singular ([58]). Since the vanishing theorem, etc., for non-singular Fano threefolds are still valid in this case, S is a K3 surface.

Definition 6.4. X is said to be **BN-general** if there exists a global section $\alpha \in H^0(\mathcal{O}_X(-K))$ such that the associated polarized K3 surface $(S, \mathcal{O}_S(-K))$ is BN-general, where $\mathcal{O}_S(-K)$ is the restriction of $\mathcal{O}_X(-K)$ to S .

The following is a generalization of Theorem 1.10.

Theorem 6.5. *If X is BN-general, then its genus g satisfies $g \leq 10$ or $g = 12$. Furthermore,*

- (1) *if $g = 2$, then $\Phi_{-K} : X \rightarrow \mathbb{P}^3$ is a double covering ramified at a sextic surface;*
- (2) *if $g \geq 3$, then Φ_{-K} is an embedding and Theorem 1.10 (2.a) and (3) hold for the anti-canonical model X_{2g-2} .*

We will prove statement (2) of the theorem for $g = 7, 8, 9, 10$ using Theorem 4.7. It is known that a hyperplane section $S_{2g-2} \subset \mathbb{P}^g$ of the anti-canonical model $X_{2g-2} \subset \mathbb{P}^{g+1}$ is a linear section of the homogeneous projective variety $\Sigma_{2g-2} \subset \mathbb{P}$. We will show, by a Lefschetz-type argument, that the same is true for $X_{2g-2} \subset \mathbb{P}^{g+1}$ itself. (By Theorem 5.5, the same argument works also for the case $g = 12$).

First, we extend the canonical embedding $i : S \rightarrow \Sigma$ to the formal neighbourhood \hat{S}_X of X along S . Specifically, we expand along S the expected regular map $j : X \rightarrow \Sigma$ into a power series and determine its coefficients inductively from a lower term. The coefficient of the term of degree 1 is equal to the global section of the vector bundle $T_\Sigma|_S \otimes N_{S/X}$. The obstacles in determining the coefficients of higher terms globally are given inductively by the first cohomology classes of $T_\Sigma|_S \otimes N_{S/X}^{\otimes n}$ for $n \geq 1$. Fortunately, by Bott's vanishing theorem [6], these cohomology groups disappear and we obtain an extension $\hat{i} : \hat{S}_X \rightarrow \Sigma$ of i .

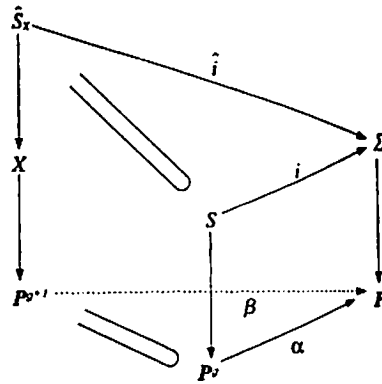


FIGURE 2

\hat{i} induces a linear map $H^0(\mathcal{O}_\Sigma(1)) \rightarrow H^0(\hat{S}_X, \mathcal{O}(1))$, and the projectification $\beta : \mathbf{P}^{g+1} \rightarrow \mathbf{P}$ of (the dual of) this map makes the diagram in Figure 2 commutative. As \hat{S}_X is mapped to Σ by β , so is X . Besides, $\beta|_X$ is an extension of i to X . Suppose that β is not an embedding. Then it becomes the composite of the projection $\mathbf{P}^{g+1} \rightarrow \mathbf{P}^g$ from a point p with the natural inclusion α . But this implies that X is the cone over S with vertex p , which is a contradiction. Therefore β is an embedding. As S is a complete linear section of Σ , so is $\beta(X) \cong X$.

