ON THE MODULI SPACE OF BUNDLES ON $K3$ SURFACES, I

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IN [12], WE have shown that the moduli space M_S of stable sheaves on a $K3$ or abelian surface S is smooth and has a natural symplectic structure. In this article, we shall study M_S more precisely in the case S is of type $K3$. We shall show that every compact 2 dimensional component of M_S is a $K3$ surface isogenous to S (Definition 1.7 and 1.8) and describe its period explicitly (Theorem 1.4). As an application of this result, we shall show that certain Hodge cycles on a product of two $K3$ surfaces are algebraic (Theorem 1.9). As a corollary, we have that two $K3$ surfaces with Picard number ≥ 11 are isogeneous in our sense if and only if their transcendental Hodge structures T_S and $T_{S'}$ are isogenous, i.e., isomorphic over $\mathbb{Q}$ (Corollary 1.10).

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§ 1. Introduction

Let S be an algebraic $K3$ surface over the complex number field $\mathbb{C}$. The cohomology group $H^2(S, \mathbb{Z})$ with the cup product pairing is an even unimodular lattice and isomorphic to $\Lambda = U^{\perp 3} \perp E_8^{\perp 2}$ which we call a $K3$ lattice, where U is the hyperbolic lattice $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and E_8 is an even unimodular negative definite lattice of rank 8. We define a bilinear form and a Hodge structure of weight 2 on the cohomology ring $H^*(S, \mathbb{Z})$. The integral bilinear form $(\ .\ )$ on $H^*(S, \mathbb{Z})$ is defined by

$$(1.1)\quad (\alpha . \beta) = -\alpha^0 \cup \beta^4 + \alpha^2 \cup \beta^2 - \alpha^4 \cup \beta^0 \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}$$

for every $\alpha = (\alpha^0, \alpha^2, \alpha^4)$ and $\beta = (\beta^0, \beta^2, \beta^4)$ in $H^*(S, \mathbb{Z})$, where we identify $H^4(S, \mathbb{Z})$ with $\mathbb{Z}$ by the fundamental cocycle $w \in H^4(S, \mathbb{Z})$. The Hodge decomposition of $H^*(S, \mathbb{C}) = H^*(S, \mathbb{Z}) \otimes \mathbb{C}$ is defined by

$$(1.2)\qquad H^{*,2,0}(S, \mathbb{C}) = H^{2,0}(S, \mathbb{C}),$$
$$H^{*,0,2}(S, \mathbb{C}) = H^{0,2}(S, \mathbb{C})$$

and

$$H^{*,1,1}(S, \mathbb{C}) = H^0(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^4(S, \mathbb{C}).$$

$H^*(S, \mathbb{Z})$ with the bilinear form (1.1) and the Hodge structure (1.2) is denoted by $\tilde{H}(S, \mathbb{Z})$. $H^2(S, \mathbb{Z})$ is a sublattice and a Hodge substructure of $\tilde{H}(S, \mathbb{Z})$.

Let E be a sheaf on S. Since $H^2(S, \mathbb{Z})$ is an even lattice, the

Chern character $ch(E)$ of E belongs to $H^*(S, \mathbb{Z})$. We denote $ch(E)\cdot\sqrt{td_S} \in H^*(S, \mathbb{Z}) = \tilde{H}(S, \mathbb{Z})$ by $v(E)$ (Definition 2.1) The $H^0(S)$-component of $v(E)$ is the rank $r(E)$ of E (at the generic point) and $H^2(S)$-component is the 1st Chern class $c_1(E)$. The $H^4(S)$-component of $v(E)$ is denoted by $s(E)$. By the Riemann-Roch theorem, we have $s(E) = r(E) + ch^2(E) = \chi(E) - r(E)$. $v(E)$ is of type (1,1) with respect to the Hodge structure defined in (1.2). For sheaves E and F on S, $\chi(E, F)$ denotes the alternating sum $\sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}_S}(E, F)$. By the Riemann-Roch theorem, we have (see Proposition 2.2)

$$\chi(E, F) = -(v(E) \cdot v(F)).$$

Let v be a vector of $\tilde{H}(S, \mathbb{Z})$ of Hodge type (1, 1), and let $M_A(v)$ be the moduli space of stable sheaves E on S with $v(E) = v$ which are stable with respect to A in the sense of [2]. Then $M_A(v)$ is smooth and each component has dimension $(v^2)+2$. Assume that v is isotropic, i.e., $(v^2) = 0$ and that v is primitive, i.e., not divisible by any integer ≥ 2. Then $M_A(v)$ is 2-dimensional. The orthogonal complement $v^\perp$ of v in $\tilde{H}(S, \mathbb{Z})$ contains v and the quotient $v^\perp/\mathbb{Z}v$ is a free $\mathbb{Z}$-module of rank 22. The quadratic form on $\tilde{H}(S,\mathbb{Z})$ defined in (1.1) induces a quadratic form on $v^\perp/\mathbb{Z}v$ with signature (3, 19). Since v is of type (1.1), the Hodge decomposition of $\tilde{H}(S,\mathbb{C})$ induces that of $(v^\perp/\mathbb{Z}v) \otimes \mathbb{C}$. Hence $v^\perp/\mathbb{Z}v$ carries the polarized Hodge structure of the same kind as $H^2(S,\mathbb{Z})$.

THEOREM 1.4. *Let S be an algebraic $K3$ surface and v a primitive isotropic vector of $\tilde{H}(S,\mathbb{Z})$. Assume that the moduli* *space* $M_A(v)$ *is nonempty and compact. Then* $'M_A(v)$ *is irreducible and is a (minimal) K3 surface. Moreover, there is an isomorphism of Hodge structures between* $H^2(M_A(v), \mathbb{Z})$ *and* $v^{\perp}/\mathbb{Z}v$ *which is compatible with the cup product pairing on* $H^2(M_A(v))$ *and the bilinear form on* $v^{\perp}/\mathbb{Z}v$ *induced by that on* $\tilde{H}(S,\mathbb{Z})$.

The above theorem and the Torelli theorem for $K3$ surfaces ([7], [20]) determine the isomorphism class of $M_A(v)$ uniquely There are many pairs of v and A for which the moduli spaces $M_A(v)$ are compact (Proposition 4.1 and 4.3).

REMARK. Even if $M_A(v)$ is not compact, every component of $M_A(v)$ is birationally equivalent to a $K3$ surface M and the period of M is isomorphic to $v^{\perp}/\mathbb{Z}v$.