7. GENUS BOUND AND NO MOVING DECOMPOSITION

We will complete a proof of Theorem 6.5 by proving the genus bound “ $g \leq 10$ or $g = 12$ ”. We begin by explaining the idea of proof in the case of dimension 2. Let $S \subset \mathbf{P}^d$ be the anti-canonical model of a non-singular del Pezzo surface and let $\beta : S_p \rightarrow S$ be the blow-up of S at a point p . There is a one-to-one correspondence between the set of anti-canonical forms on S vanishing at p and the set of anti-canonical forms on S_p . This gives rise to the commutative diagram

$$(7.1) \quad \begin{array}{ccc} S_p & \xrightarrow{\Phi_{-\kappa}} & \mathbf{P}^{d-1} \\ \downarrow \beta & & \uparrow \pi_p \\ S & \xrightarrow{\Phi_{-\kappa}} & \mathbf{P}^d \end{array}$$

where π_p is the projection from p .

Since $S \subset \mathbf{P}^d$ has only a finite number of lines, we may choose p so that it lies on none of these lines. Then S_p is a del Pezzo surface of degree one less than the degree of S .

Next, let $X \subset \mathbf{P}^{g+1}$ be the anti-canonical model of a Gorenstein Fano threefold and again let $\beta : X_p \rightarrow X$ be the blow-up of X at a non-singular point p . In this case, the set of anti-canonical forms on X vanishing twice at p is in one-to-one correspondence with those on X_p . Let π_{2p} be the projection from the tangent

(projective) space at p . Then we obtain a commutative diagram

$$(7.2) \quad \begin{array}{ccc} X_p & \xrightarrow{\Phi_{-K}} & \mathbf{P}^{g-3} \\ \downarrow \beta & & \uparrow \pi_{2p} \\ X & \xrightarrow{\Phi_{-K}} & \mathbf{P}^{g+1}. \end{array}$$

As in the case of del Pezzo surfaces, we may choose p so that it will not lie on any lines on $X \subset \mathbf{P}^{g+1}$. But, unlike π_p , the map π_{2p} contracts conics passing through p to a point. Further, such a conic usually exists (similar to the existence of a quadrisecant in the previous section). Hence X_p itself is not a Fano threefold. However, if $c_1(X_p)^3 = c_1(X)^3 - 8 > 0$, then X_p becomes “almost” Fano ([56]). About this, we summarize the results relevant only to the genus bound.

Proposition 7.3. *Let X be a BN-general Fano threefold with at most Gorenstein terminal singularities. Let $p \in X$ be a generic point. Then the following assertions hold.*

(1) *If X has genus $g \geq 7$, then the image $\overline{X}_p := \Phi_{-K}(X_p)$ is also a BN-general Fano threefold with at most Gorenstein terminal singularities.*

(2) *If $g \geq 11$, then $\overline{X}_p \subset \mathbf{P}^{g-3}$ contains a Veronese surface P given by the image of the exceptional divisor of the blow-up β . If $\langle P \rangle \cong \mathbf{P}^5$ denotes its linear envelope, then $\langle P \rangle \cap \overline{X}_p = P$.*

Example 7.4. When $g = 7$, \overline{X}_p is a singular quartic hypersurface in \mathbf{P}^4 containing a quartic surface P isomorphic to \mathbf{P}^2 . There exists exactly one trisecant of $P \subset \mathbf{P}^4$ that passes through generic points of \overline{X}_p , and of a hyperplane \mathbf{P}^3 . Hence \overline{X}_p is rational ([66]). Therefore X itself is also rational.

Example 7.5. When $g = 12$, \overline{X}_p is isomorphic to the image \overline{Y} of Φ_{-K} in Example 6.2. The Veronese surface P in $\overline{X}_p \subset \mathbf{P}^9$ is the image of a cubic surface containing the rational sextic curve R_6 .

There is an inductive structure on Fano threefolds with terminal singularities: the genus decreases by 4 in the process of obtaining \overline{X}_p from X . When they are BN-general, we can prove, by contradiction, the non-existence of higher genera, based on the properties of Fano threefolds of genus 7, 9, 10, 12 that we have already established.

Proof of Theorem 6.5. By part (1) of the proposition above, it suffices to prove the impossibility of genus 11, 13, 14, 16. We consider only the last two cases, which are rather easy. Suppose that X has genus 16. Then \overline{X}_p has genus 12 and contains a Veronese surface. But, from what we established about Theorem 6.5 (§6), we find that \overline{X}_p is non-singular and its Picard group is generated by $\mathcal{O}_X(-K)$ (Theorem 5.4); this is a contradiction. Suppose now that X has genus 14. Then \overline{X}_p has genus 10. It is a linear section of a G_2 -variety $\Sigma_{18} \subset \mathbf{P}^{13}$. But we see, using a cellular decomposition, that Σ_{18} does not contain any quartic surface. This is again a contradiction. \square

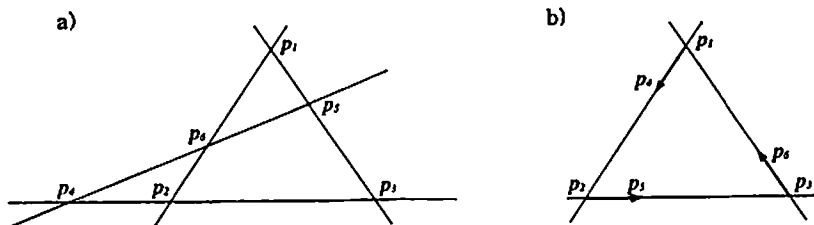
Finally, we discuss canonical singularities and moving decomposition. This completes our explanation about our classification of Fano threefolds by vector bundles.

Canonical singularities are singularities defined by the same condition as in Definition 6.3 except that $(*)$ is weakened to be $a_i \geq 0$. They are a higher dimensional analogue of rational double points of surfaces. For example, in dimension 3, rational double points along a line and the vertex of a cone in a del Pezzo surface are canonical singularities. Del Pezzo surfaces with rational double points have been well investigated ([9]).

Example 7.6. Let $S(p_1, \dots, p_6)$ be a blow-up of P^2 at 6 points. Consider the image $S_3 \subset P^3$ of the anti-canonical morphism Φ_{-K} associated to $S(p_1, \dots, p_6)$.

(a) If the 6 points are vertices of a complete quadrangle, then S_3 is a cubic (del Pezzo) surface with 4 ordinary double points (=rational double points of type A_1).

(b) If p_1, p_2, p_3 are not collinear and if p_4, p_5, p_6 are their infinitely near points lying in the direction of $\overrightarrow{p_1 p_2}, \overrightarrow{p_2 p_3}, \overrightarrow{p_3 p_1}$, respectively, then S_3 has 3 rational double points of type A_2 .



Hyperplane sections of the anti-canonical model of a Fano threefold with only Gorenstein canonical singularities are K3 surfaces with at most rational double points. We can deal with them in the same way as with non-singular K3 surfaces. In addition, the process $X \Rightarrow \overline{X}_p$ (the image of double projection from a point) yields only terminal singularities. Hence Theorem 6.5 and its proof remain valid if we replace “terminal” by “canonical”.

Denote by $| -K_X |$ the family of the zero locus (divisors) of all non-zero global sections $s \in H^0(\mathcal{O}_X(-K_X))$ of the anti-canonical line bundle. For a Weil divisor D , let $|D|$ be the set of positive divisors linearly equivalent to D (complete linear system).

Definition 7.7. If $| -K_X |$ contains a sum of two divisors $A + B$ such that both $\dim |A|$ and $\dim |B|$ are positive, then $-K_X \sim A + B$ is said to be **moving decomposition**.

Proposition 7.8. Let X be a Fano threefold with at most Gorenstein canonical singularities. Assume that $-K_X$ has no moving decomposition. Then X is BN-general, except for the case:

(**) X has genus 3 and Φ_{-K} is a double covering of a quadric hypersurface in P^4 .

For a non-singular Fano threefold with $B_2 = r = 1$, $H^2(X, Z)$ is generated by $c_1(X)$. Hence every element of $| -K_X |$ is irreducible. Therefore Theorem 1.10 follows from the proposition and Theorem 6.5.