Now we show how the isomorphism between $H^2(M_A(v),\mathbb{Z})$ and $v^{\perp}/\mathbb{Z}v$ is obtained. The isomorphism is induced by a natural algebraic cycle on $S\times M_A(v)$. There exists a sheaf $\mathscr{E}$ on $S\times M_A(v)$ which we call a quasi-universal sheaf (Definition A. 4 and Theorem A.5). $\mathscr{E}$ is flat over $M_A(v)$ and the restriction to $S\times m$ is isomorphic to $E_m^{\oplus\sigma}$ for every point $m \in M_A(v)$, where E_m is a stable sheaf in $M_A(v)$ corresponding to m. The integer $\sigma = \sigma(\mathscr{E})$ does not depend on m and is called the similitude of E. Let $ch(\mathscr{E}) \in H^*(S\times M_A(v), \mathbb{Q})$ be the Chern character of $\mathscr{E}$. Put $Z_{\mathscr{E}} = (\pi_S^* \sqrt{td_S}) \cdot ch(\mathscr{E}) \cdot (\pi_M^* \sqrt{td_M}/\sigma(\mathscr{E})$, where td_S is the Todd classs of S and $M = M_A(v)$. $Z_{\mathscr{E}}$ is an algebraic cycle o1 $S\times M_A(v)$ (with $\mathbb{Q}$-coefficient) and induces the homomorphism

$$\begin{array}{ccc} f_{Z_{\mathscr{E}}} : \tilde{H}(S, \mathbb{Q}) & \to & \tilde{H}(M_A(v), \mathbb{Q}) \\ \| & & \| \\ H^*(S, \mathbb{Q}) & \to & H^*(M_A(v), \mathbb{Q}). \\ \Psi & & \Psi \\ t & \longrightarrow & \pi_{M,*}(Z_{\mathscr{E}} \cdot \pi_S^* t) \end{array}$$

$f_{Z_{\mathscr{E}}}$ is a homomorphism of Hodge structures. $f_{Z_{\mathscr{E}}}$ sends v to the fundamental cocycle $w \in H^4(M_A(v), \mathbb{Z})$ (Lemma 4.11) and maps $v^{\perp}$ into $H^0(M_A(v), \mathbb{Q}) \oplus H^2(M_A(v), \mathbb{Q})$. Hence $f_{Z_{\mathscr{E}}}$ induces the homomorphism $\varphi_{\mathbb{Q}} = (v^{\perp} \otimes \mathbb{Q})/\mathbb{Q} v \to H^2(M_A(v), \mathbb{Q})$.

THEOREM 1.5. *Assume that v is an isotropic vector and that $M_A(v)$ is nonempty and compact. Then we have*

1) *$\varphi_{\mathbb{Q}}$ does not depend on the choice of a quasi-universal family $\mathscr{E}$ on $S \times M_A(v)$,*

2) *$\varphi_{\mathbb{Q}}$ is an isomorphism of Hodge structures and compatible with the bilinear forms on $(v^{\perp} \otimes \mathbb{Q})/\mathbb{Q}v$ and $H^2(M_A(v), \mathbb{Q})$, and*

3) *$\varphi_{\mathbb{Q}}$ is defined over $\mathbb{Z}$, i.e., $\varphi_{\mathbb{Q}}(v^{\perp}/\mathbb{Z}v) = H^2(M_A(v), \mathbb{Z})$.*

If $\mathscr{E}$ is a universal family (i.e., $\sigma(\mathscr{E}) = 1$), then $Z_{\mathscr{E}}$ is integral and $f_{Z_{\mathscr{E}}}$ gives an Hodge isometry of between $\tilde{H}(S, \mathbb{Z})$ and $\tilde{H}(M, \mathbb{Z})$ (Theorem 4.9).

REMARK 1.6. The relation between the periods of a variety X and the moduli space of bundles on X was studied in the case X is a curve in [16] : Let M be the moduli space of stable rank 2 bundles with a fixed determinant ξ. If deg ξ is odd, then M is compact and the two polarized Hodge structures $H^1(C, \mathbb{Z})$ and $H^3(M, \mathbb{Z})$ are isomorphic and the isomorphism is given by using the Chern class of a universal family on $C \times M$. (Since the weights are odd, in this case, the polarization is not symmetric but skew symmetric).

The following is a natural analogue of the notion of isogeny of abelian surfaces.

DEFINITION 1.7. An algebraic cycle $Z \in H^4(S \times S', \mathbb{Q})$ on a product of two $K3$ surfaces S and S' is an *isogeny*, if the homomorphism $f_z : H^2(S, \mathbb{Q}) \to H^2(S', \mathbb{Q}), t \to \pi_{S',*}(Z.\pi_S^* t)$, is an isometry, *i.e.* an isomorphism compatible with cup product pairings.

f_Z is an isometry if and only if so is the homomorphism $f'_Z : H^2(S', \mathbb{Q}) \to H^2(S, \mathbb{Q}),\ t' \to \pi_{S,*}(Z, \pi_{S'}^* t')$ because f_Z and f'_Z are adjoint to each other with respect to the cup product pairings. In fact, we have $(t' \cdot f_Z(t)) = (\pi_{S'}^* t' \cdot Z \cdot \pi_S^* t) = (f'_Z(t) \cdot t)$ for every $t \in H^2(S, \mathbb{Q})$ and $t' \in H^2(S', \mathbb{Q})$.

DEFINITION 1.8. Two $K3$ surfaces S and S' are *isogenous* if there exists an isogeny $Z \in H^4(S \times S', \mathbb{Q})$ on $S \times S'$.

Let N_S be the Néron-Severi group of S. N_S is canonically isomorphic to $H^{1,1}(S, \mathbb{Z})$ and is a primitive sublattice of $H^2(S, \mathbb{Z})$. The orthogonal complement T_S of N_S is called the *transcendental lattice* of S. Every cohomology class in N_S is of type (1,1) and any cohomology class in T_S is not so. $H^2(S, \mathbb{Z})$ contains $N_S \perp T_S$ as a sublattice of a finite index and $H^2(S, \mathbb{Q})$ is isomorphic to $(N_S \otimes \mathbb{Q}) \perp (T_S \otimes \mathbb{Q})$. Hence the cohomology group $H^4(S \times S', \mathbb{Q})$ is the direct sum of 4 vector spaces $N_S \otimes N_{S'} \otimes \mathbb{Q}$, $N_S \otimes T_{S'} \otimes \mathbb{Q}$, $T_S \otimes N_{S'} \otimes \mathbb{Q}$ and $T_S \otimes T_{S'} \otimes \mathbb{Q}$. Neither $N_S \otimes T_{S'} \otimes \mathbb{Q}$ nor $T_S \otimes N_{S'} \otimes \mathbb{Q}$ contains a cohomology class of type (2, 2). Hence if $Z \in H^4(S \times S', \mathbb{Q})$ is a Hodge cycle, then Z is the sum of $Z_\nu \in N_S \otimes N_{S'} \otimes \mathbb{Q}$ and $Z_\tau \in T_S \otimes T_{S'} \otimes \mathbb{Q}$. Z_ν is always an algebraic cycle. Hence a Hodge cycle Z is algebraic if and only if so is Z_τ. Z_τ induces the homomorphism $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$. In particular, S and S' are isogeneous if and only if there exists an algebraic cycle Z on $S \times S'$ such that $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$ is an isometry. By Theorem 1.5, $Z_{\mathscr{E}}$ is an isometry and S and $M_A(v)$ are isogeneous. As an application of this fact, we have

THEOREM 1.9. *Let S and S' be algebraic $K3$ surfaces and $Z \in H^4(S \times S', \mathbb{Q})$ a Hodge cycle on $S \times S'$. Assume that $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$ is an isometry and that the lattice $T = T_S \cap (f^\tau{}_Z)^{-1} T_{S'}$ can be primitively embedded into a $K3$ lattice Λ. Then Z is an algebraic cycle.*

If $\rho(S) \geqq 11$, then rank $T \leqq 11$ and T can be primitively embedded into Λ by Corollary 1.12.3 in [17]. Hence we have

COROLLARY 1.10. *If $\rho(S) \geqq 11$ and if $f_Z^\tau : T_S \otimes \mathbb{Q} \to$*

$T_{S'} \otimes \mathbb{Q}$ is an isometry, then the Hodge cycle Z is algebraic.

REMARK 1.11. By the corollary, two $K3$ surfaces S and S' with $\rho \geq 11$ are isogenous if and only if the Hodge structures T_S and $T_{S'}$ are so. This partially answers to the question posed in [21]. For $K3$ surfaces with $\rho = 20$, this has been proved by Shioda-Inose [22]. Moreover, Inose [4] has proved that if T_S and $T_{S'}$ are isogenous for such two $K3$ surfaces S and S', then there exist rational maps of finite degree from S to S' and from S' to S.