8. FANO THREEFOLDS AS MODULI SPACES OF ABELIAN SURFACES

Quotients of the Poincaré upper half-plane by congruence subgroups $\Gamma_0(N)$ or $\Gamma(N)$ are rational curves when N is small. They become isomorphic to P^1 by

adding cusps. Similarly, let

$$\mathfrak{H}_2 = \left\{ Z = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} \in \text{Sym}_2 \mathbb{C} \mid \text{Im} Z \text{ is positive definite} \right\}$$

be the 3-dimensional Siegel upper half-space. Then several Fano threefolds can be obtained as suitable compactifications of the quotient spaces $\Gamma \backslash \mathfrak{H}_2$ of \mathfrak{H}_2 by some arithmetic subgroup Γ . First, we look at the case where principally polarized abelian surfaces are parametrized.

Example 8.1. Segre cubic hypersurface: it has 10 ordinary double points and it can be defined by

$$V_3 : \sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0 \text{ in } \mathbb{P}^5.$$

Analogously, write $V_4(a) \subset \mathbb{P}^4$ for the quartic hypersurface in \mathbb{P}^5 defined by the two symmetric equations

$$\sum_{i=0}^5 x_i = \left(\sum_{i=0}^5 x_i^2 \right)^2 - a \sum_{i=0}^5 x_i^4 = 0.$$

Example 8.2. $V_4(4) \subset \mathbb{P}^4$ is singular along 15 lines.

Example 8.3. $V_4(2) \subset \mathbb{P}^4$ has 45 ordinary double points.

Let $|2h - p_1 - \dots - p_5|$ be the linear system of a quadric surface in \mathbb{P}^3 passing through 5 points in general position. The threefold in Example 8.1 is the image of the rational map $\mathbb{P}^3 \dots \rightarrow \mathbb{P}^4$ associated to the linear system ([63], Chap. 8). It contains 15 planes $P_{\mu\nu}, 1 \leq \mu < \nu \leq 6$, which are the images of exceptional divisors given by blowing up the 5 points and of the planes passing through 3 points. We will explain that the cubic hypersurface V_3 parametrizes Kummer surfaces.

Let (A, L) be a principally polarized abelian surface. The image S of $\Phi_{|2L|} : A \rightarrow \mathbb{P}^3$ is called a Kummer quartic surface. S is isomorphic to the quotient of A by its automorphism -1_A , and it has 16 ordinary double points. As S is fixed by an action of a Heisenberg group, its defining equation can be written as

$$(8.4) \quad \begin{aligned} &A(x^4 + y^4 + z^4 + t^4) + B(x^2y^2 + z^2t^2) \\ &+ C(x^2z^2 + y^2t^2) + D(x^2t^2 + y^2z^2) + Exyzt = 0 \end{aligned}$$

(e.g. [51], §5). The discriminant Δ of the equation vanishes at the singularities of S .

Proposition 8.5. *The discriminant Δ of the quartic equation (8.4) is a homogeneous polynomial of degree 108 in A, B, C, D, E . It is factored into*

$$\Delta = \Delta_K^{16} \prod_{1 \leq i < j \leq 6} \Delta_{ij}^4,$$

where Δ_K is cubic and each Δ_{ij} is linear¹³.

¹³The claim of [51], I, p.354 is false.

If we choose $(A : B : C : D : E)$ as coordinates of \mathbf{P}^4 , then $\Delta_K = 0^{14}$ gives the Segre cubic hypersurface V_3 in \mathbf{P}^4 . Equation (8.4) becomes a Kummer surface if and only if $\Delta_K = 0$ and $\Delta_{ij} \neq 0$. The union of 15 hyperplane sections $\Delta_K = \Delta_{ij} = 0$ coincides with the union of 15 planes $P_{\mu\nu}$ in the above. In particular, we obtain a morphism from $V_3 \setminus \bigcup P_{\mu\nu}$ to the moduli space $\Gamma(2) \backslash \mathfrak{H}_2$ of principally polarized abelian surfaces $(A, L, (\mathbf{Z}/2)^{\oplus 4} \xrightarrow{\sim} A_2)$ with level 2 structure, where A_2 is the abelian group of 2-torsion points in A . Analyzing the boundary correspondence, we see that the blow-up of V_3 at the 10 ordinary double points is the Igusa compactification¹⁵ of $\Gamma(2) \backslash \mathfrak{H}_2$.