In [10], Morrison has proved that if T_S has a primitive embedding $T_S \hookrightarrow U^{\perp 3}$, then there exist an abelian surface A and a certain algebraic correspondence on $S \times A$ which induces $T_S \cong T_A$. By this result and the above corollary, we have

THEOREM 1.12. *Let S be an algebraic $K3$ surface. If $T_S \otimes \mathbb{Q}$ can be embedded into $(U \otimes \mathbb{Q})^{\perp 3}$ as a lattice, then there exists an algebraic cycle on $S \times A$ which induces an isometry between $T_S \otimes \mathbb{Q}$ and $T_A \otimes \mathbb{Q}$*

This was conjectured in [10] by modifying Oda's conjecture in [19].

NOTATION *A $K3$ surface* always means a minimal algebraic $K3$ surface over $\mathbb{C}$, throughout this article. For a complex manifold X over $\mathbb{C}$, $H^*(X, \mathbb{Z})$ is the cohomology ring of X. The even (resp. odd) part of $H^*(X, \mathbb{Z})$ is denoted by $H^{ev}(X, \mathbb{Z})$ (resp. $H^{odd}(X, \mathbb{Z})$. $*$ is the involution of $H^{ev}(X, \mathbb{Z})$ which is +1 on $\oplus_n H^{4n}(X)$ and −1 on $\oplus_n H^{4n+2}(X)$.

A *sheaf* on X is a choerent $\mathcal{O}_X$-module. $h^i(E)$ is the dimension of the cohomology group $H^i(X, E)$ and $\chi(E)$ is the alternating sum $\Sigma(-1)^i h^i(E)$. For an ample line bundle A and a nontorsion sheaf E, the rational number $(c_1(E) \cdot A^{\dim X - 1})/r(E)$ is called the *slope* of E with respect to A and denoted by $\mu_A(E)$. A torsion free sheaf E is μ-stable (resp. *μ-semi-stable*) with respect ot A, if $\mu_A(F) > \mu_A(E)$ (resp. $\mu_A(F) \geq \mu_A(E)$) for every proper nontorsion quotient sheaf F of E. The set of isomorphism classes of all μ-stable (resp. μ-semi-stable) sheaves on X is denoted by M_X^μ (resp. SM_X^μ). M_X^μ is an open subset of the moduli space M_X of stable (in Gieseker's sense) sheaves on X. For a sheaf E on X, $E^\vee$ denotes the dual sheaf $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$. $ch(E) \in H^{ev}(X, \mathbb{Q})$ is the Chern character of E. If E is locally free, then we have $ch(E^\vee) = ch(E)^*$.

A *lattice* over a ring R is a free R-module L with a symmetric bilinear form $(\ \cdot\ ) : L \times L \to R$ and a *lattice* means a lattice over $\mathbb{Z}$. A sublattice L_0 of L is *primitive* if L/L_0 has no torsion and a vector v of L is *primitive* if $\mathbb{Z}v$ is a primitive sublattice. An isomorphism $f : L \xrightarrow{\sim} L'$ between two lattices L and L' is an *isometry* if f is compatible with the bilinear forms on L and L

For an algebraic variety X, the Néron-Severi group N_X is the Picard group Pic (X) modulo algebraic equivalence. The Picard number $\rho(X)$ is the rank of N_X. If S is a $K3$ surface, then the natural map Pic $(S) \to N_S$ is a bijection. For $\ell \in N_S$, we denote by $\mathcal{O}_S(\ell)$ the line bundle corresponding to ℓ.

§ 2. Generalities

In this section, we assume that S is an abelian or $K3$ surface. The Todd class td_S of S is equal to $1 + 2\epsilon w$, where $1 \in H^0(S, \mathbb{Z})$ is the unit element of the cohomology ring $H^*(S, \mathbb{Z})$, $w \in H^4(S, \mathbb{Z})$ is the fundamental cocycle of S and ϵ is equal to 0 or 1 according as S in abelian or of type $K3$. The positive square root $\sqrt{td_S} = 1 + \epsilon w$ lies in the even part $H^{ev}(S, \mathbb{Z})$ of $H^*(S, \mathbb{Z})$. Let E be a sheaf on S. Then the Chern character $ch(E)$ belongs to $H^{ev}(S, Z)$.

DEFINITION 2.1. For a sheaf E, we put $v(E) = ch(E)\cdot \sqrt{td_S} \in H^{ev}(S, \mathbb{Z})$ and call it the *vector associated* to E.

We define a symmetric integral bilinear form (.) on $H^{ev}(S, \mathbb{Z})$ by

$$(u.u') = \alpha \cup \alpha' - r \cup s' - s \cup r' \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}$$

for every $u = (r, \alpha, s)$ and $u' = (r', \alpha', s') \in H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. We denote $H^{ev}(S, \mathbb{Z})$ with this inner product (.) by

$\tilde{H}(S, \mathbb{Z})$. $\tilde{H}(S, \mathbb{Z})$ is an even lattice of rank $8(1+2\epsilon)$ and isomorphic to $U^{14} \perp E_8^{12\epsilon}$ as an abstract lattice. The inner product $(u \,.\, u')$ is equal to the $H^4(S, \mathbb{Z})$-component of $-u^{*}\cdot u \in H^{ev}(S, \mathbb{Z})$. Hence, for sheaves E and F on S, $(v(E)\cdot v(F))$ is equal to the $H^4(S)$-component of $-ch(E)^{*}\cdot ch(F)\cdot td_S$. Therefore, by the Riemann-Roch theorem, we have

PROPOSITION 2.2. *Let E and F be sheaves on S and put* $\chi(E, F) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}_S}(E, F)$. *Then we have* $\chi(E, F) = -(v(E) \,.\, v(F))$.

PROOF. If E is locally free, then $\mathrm{Ext}^i_{\mathcal{O}_S}(E, F)$ is canonically isomorphic to $H^i(S, E^{\vee}\otimes F)$ for every i and $-ch(E)^{*}\cdot ch(F)\cdot td_S$ is equal to $-ch(E \otimes F)\cdot td_S$. Hence our assertion follows from the usual Riemann-Roch theorem. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then $\chi(E, F)$ and $(v(E). v(F))$ are equal to $\chi(E', F) + \chi(E'', F)$ and $(v(E')\cdot v(F)) + (v(E'')\cdot v(F))$, respectively. Since E has a resolution by locally free sheaves, we have our assertion for every sheaves E and F. q.e.d.

The dualizing sheaf ω_S of S is trivial. Hence the Serre duality is simple in form and is a very effective tool of our study.

PROPOSITION 2.3. *Let E and F be sheaves on S. Then the pairing* $\mathrm{Ext}^i_{\mathcal{O}_S}(E, F) \times \mathrm{Ext}^{2-i}_{\mathcal{O}_S}(F, E) \longrightarrow H^2(\mathcal{O}_S), (\alpha, \beta) \longrightarrow \mathrm{tr}^2(\alpha\circ\beta)$

is nondegenerate for every i, where $tr^2 : Ext^2_{\mathcal{O}_S}(F, F) \to H^2(\mathcal{O}_S)$ *is the trace homomorphism of* $Ext^2_{\mathcal{O}_S}(F, F)$. *In particular we have* $\dim Ext^2_{\mathcal{O}_S}(E, F) = \dim \mathrm{Hom}_{\mathcal{O}_S}(F, E)$ *and* $\dim \mathrm{Ext}^1(E, F) = \dim \mathrm{Ext}^1_{\mathcal{O}_S}(F, E)$.