The quartic hypersurface in Example 8.2 is the projective dual of that in Example 8.1, i.e. the image of the following rational map:

$$(8.6) \quad V_3 \ni p \longmapsto [\text{tangent hyperplane of } V_3 \text{ at } p] \in \mathbf{P}^4.$$

The 15 planes $P_{\mu\nu} \subset V_3$ correspond to the 15 lines $l_{\mu\nu}$ that are the singular locus of $V_4(4)$. On the other hand, (8.6) induces a rational map from $\Gamma(2) \backslash \mathfrak{H}_2$ to $V_4(4)$. It gives an isomorphism between the Satake compactification of $\Gamma(2) \backslash \mathfrak{H}_2$ and $V_4(4)$ (see [14], where the morphism is constructed in a different way). The boundary component of dimension 1 is mapped to the union of lines $l_{\mu\nu}$. The symmetric group \mathfrak{S}_6 on six letters acts on both V_3 and $V_4(4)$ by permuting the coordinates, and $Sp_4(\mathbf{F}_2)$ acts on $\Gamma(2) \backslash \mathfrak{H}_2$. The above correspondence is equivariant with respect to these actions. In particular, the Satake compactification of $Sp_4(\mathbf{Z}) \backslash \mathfrak{H}_2$ is $\mathfrak{S}_6 \backslash V_4(4)$, and is isomorphic to a weighted projective space $\mathbf{P}(2 : 3 : 5 : 6)$ (cf. [22]).

The threefold in Example 8.3 is called a Brückhardt quartic hypersurface and is birational to $\Gamma(3) \backslash \mathfrak{H}_2$ ([3], [15]). The simple group $Sp_4(\mathbf{E}_3)$ of order 15,200 (which is a subgroup of index 2 in a Weyl group $W(\mathbf{E}_6)$) acts on it. Besides, 45 is the maximal number of ordinary double points that a quartic hypersurface may take. It is also equal to the moduli number of quartic hypersurfaces ([11]).

Next, we consider the case with non-principal polarization. Given a prime p , set

$$\Gamma_{1,p} = \left\{ g \in Sp_4(\mathbf{Z}) \mid g - 1_4 \in \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ p\mathbf{Z} & p\mathbf{Z} & p\mathbf{Z} & p^2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \end{pmatrix} \right\}.$$

Then the quotient space of \mathfrak{H}_2 by the congruence subgroup $\Gamma_{1,p}$ is the moduli space of triples consisting of the following data:

- (i) an abelian surface A ,
- (ii) a line bundle L on A of degree $2p$, and
- (iii) a level structure $(\mathbf{Z}/p\mathbf{Z})^{\oplus 2} \xrightarrow{\sim} \{a \in A | T_a^* L \cong L\}$.

It is known that the Horrocks-Mumford bundle E_{HM} is an indecomposable vector bundle on \mathbf{P}^4 of rank 2. The zero locus $(s)_0 \subset \mathbf{P}^4$ of a general global section s of E_{HM} is an abelian surface of degree 10. By the correspondence $\mathcal{C}s \longmapsto (s)_0$, we obtain a birational map ([20])

$$\mathbf{P}^3 = \mathbf{P}_* H^0(\mathbf{P}^4, E_{HM}) \cdots \longrightarrow \Gamma_{1,5} \backslash \mathfrak{H}_2.$$

¹⁴As in Maschke [35], if we express (8.4) as $\sum_{i=1}^6 a_i \phi_i = 0$ with $\sum_{i=1}^6 a_i = 0$, then $\Delta_K = \sum_{i=1}^6 a_i^3$ and

$\Delta_{ij} = a_i - a_j$.

¹⁵This compactification is given in [22]. It coincides with the toroidal compactification of [2].

The birational map has been investigated very well, and its properties are summarized in [21]. To simplify our description, it is convenient to use a Fano threefold $U_{22} \subset \mathbf{P}^{13}$ of genus 12 (Remark 5.6) that is the equivariant compactification of $SL(2)/\text{Icosa}$. Let p be an interior point in U_{22} . As in Section 7, consider a double projection $\pi_{2p} : U_{22} \dashrightarrow \mathbf{P}^9$ at p and let U'_{14} denote the image. Then U'_{14} is a Fano threefold of genus 8 with six ordinary double points, say p_1, \dots, p_6 , at the images of the six conics passing through p ; it contains a Veronese surface $P \cong \mathbf{P}^2$ (a special case of Example 6.2).

Proposition 8.7. *Let $B' \subset U'_{14}$ be the image of π_{2p} at the boundary B of U_{22} . Then the complement of B' is isomorphic to the open subset of the moduli space $\Gamma_{1,5} \setminus \mathfrak{H}_2$ of polarized abelian surfaces of degree 10 defined by removing $\mathbf{P}^1 \times \mathbf{P}^1$ that parametrizes the direct products of elliptic curves (with product polarization). The isomorphism is equivariant with respect to the action of the icosahedral group on U'_{14} and with respect to that of $SL(2, \mathbf{F}_5)$ on the moduli space. Furthermore, the isomorphism induces a parametrization of abelian surfaces with real multiplication in $\mathbf{Q}(\sqrt{5})$ (Comesatti surfaces) by the complement of the six singularities p_1, \dots, p_6 in the Veronese surface P .*

In the case $p = 7$, the moduli space $\Gamma_{1,p} \setminus \mathfrak{H}_2$ is also related closely with Fano threefolds of genus 12 ([34]), especially with a description by the net of quadric surfaces ([41], §3; [72]).

In addition to abelian surfaces, some vector bundles can be parametrized by Fano threefolds. See [47] and [71] for a relation between Fano threefolds of genus 7 and the moduli space of special stable vector bundles on a curve.

9. MODULI SPACE OF FANO THREEFOLDS

We discuss the moduli spaces of del Pezzo surfaces of degree 2,3 and of del Pezzo threefolds of degree 4 (B_4 in Table 1.4). We describe also their compactifications. In the case of dimension 3, there is an interesting relation between degeneration and period, whereas more general cases are left as a subject for further investigation.

I. Let $F_3(x, y, z, t) = 0$ be a defining equation for a cubic (del Pezzo) surface $S_3 \subset \mathbf{P}^3$. Two cubic surfaces S_3 and S'_3 are isomorphic if and only if their defining equations $F_3, F'_3 \in S^3C^4$ belong to the same orbit with respect to the action of the general linear group $GL(4)$. Hence the orbit space $(S^3C^4)^{\text{smooth}}/GL(4)$ gives the moduli space of non-singular cubic surfaces¹⁶, where $(S^3C^4)^{\text{smooth}}$ denotes the open subset of S^3C^4 defined by the non-vanishing of the discriminant $\Delta \neq 0$.

Definition 9.1. Let $F \in S^dC^n$ be a homogeneous polynomial of degree d in n variables. Write $O(F)$ for the $SL(n)$ orbit of F . Then:

- (i) F is **semi-stable** \iff the closure of $O(F)$ does not contain 0;
- (ii) F is **weakly stable**¹⁷ $\iff O(F)$ is a closed set;
- (iii) F is **stable** $\iff F$ is weakly stable and its stabilizer is finite.

Using a numerical criterion of Hilbert and Mumford, we obtain the following.