PROOF. The usual Serre duality says that the natural pairing $H^i(S, G) \times \mathrm{Ext}^{2-i}_{\mathcal{O}_S}(G, \omega_S) \to H^2(S, \omega_S)$ is nondegenerate for every sheaf G on S. In the case where E is locally free, applying this Serre duality for $G = E^\vee \otimes F$, we have our proposition. In the general case, take locally free resolutions $0 \to E^m \to E^{m-1} \to \ldots \to E^0 \to E \to 0$ and $0 \to F^n \to F^{n-1} \to \ldots \to F^0 \to F \to 0$ of E and F, and apply the Serre duality for $\mathrm{Hom}^{\cdot}_{\mathcal{O}_S}(E^{\cdot}, F^{\cdot})$ in the derived category $D(S)$ of S ([3]), where $E^{\cdot} = [0 \to E^m \to E^{m-1} \to \ldots \to E^0 \to 0]$ and $F^{\cdot} = [0 \to F^n \to F^{n-1} \to \ldots \to F^0 \to 0]$. Then we have our proposition. q.e.d.

In the special case where $E = F$, the Serre pairing is a nondegenerate bilinear form on $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ which we call the Serre bilinear form. This form is skew symmetric.

By Proposition 2.2 and 2.3, we have

PROPOSITION 2.4. $(v(E).\ v(F)) = \dim\ \mathrm{Ext}^1_{\mathcal{O}_S}(E, F)$ $\dim \mathrm{Hom}_{\mathcal{O}_S}(E, F) - \dim \mathrm{Hom}_{\mathcal{O}_S}(F, E)$.

COROLLARY 2.5. $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = (v(E)^2) + 2 \dim \mathrm{End}_{\mathcal{O}_S}(E)$ *for every sheaf E on S. In particular,* $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ *is always an even integer. If E is simple, then* $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = (v(E)^2 + 2$ *and hence* $(v(E)^2) \geqq -2$.

The tangent space of Spl_S (or M_A) at the point $[E] \in \mathrm{Spl}_S$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$. Since Spl_S is smooth ([12]), we have

COROLLARY 2.6. *Let v be a vector of* $\tilde{H}(S, \mathbb{Z})$. *Then every component of* $\mathrm{Spl}_S(v)$ *is smooth and has dimension* $(v^2) + 2$.

Next we prove some inequalities for $(v(E)^2)$ and $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ which play an important role for our study of sheaves on S.

PROPOSITION 2.7. *Let* $X : 0 \to F \xrightarrow{f} E \xrightarrow{g} G \to 0$ *be an exact sequence of sheaves on S such that* $\mathrm{Hom}_{\mathcal{O}_S}(F, G) = 0$. *Define* $i : \mathrm{Ext}^1_{\mathcal{O}_S}(G, F) \to \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ and $j : \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \to \mathrm{Ext}^1_{\mathcal{O}_S}(F, G)$ by $i(\alpha) = f \circ \alpha \circ g$ *and* $j(\beta) = g \circ \beta \circ f$. *Let I be the image of i and J the kernel of j. Then we have*

(1) $I \subset J$ *and the quotient J/I is isomorphic to* $\mathrm{Ext}^1_{\mathcal{O}_S}(F, E) \oplus \mathrm{Ext}^1_{\mathcal{O}_S}(G, G)$,

(2) *Let* $e \in \mathrm{Ext}^1_{\mathcal{O}_S}(G, F)$ *be the extension class of X and define the homomorphism* $h : \mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G) \to$ $\mathrm{End}_{\mathcal{O}_S}(G, F)$ by $h(e_F, e_G) = e_F \circ e - e \circ e_G$. *Then the sequence*

(2.7.1) $0 \to \mathrm{End}_{\mathcal{O}_S}(E) \to \mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G) \xrightarrow{h}$
$\mathrm{Ext}^1_{\mathcal{O}_S}(G, F) \xrightarrow{i} \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$

is exact (Since $\mathrm{Hom}_{\mathcal{O}_S}(F, G) = 0$, *every endomorphism of E preserves X and induces endomorphisms of F and G.), and*

(3) *J is the orthogonal complement* $I^{\perp}$ *of I with respect to the Serre bilinear form on* $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ *and I is totally isotropic.*

PROOF. (1) since $g \circ f = 0$, $j \circ i = 0$ and J contains I. We show that J/I is isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_S}(F, F) \oplus \mathrm{Ext}^1_{\mathcal{O}_S}(G, G)$. If $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ belongs to J, then $(g \circ \alpha) \circ f = 0$. Hence there exists $\alpha_G \in \mathrm{Ext}^1_{\mathcal{O}_S}(G, G)$ such that $g \circ \alpha = \alpha_G \circ g$. Since $\mathrm{Hom}_{\mathcal{O}_S}(F, G) = 0$, such an α_G is unique. In a similar way, there exists a unique $\alpha_F \in \mathrm{Ext}^1_{\mathcal{O}_S}(F, F)$ such that $\alpha \circ f = f \circ \alpha_F$. It is easy to see that the map $\varphi : J \to \mathrm{Ext}^1_{\mathcal{O}_S}(F, F) \oplus \mathrm{Ext}^1_{\mathcal{O}_S}(G, G)$, $\alpha \mapsto (\alpha_F, \alpha_G)$ is a homomorphism.

CLAIM: $\mathrm{Ker}\, \varphi = I$.

If $\alpha \in I$, then $g \circ \alpha = \alpha \circ f = 0$. Hence $\alpha_F = \alpha_G = 0$ and I is contained in $\mathrm{Ker}\, \varphi$. Assume that α belongs to $\mathrm{Ker}\, \varphi$. Then we have $\alpha \circ f = g \circ \alpha = 0$. Hence there exists $\beta \in \mathrm{Ext}^1_{\mathcal{O}_S}(E, F)$ such that $\alpha = f \circ \beta$. Since $f \circ (\beta \circ f) = \alpha \circ f = 0$ and since $\mathrm{Ext}^1_{\mathcal{O}_S}(F, F) \xrightarrow{f \circ *} \mathrm{Ext}^1_{\mathcal{O}_S}(F, E)$ is injective, we have $\beta \circ f = 0$. Hence $\beta = \gamma \circ g$ for some $g \in \mathrm{Ext}^1_{\mathcal{O}_S}(F, G)$. Therefore, α is equal to $f \circ \gamma \circ g$ and belongs to I.

CLAIM : φ is surjective.

By the Serre duality and by our assumption, we have $\mathrm{Ext}^2_{\mathcal{O}_S}(G, F) = 0$. Hence the homomorphism $\mathrm{Ext}^1_{\mathcal{O}_S}(E, F) \xrightarrow{*\circ f} \mathrm{Ext}^1_{\mathcal{O}_S}(F, F)$ is surjective. Therefore, for every $\alpha_F \in \mathrm{Ext}^1_{\mathcal{O}_S}(F, F)$, there exists $\beta \in \mathrm{Ext}^1_{\mathcal{O}_S}(E, F)$ such that $\alpha_F = \beta \circ f$. Put $\alpha = f \circ \beta \in \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$. Then it is easy to see that $\varphi(\alpha) = (\alpha_F, 0)$. In a similar way, for every $\alpha_G \in \mathrm{Ext}^1_{\mathcal{O}_S}(G, G)$, we obtain $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ such that $\varphi(\alpha) = (0, \alpha_G)$. Hence φ is surjective.