Proposition 9.2 ([52], §1.14). *Let $S : F(x, y, z, t) = 0$ be a cubic surface. Then:*
 (1) F is stable $\iff S$ has only a finite number of ordinary double points as its singularity;

¹⁶See [55] for another way of describing the moduli space of cubic surfaces.

¹⁷We followed [16] for this terminology.

(2) F is weakly stable $\iff F$ is stable or it can be transformed to $xyz + t^3$ by the action of $GL(4)$.

Let $(S^d\mathbb{C}^n)^s$ (resp. $\subset (S^d\mathbb{C}^n)^{ss}$) be the open set of stable (resp. semi-stable) homogeneous polynomials F of degree d . A categorical quotient space

$$(S^d\mathbb{C}^n)^{ss} // GL(n)$$

exists as a projective algebraic variety, and the orbit space $(S^d\mathbb{C}^n)^s // GL(n)$ is contained in it as an open subset. In the case of cubic surfaces, we obtain a compactification $(S^3\mathbb{C}^4)^{ss} // GL(4)$ of the moduli space $(S^3\mathbb{C}^4)^s // GL(4)$ of cubic surfaces with only ordinary double points by adding a point $[b]$ represented by $S_b : xyz + t^3 = 0$. The cubic surface S_b is the quotient of \mathbb{P}^2 by a group of automorphisms of order 3; this coincides with (b) of Example 7.6.

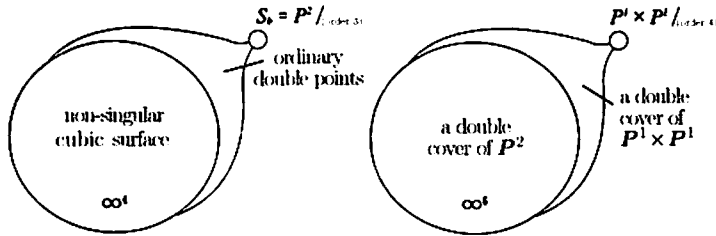


FIGURE 3. Moduli spaces of del Pezzo surfaces of degree 3 and 2.

II. Let S be a del Pezzo surface of degree 2. The anti-canonical morphism $\Phi_{-K} : S \rightarrow \mathbb{P}^2$ is a double covering ramified at a quartic curve $C : F_4(x, y, z) = 0$. Hence $(S^4\mathbb{C}^3)^{\text{smooth}} // GL(3)$ is the moduli space of non-singular del Pezzo surfaces of degree 2.

Proposition 9.3. *Let $S : \tau^2 = F(x, y, z)$ be a quadric del Pezzo surface. Then:*

- (1) F is stable \iff the singularities of S are only rational double points of type A_1 or A_2 ;
- (2) F is weakly stable $\iff F$ is stable or it can be transformed to $(xz - y^2)^2$ by the action of $GL(3)$.

Here $(S^4\mathbb{C}^3)^s // GL(3)$ is the moduli space of quadric del Pezzo surfaces with at most rational double points of type A_1, A_2 . By adding a point b represented by the surface $S_b : \tau^2 = (xz - y^2)^2$, we thus obtain a compactification of it. S_b is, however, reducible. A better compactification is given by blowing up the point b .

Proposition 9.4. *Let \mathcal{M} be the blow-up of $(S^4\mathbb{C}^3)^{ss} // GL(3)$ with center b and let E denote its exceptional divisor.*

- (1) E is isomorphic to the moduli space $(S^8\mathbb{C}^2)^{ss} // GL(2)$ of polynomials $f_8(x, y)$ of degree 8 in two variables.
- (2) There exists a family of surfaces $\{S_m : m \in \mathcal{M}\}$ parametrized by \mathcal{M} such that the following two conditions hold:
 - (i) $\mathcal{M} \setminus E \cong (S^4\mathbb{C}^3)^s // GL(3)$ parametrizes double covers of \mathbb{P}^2 in the same way as before.
 - (ii) For $m \in E$, let $f_8(x, y)$ be the corresponding polynomial and let C be the hyperelliptic curve of genus 3 defined by $z^2 = f_8(x, y)$. Then the image $C_8 \subset \mathbb{P}^5$

of Φ_{2K} at C is contained in the cone \bar{F}_4 of a quartic rational curve. Each S_m is a double cover of \bar{F}_4 ramified at C_8 .