(2) If $h(e_F, e_G) = 0$, then $e_F \circ e = e \circ e_G$ which means that two endomorphisms e_F and e_G of F and G are compatible with respect to the extension class of X. Hence there exists an endomorphism of E which induces e_F and e_G. Therefore, the sequence (2.7.1) is exact at $\mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G)$. Since $f \circ e = e \circ g = 0$, we have $h \circ i = 0$. Assume that $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_S}(G, F)$ and $i(\alpha) = 0$, i. e., $f \circ (\alpha \circ g) = 0$. Then there exists $\beta \in \mathrm{Hom}_{\mathcal{O}_S}(E, G)$ such that $\alpha \circ g = e \circ \beta$. Since $\mathrm{Hom}_{\mathcal{O}_S}(F, G) = 0$, there exists an endomorphism γ_G of G such that $\beta = \gamma_G \circ g$. Since $(\alpha - e \circ \gamma_G) \circ g = 0$, there exists an endomorphism γ_F of F such that $\alpha - e \circ \gamma_G = \gamma_F \circ e$. Therefore, α lies in the image of h and the sequence (2.7.1) is exact at $\mathrm{Ext}^1_{\mathcal{O}_S}(G, F)$.

(3) Since ω_S is trivial, the homomorphisms i and j are dual to each other by the Serre duality. Hence I and $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)/J$ are dual to each other. If $\alpha \in I$ and $\beta \in J$, then $\alpha \circ \beta \in \mathrm{Ext}^2_{\mathcal{O}_S}(E, E)$ is zero. Hence I and F are perpendicular with respect to the Serre bilinear form on $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$. Since the Serre bilinear form is nondegenerate, J coincides with $I^{\perp}$. q.e.d.

COROLLARY 2. 8. ([11]) *Let X be same as above. Then we have* $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(F, F) + \dim \mathrm{Ext}^1_{\mathcal{O}_S}(G, G) \leq \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$.

REMARK 2.9. If S is a surface and $|-K_S| \neq \phi$, then $\mathrm{Hom}_{\mathcal{O}_S}(F, G) = 0$ implies $\mathrm{Ext}^2_{\mathcal{O}_S}(G, F) = 0$. Hence (1) and (2) of the proposition and the corollary are true for such surfaces (1) of the proposition says that every infinitesimal deformation of F and G can be lifted to an infinitesimal deformation of E.

The following proposition and its proof are quite similar to above ones. In fact, these propositions are equivalent if one consider them in the derived category $D(S)$ of S.

PROPOSITION 2. 10. *Let* $X : 0 \to E \xrightarrow{g} G \xrightarrow{f} F \to 0$ *be an exact sequence of sheaves on S such that* $\mathrm{Ext}^1_{\mathcal{O}_S}(F, G) = 0$. *Let* $f \in \mathrm{Ext}^1_{\mathcal{O}_S}(F. E)$ *be the extension class of X. Define* $i : \mathrm{Hom}_{\mathcal{O}_S}(G, F) \to \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ *and* $j : \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \to \mathrm{Ext}^2_{\mathcal{O}_S}(F, G)$ *by* $i(\alpha) = f \circ \alpha \circ g$ *and* $j(\beta) = g \circ \beta \circ f$. *Let I be the image of i and J the kernel of j. Then we have (1) and (3) in Proposition 2.7 and*

(2) *define the homomorphism* $h : \mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G) \to \mathrm{Hom}_{\mathcal{O}_S}(G, F)$ *by* $h(e_F, e_G) = e_F \circ e - e \circ e_G$ *for* $e_F \in \mathrm{End}_{\mathcal{O}_S}(F)$ *and* $e_G \in \mathrm{End}_{\mathcal{O}_S}(G)$. *Every endomorphism of E is induced by that of G and the sequence*

$$0 \to \mathrm{End}_{\mathcal{O}_S}(E) \to \mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G) \xrightarrow{h} \mathrm{Ext}^1_{\mathcal{O}_S}(G, F)^{i} \to \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$$

is exact. In particular, if $I = 0$, then h is surjective.

PROOF. By the Serre duality, we have $\mathrm{Ext}^1_{\mathcal{O}_S}(G, F) = 0$. (1) and (3) can be proved in a similar way to Proposition 2.7. Since $\mathrm{Ext}^1_{\mathcal{O}_S}(F, G) = 0$, the map $\mathrm{End}_{\mathcal{O}_S}(G) \to \mathrm{Hom}_{\mathcal{O}_S}(E, G)$ is surjective. Hence every endomorphism of E is a restriction of an endomorphism of G. Hence the homomorphism $\mathrm{End}_{\mathcal{O}_S}(E) \to \mathrm{End}_{\mathcal{O}_S}(F) \oplus \mathrm{End}_{\mathcal{O}_S}(G)$ is well defined. The exactness of the sequence can be proved in a similar way to Proposition 2.7. q.e.d.

COROLLARY 2. 11. *Let X be same as above. Then we have*

$$\dim \mathrm{Ext}^1_{\mathcal{O}_S}(F, F) + \dim \mathrm{Ext}^1_{\mathcal{O}_S}(G, G) \leq \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E).$$

Let E be a torsion free sheaf and $\tilde{E}$ the double dual of E. Then the natural homomorphism $E \to \tilde{E}$ is injective and the cokernel M is of finite length. We have the exact sequence

$$0 \to E \to \tilde{E} \xrightarrow{s} M \to 0.$$

Since $\tilde{E}$ is locally free, we have $\mathrm{Ext}^1_{\mathcal{O}_S}(M, \tilde{E}) \cong \mathrm{Ext}^1_{\mathcal{O}_S}(\tilde{E}, M)^{\vee} = 0$. Since $(v(M)^2) = 0$, $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ is equal to $2 \dim \mathrm{End}_{\mathcal{O}_S}(E)$ by Corollary 2.5. Hence we have

COROLLARY 2. 12. *Let E be a torsion free sheaf on S and $\tilde{E}$ and M be as above. Then we have*

$$\dim \mathrm{Ext}^1_{\mathcal{O}_S}(\tilde{E}, \tilde{E}) + 2 \dim \mathrm{End}_{\mathcal{O}_S}(M) \leq \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E).$$

If equality holds in the above relation, then the natural homomorphism $\mathrm{End}_{\mathcal{O}_S}(\tilde{E}) \oplus \mathrm{End}_{\mathcal{O}_S}(M) \to \mathrm{Hom}_{\mathcal{O}_S}(\tilde{E}, M)$, $(\alpha, \beta) \mapsto e\circ\alpha - \beta\circ e$, *is surjective.*

LEMMA 2.13. *Let $(R, \mathfrak{m})$ be a local ring and M an artinian R-module. Then we have* length $(\mathrm{End}_R(M)) \geq$ length (M). *If equality holds, then M is isomorphic to R/I for an ideal I of R.*

PROOF. We prove by induction on length. (M). Let M_0 be the submodule $\{x \in M;\ \mathfrak{m}x = 0\}$ of M. Every endomorphism of M maps M_0 into itself. Hence we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, M_0) \to \mathrm{End}_R(M) \to \mathrm{End}_R(M/M_0) \to 0.$$

Since M is artinian, M_0 is nonzero. Hence by induction hypothesis, we have length $(\mathrm{End}_R(M/M_0)) \geq \mathrm{length}(M/M_0)$. Since $\mathfrak{m}M_0 = 0$, every homomorphism from M to M_0 factors through $M/\mathfrak{m}M$. Hence $\mathrm{Hom}_R(M, M_0)$ is isomorphic to the vector space $\mathrm{Hom}_{R/\mathfrak{m}}(M/\mathfrak{m}M, M_0)$. Therefore, we have

$$\text{length}(\mathrm{End}_R(M)) =$$
$$\text{length}(\mathrm{End}_R(M/M_0)) + \text{length}(\mathrm{Hom}_R(M, M_0))$$

$$\geq \text{length}(M/M_0) + \text{length}(M/\mathfrak{m}M)\,\text{length}(M_0)$$

$$\geq \text{length}(M)$$

which shows the first half of the lemma. If equalities hold in the above relations, then we have length $(\text{End}_R(M/M_0))$ = length(M/M_0) and length$(M/\mathfrak{m}M) = 1$. By the latter equality and the Nakayama's lemma, M is generated by one element. Hence M is isomorphic to R/I for an ideal I. q.e.d.

By Corollary 2.12 and the above lemma, we have

PROPOSITION 2.14. *Let E be a torsion free sheaf on S, $\tilde{E}$ the double dual of E and $M = \tilde{E}/E$. Then we have*

$$\dim \text{Ext}^1_{\mathcal{O}_S}(\tilde{E}, \tilde{E}) + 2\,\text{length}(M) \leq \dim \text{Ext}^1_{\mathcal{O}_S}(E, E).$$

If equality holds, then the natural map $\text{End}_{\mathcal{O}_S}(\tilde{E}) \oplus \text{End}_{\mathcal{O}_S}(M) \to \text{Hom}_{\mathcal{O}_S}(\tilde{E}, M)$ is surjective and *M is isomorphic to $\mathcal{O}_S/\mathcal{J}$ for an ideal $\mathcal{J}$ of $\mathcal{O}_S$.*

REMARK 2.15. Since $\tilde{E}$ is locally free, $\text{Ext}^1_{\mathcal{O}_S}(\tilde{E}, M) = \text{Ext}^1_{\mathcal{O}_S}(M, \tilde{E}) = 0$ for any surface S. Hence Corollary 2.12 and the above proposition are true for any (smooth) surface.

Let $0 \to F \to E \to G \to 0$ be an exact sequence of nontorsion sheaves on S. Since $v(E) = v(F) + v(G)$ and $r(E) = r(F) + r(G)$, we have

$$\frac{(v(F)^2)}{r(F)}+\frac{(v(E)^2)}{r(G)}-\frac{(v(E)^2)}{r(E)}=\frac{r(F)r(G)}{r(E)}\left(\frac{v(F)}{r(F)}\quad\frac{v(G)}{r(G)}\right)^2 .$$

Since $\dfrac{v(F)}{r(F)}-\dfrac{v(G)}{r(G)}=\left(0,\ \dfrac{c_1(F)}{r(F)}-\dfrac{c_1(G)}{r(G)},\ \dfrac{s(F)}{r(F)}-\dfrac{s(G)}{r(G)}\right)$,

the right hand side of the above equality is equal to

$\dfrac{r(F)r(G)}{r(E)}\left(\dfrac{c_1(F)}{r(F)}-\dfrac{c_1(G)}{r(G)}\right)^2$. Hence we have

PROPOSITION 2.16. *Let* $0\to F\to E\to G\to 0$ *be an exact sequence of nontorsion sheaves. Then we have*

$$\frac{(v(F)^2)}{r(F)}+\frac{(v(G)^2)}{r(G)}-\frac{(v(E)^2)}{r(E)}=\frac{r(F)r(G)}{r(E)}\left(\frac{c_1(F)}{r(F)}-\frac{c_1(G)}{r(G)}\right)^2$$

If $\rho(S)=1$, then the right hand side is always nonnegative because we are assuming that S is algebraic. Hence we have

COROLLARY 2.17. *If (S is algebraic and)* $\rho(S)=1$, *then* $\dfrac{(v(F)^2)}{r(F)}+\dfrac{(v(G)^2)}{r(G)}\geq\dfrac{(v(E)^2)}{r(E)}$. *Here equality holds if and only if* $c_1(F)/r(F)=c_1(G)/r(G)$.

If F and G have the same slope with respect to an ample line bundle A, i.e., $\mu_A(F)=\mu_A(G)$, then we have

$\left(A.\dfrac{c_1(F)}{r(F)}-\dfrac{c_1(G)}{r(G)}\right)=0$. Hence, by the Hodge index theorem $\left(\dfrac{c_1(F)}{r(F)}-\dfrac{c_1(G)}{r(G)}\right)^2$ is always nonpositive and is equal to zero if and only if $c_1(F)/r(F)=c_1(G)/r(G)$. Hence we have

COROLLARY 2.18. *Assume that F and G have the same slope with respect to an ample line bundle. Then we have*

$$\frac{(v(F)^2)}{r(F)} + \frac{(v(G)^2)}{r(G)} \leq \frac{(v(E)^2)}{r(E)}.$$

and equality holds if and only if $c_1(F)/r(F) = c_1(G)/r(G)$.

Let E be a μ-semi-stable sheaf. Then there is a filtration

$$E_* : 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that every successive quotient $F_i = E_i/E_{i-1}$ is μ-stable and has the same slope as E. Such a filtration E_* is called a μ-*JHS* filtration of E. Applying the above corollary repeatedly for this filtration, we have the following:

PROPOSITION 2.19. *Let E be a μ-semi-stable sheaf and F_i $(1 \leq i \leq n)$ the successive quotients of a μ-JHS filtration of E. Then we have*

$$\sum_{i=1}^{n} \frac{(v(F_i)^2)}{r(F_i)} \leq \frac{(v(E)^2)}{r(E)}$$

Equality holds if and only if $c_1(F_i)/r(F_i)$ is equal to $c_1(E)/r(E)$ for every $1 \leq i \leq n$.

REMARK 2.20 If E is a semi-stable sheaf. Then there is a filtration

$$0 = E_0 \subset E_1 \subset \ldots . \subset E_n = E$$

such that F_i is stable, has the same slope as E and $s(F_i)/r(F_i) = s(E)/r(E)$ for every $i = 1, \ldots, n$. Such a filtration is called a *JHS* filtration of E. The above propsotion is also true for a semi-stable sheaf E and its *JHS* filtration.

Now we assume that S is a $K3$ surface and prove a result which we shall need in §5. Let F be a sheaf on S which satisfies

(2.21) the canonical homomorphism $f : H^0(S, F) \otimes \mathcal{O}_S \rightarrow F$ is injective and $H^2(S, F) = 0$.

We construct a sheaf E on S from F, which we call the *reflection* of E (from the left), such that $r(E) = -s(F)$, $c_1(E) = c_1(F)$ and $s(E) = -r(F)$. We show that E is simple if and only if F is so. This result is a very special case of the theory of the reflection functor of S, which we will discuss systematically in [14].