The surface S_m in (ii) above has 2 singularities other than rational double points. They belong to a class of singularities called **log terminal singularities** (this is always expected even in the case of dimension 3). If $f_8(x, y) = x^4y^4$ in (ii), then S_m is isomorphic to the quotient of $P^1 \times P^1$ by a group of automorphisms of order 4. The proposition can be proved in the same way as in [64].

Remark 9.5. We may regard \mathcal{M} as a compactification of the moduli space of curves of genus 3. It is similar to the compactification of Schubert [62] rather than that of Deligne and Mumford [7].

III. A del Pezzo threefold of degree 4 is a complete intersection of two quadric hypersurfaces in P^5 which may be defined by

$$V = V(a : b) \begin{cases} a_1x_1^2 + a_2x_2^2 + \dots + a_6x_6^2 = 0, \\ b_1x_1^2 + b_2x_2^2 + \dots + b_6x_6^2 = 0. \end{cases}$$

The sextic homogeneous polynomial $f_V(\xi, \eta) = \prod_{i=1}^6 (a_i\xi + b_i\eta)$ in two indeterminates ξ, η is called the characteristic polynomial of V . All the non-singular del Pezzo threefolds of degree 4 can be expressed in this way. Further, $V(a : b)$ and $V(a' : b')$ are isomorphic if and only if their characteristic polynomials belong to the same orbit with respect to the action of $GL(2)$. Hence $\mathcal{N} = (S^6C^2)^{ss} // GL(2)$ gives a compactification of their moduli space. Depending on the type of a weakly stable characteristic polynomial f_V , the singularities of V are described as follows:

(i) if $f_V(\xi, \eta) = 0$ has no triple root, then there are twice as many singularities on V as double roots of the equation, and all the singularities are ordinary double points (in particular, terminal singularities);

(ii) if $f_V(\xi, \eta)$ is equivalent to $\xi^3\eta^3$, then V has ordinary double points along 2 quadric curves (these are canonical singularities).

In the case where V is non-singular, the analytic genus p defined in (1.3) is equal to 2 and its intermediate Jacobian variety is isomorphic to the Jacobian variety of the curve $\tau^2 = f_V(\xi, \eta)$ of genus 2. Hence the period map can be extended naturally to \mathcal{N} except at the point corresponding to (ii). By this correspondence, \mathcal{N} is isomorphic to the quotient of the cubic hypersurface in Example 8.1 by an action of the symmetric group on 6 letters; in other words, it is isomorphic¹⁸ to the weighted projective space $P(2 : 4 : 5 : 6)$.

In every case of I ~ III above, the moduli space can be compactified by adding **irreducible** Fano threefolds with at most mild singularities¹⁹. It seems interesting to me to study, using the new classification explained earlier in this article, whether or not this type of compactification can also be achieved in other cases ([49]). On the other hand, there is a close relation between the Einstein-Kähler metric and a compactification of the moduli space of del Pezzo surfaces (see [32] and its

¹⁸It is known that the degree-even piece of the invariant ring of binary sextic polynomials is generated by the invariant polynomials of degree 2,4,6,10 (Clebsch-Gordan Theorem, e.g. [38], p.66; Igusa [70]; Schur [73]). $P(2 : 4 : 5 : 6)$ is isomorphic to $P(2 : 4 : 6 : 10)$. See [69].

¹⁹The statement is true also for 3-dimensional cubic hypersurfaces. The moduli space of those with only terminal singularities (which is of dimension 10) can be compactified by adding a point represented by the secant variety of a rational quartic curve (work of Mutsumi Yokoyama [74] and Allcock [68]).

references). Another interesting direction to pursue would be to investigate such phenomena for Fano threefolds.

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