Let $\bar{F}$ be the cokernel of the canonical homomorphism f : $H^0(S, F) \otimes \mathcal{O}_S \rightarrow F$. We have the exact sequence

(2.22) $$0 \rightarrow H^0(S, F) \otimes \mathcal{O}_S \xrightarrow{f} F \rightarrow \bar{F} \rightarrow 0.$$

Since $H^1(S, \mathcal{O}_S) = H^2(S, F) = 0$, the above sequence induces the exact sequence

$$(2.23)\ 0 \to H^1(S, F) \overset{\alpha}{\to} H^1(S, \bar{F}) \to H^0(S, F) \underset{k}{\otimes} H^2(S, \mathcal{O}_S) \to 0.$$

$$H^0(S, F)'$$

Construct an exact sequence

$$(2.24) \qquad 0 \to \bar{F} \to E \to H^1(S, F) \to H^1(S, F) \otimes \mathcal{O}_S \to 0$$

so that the coboundary map $\delta : H^1(S, F) \otimes H^0(S, \mathcal{O}_S) \to H^1(S, \bar{F})$ is equal to α. We call this extension E of $H^1(S, F) \otimes \mathcal{O}_S$ by $\bar{F}$ the reflection of F (from the left). Since $H^2(S, F) = 0$ by our assumption, $\chi(F)$ is equal to $h^0(F) - h^1(F)$. Hence we have.

$$\begin{aligned} v(E) &= v(F) + h^1(F)v(\mathcal{O}_S) \\ &= v(F) - h^0(F)v(\mathcal{O}_S) + h^1(F)v(\mathcal{O}_S) \\ &= v(F) - \chi(F)v(\mathcal{O}_S). \end{aligned}$$

Since $\chi(F) = r(F) + s(F)$ and $v(\mathcal{O}_S) = (1, 0, 1)$, we have $r(E) = -s(F)$, $c_1(E) = c_1(F)$ and $s(E) = -r(F)$. (By our assumption, $\chi(F) \leq h^0(F) \leq r(F)$. Hence $s(F)$ is nonpositive.)

PROPOSITION 2.25. *Assume that F satisfies* (2.21)*and let E be the reflection of F. Then we have* $\mathrm{End}_{\mathcal{O}_S}(E) \cong \mathrm{End}_{\mathcal{O}_S}(F)$.

PROOF. We have constructed E canonically from F. It is almost clear that every endomorphism of F induces an endomorphism of E. Let φ be an endomorphism of E. We show that φ is induced by an endomorphism of F. Since $\mathrm{Hom}_{\mathcal{O}_S}(F, \mathcal{O}_S) = 0$ by our assumption and the Serre duality, we have $\mathrm{Hom}_{\mathcal{O}_S}(F, \mathcal{O}_S) = 0$. Hence φ preserves the exact sequence (2.24) and induces an endomorphism $\bar{\psi}$ of $\bar{F}$ and f_1 of $H^1(S, F)$. Since $\bar{\psi}$ and f_1 are induced by φ, the following diagram

$$\begin{array}{ccc} H^1(S, F) \otimes H^0(S, \mathcal{O}_S) & \xrightarrow{\delta = \alpha} & H^1(S, \bar{F}) \\ \downarrow f_1 & & \downarrow H^1(\bar{\psi}) \\ H^1(S, F) \otimes H^0(S, \mathcal{O}_S) & \xrightarrow[\delta = \alpha]{} & H^1(S, \bar{F}) \end{array}$$

is commutative. Hence f_1 preserves the exact sequence (2.23), and induces an endomorphism f_0 of H^0 (S, F). From the long exact sequence $\mathrm{Ext}^*_{\mathcal{O}_S}((2.22), \mathcal{O}_S)$, we obtain the exact sequence

$$0 \to H^0\ (S, F)^\vee \xrightarrow{\delta'} \mathrm{Ext}^1_{\mathcal{O}_S}(\bar{F}, \mathcal{O}_S) \to \mathrm{Ext}^1_{\mathcal{O}_S}(F, \mathcal{O}_S) \to 0.$$

This sequence is the dual of the exact sequence (2.22) via the Serre duality. Hence we have the following commutative diagram:

$$\begin{array}{ccc} H^0(S, F)^\vee & \xrightarrow{\delta} & \mathrm{Ext}^1_{\mathcal{O}_S}(\bar{F}, \mathcal{O}_S) \\ \downarrow f_0^\vee & & \downarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\bar{\psi}, \mathcal{O}_S) \\ H^0(S, F)^\vee & \xrightarrow{\delta'} & \mathrm{Ext}^1_{\mathcal{O}_S}(\bar{F}, \mathcal{O}_S) \end{array}$$

Therefore, there exists an endomorphism ψ of F which preserves the exact sequenee (2.22) and induces $\bar{\psi}$ on F and f_0 on H^0 (S, F). By our construction, this ψ induces φ.

For our requirements in §4, we show a vanishing of higher direct image sheaf $R^i f_* F$, which was essentially proved in [15].

PROPOSITION 2.26 *Let* $f : X \to Y$ *be a proper morphism of* *noetherian schemes and F a Y-flat coherent $\mathcal{O}_X$-module. Let Z be a closed subscheme which is locally complete intersection in Y. For $y \in Y$, let F_y be the restriction of F to the fibre $X_y = f^{-1}(y)$. Assume that $H^i(X_y, F_y)$* vanishes for *every $i <$* codim *Z and $y \in Y\text{-}Z$. Then $R^i f_*F = 0$ for every $i <$* codim *Z.*

PROOF. We may assume that $Y = \operatorname{Spec} A$ is affine and Z is defined by a regular sequence $x_1, \ldots, x_n \in A$, $n = \operatorname{codim} Z$. By the theorem in §5 [15], there exists a finite complex $K^\bullet$ of finitely generated projective A-modules such that $H^i(K^\bullet) \cong R^i f_* F$. By the base change theorem and by our assumption $R^i f_* F$ has a support on Z for every $i < n$. Hence there exists an integer N such that $\mathfrak{a}^N H^i(K^\cdot) = 0$ for every $i < N$, where $\mathfrak{a} = (x_1, \ldots, x_n) A$. Our proposition follows from the following:

LEMMA. *Let $K_\cdot$ be a finite complex of finitely generated projective A-module and $\mathfrak{a}$ an ideal of A generated by a regular sequence $x_1, \ldots, x_n$ of A. If $\mathfrak{a}^N H^i(K^\cdot) = 0$ for every $i < n$, then $H^i(K^\cdot) = 0$ for every $i < n$.*

This can be proved in the same way as the lemma in ([15] p.127) by using induction on n. q.e.d.

§3. Semi-rigid sheaf

In this section, we shall study sheaves E on a $K3$ surface S

with small $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$.

DEFINITION 3.1. A sheaf E on S is *rigid* if $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = 0$. By Proposition 2.5, we have

PROPOSITION 3.2. *If E is simple, then the following are equivalent:*

(1) *E is rigid,*

(2) *$(v(E)^2) = -2$, and*

(3) *$(v(E)^2) < 0$.*

By Proposition 2.14, we have

PROPOSITION 3.3. *If E is rigid and torsion free, then E is locally free.*

If E is a rigid sheaf and if $v(F) = av(E)$ for a rational number a, then $\chi(E, F)$ is equal to $a\chi(E, E)$ and is positive. Hence we have

PROPOSITION 3.4. *Let E be a rigid sheaf and F a sheaf with $v(F) = av(E)$, $a \in \mathbb{Q}$. Then either* $\mathrm{Hom}_{\mathcal{O}_S}(E, F) \neq 0$ *or* $\mathrm{Hom}_{\mathcal{O}_S}(F, E) \neq 0$.

If E is stable and F is semi-stable and if $v(E) = v(F)$, then every nonzero homomorphism between E and F is an isomorphism. Hence we have

COROLLARY 3.5. *Let E be a stable rigid bundle. If F is semi-stable and $v(F) = v(E)$, then F is isomorphic to E.*

COROLLARY 3.6. *Let v be a vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ with $(v^2) = -2$. Then the moduli space $M_A(v)$ is empty or a reduced one point.*

PROOF. By Corollary 3.5, if $M_A(v)$ is nonempty, then $M_A(v)$ is one point. The tangent space of $M_A(v)$ at the point $[E] \in M_A(v)$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = 0$. Hence $M_A(v)$ is reduced. q.e.d.

dim $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$ is always an even integer (Corollary 2.5). Hence if $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \neq 0$, then dim $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \geq 2$.

DEFINITION 3.7. A simple sheaf E on S is *semi-rigid* if E satisfies the following equivalent conditions:

(1) dim $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = 2$, and

(2) $v(E) \in \tilde{H}^{1,1}(S, \mathbb{Z})$ is isotropic, i.e. $(v(E)^2) = 0$.

Proposition 3.3 is not true for semi-rigid sheaf. In fact, there is a semi-rigid torsion free sheaf which is not locally free. The simplest example is a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$. We can construct many such semi-rigid sheaves from a rigid bundle. Let F be a simple rigid vector bundle of rank r. Take a point $s \in S$ and put $V = F \otimes k(s)$ and $\tilde{F} = F \otimes_k V^\vee$. $\tilde{F}$ is a rigid bundle of rank r^2 and $\tilde{F} \otimes k(s)$ is isomorphic to End(V). Let E be the kernel of the homomorphism $f: \tilde{F} \to \tilde{F} \otimes k(s) \cong \mathrm{End}(V) \xrightarrow{tr} k(s)$ where tr is the trace map of End(V). Every endomorphism of E is induced by an endomorphisn α of $\tilde{F}$. Since α preserves f, α is a constant multiplication and hence E is simple. It is easy to check that $v(E)$ is isotropic. We call this E the *semi-rigid sheaf associated to F*. We have proved the following:

PROPOSITION 3.8. *Let F be a simple rigid bundle of rank r. Then, for every point $s \in S$, there exists a semi-rigid sheaf E of rank r^2 and an exact sequence*

$$0 \to E \to F^{\oplus r} \to k(s) \to 0.$$

The above examples of semi-rigid torsion free sheaves are locally free except at one point. This is true in general. In fact, by Proposition 2.14, we have

PROPOSITION 3.9. *Let E be a torsion free sheaf with* dim $\mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = 2$. *Let $\tilde{E}$ be the double dual of E and assume* *that E is not locally free. Then the quotient $\tilde{E}/E$ is isomorphic to $k(s)$ for a point $s \in S$. Moreover, E is a rigid vector bundle and the natural homomorphism α : $\mathrm{End}_{\mathcal{O}_S}(\tilde{E}) \to \mathrm{Hom}_{\mathcal{O}_S}(\tilde{E}, k(s))$ induced by the exact sequence $0 \to E \to \tilde{E} \to k(s) \to 0$ is surjective.*

COROLLARY 3.10. *Let E be a μ-stable semi-rigid sheaf. If E is not locally free, then $r(E) = 1$ and E is isomorphic to $L \otimes \mathfrak{m}$ for a line bundle L and a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$.*

PROOF. Since E is μ-stable, so is E. Hence E is simple. Since α is surjective and $\dim \mathrm{End}_{\mathcal{O}_S}(\tilde{E}) = 1$, we have $\dim \mathrm{Hom}_{\mathcal{O}_S}(\tilde{E}, k(s) \leq 1$. Therefore, $\tilde{E}$ is a line bundle. q.e.d.

REMARK 3.11. If F is a stable rigid bundle, then the semi-rigid sheaf E associated to F is stable. Hence the above corollary is not true for stable semi-rigid sheaves.

If E is semi-rigid and $v(F) = v(E)$, then $\chi(E, F) = -(v(E).v(F)) = 0$. Hence, if $\mathrm{Hom}_{\mathcal{O}_S}(E, F) = \mathrm{Hom}_{\mathcal{O}_S}(F, E) = 0$, then we have $\mathrm{Ext}^1_{\mathcal{O}_S}(E, F) = 0$.

PROPOSITION 3.12. *Let E be a stable semi-rigid sheaf and F a semi-stable sheaf with $v(F) = (v(E)$. If E is not isomorphic to F, then $\mathrm{Ext}^i_{\mathcal{O}_S}(E, F)$ and $\mathrm{Ext}^i_{\mathcal{O}_S}(F, E)$ vanish for every i.*

PROOF. By the assumption of (semi-) stability of E and F, every homomorphism between E and F is either zero or an isomorphism. Hence, if $E \not\cong F$, then $\mathrm{Hom}_{\mathcal{O}_S}(E, F) = \mathrm{Hom}_{\mathcal{O}_S}(F, E) = 0$. Since $\chi(E, F) = \chi(F, E) = 0$, we have our assertion by Proposition 2.4 q.e.d.

If $M_A(v) \neq \phi$, then $M_A(av)$ is empty for every a $\neq 1$, In fact, we have

PROPOSITION 3.13. *Let E be a stable semi-rigid sheaf and F a simple semi-stable sheaf with $v(F) = av(E)$, $a \in \mathbb{Q}$. Then every nonzero homomorphism between E and F is an isomorphism.*

PROOF. Let $f : E \to F$ be a nonzero homomorphism. Then f is injective and the cokernel of F is semi-stable by our assumption on (semi-)stability of E and F.

CLAIM: F is E-potent, i.e., has a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$ such that $F_i/F_{i-1} \cong E$ for every $i = 1, \ldots, n$.

We define $F_1 = \mathrm{Im}(f)$ and F_i inductively for $i \geqq 2$. Assume that F_i has been defined and $F_i \neq F$. Let G_i be the quotient F/F_i, Since E is simple, $\mathrm{Hom}_{\mathcal{O}_S}(G_i, E) = 0$. Since $G_i \neq 0$ and F is simple, the exact sequence $0 \to F_i \to F \to G_i \to 0$ does not split. Hence $\mathrm{Ext}^1_{\mathcal{O}_S}(G_i, F_i) \neq 0$. Since F_i is E-potent and $\mathrm{Ext}^1_{\mathcal{O}_S}$

is an additive functor, we have $\mathrm{Ext}^1_{\mathcal{O}_S}(G_i, E) \neq 0$. Since $\chi(G_i, E) = -(v(G_i).\ v(E)) = (i-a)\,(v(E)^2) = 0$, we have dim $\mathrm{Hom}_{\mathcal{O}_S}(E, G_i) = \dim \mathrm{Ext}^1_{\mathcal{O}_S}(G_i, E) - \dim \mathrm{Hom}_{\mathcal{O}_S}(G_i, E) > 0$. Hence there exists a nonzero homomorphism $f_i : E \to G_i$. Let F_{i+1} be the pull-back of $\mathrm{Im}(f_i)$ by $F \twoheadrightarrow G_i$. Since G_i is semi-stable, f_i is injective and F_{i+1}/F_i is isomorphic to E. So F_{i+1} is well defined.

If $g : F \to E$ is a nonzero homomorphism, then g is surjective. By the same argument, we have our claim in this case. Since F is simple, F is isomorphic to E by the above and f and g are isomorphisms. q.e.d.

Next we investigate the stability of semi-rigid sheaves.

PROPOSITION 3.14. *Let S be an algebraic $K3$ surface with Picard number 1 and E a simple torsion free sheaf on S. Assume that E is rigid or semi-rigid and that $v(E)$ is primitive in $\tilde{H}^{1,1}(S, \mathbb{Z})$. Then E is stable.*

PROOF. Since $\rho(S) = 1$ and $v = v(E)$ is primitive, every semi-stable sheaf E' with $v(E') = v$ is stable. Hence it suffices to show that E is semi-stable. Assume that E is not so. Let F_1 be the β-subsheaf of E, *i.e.*, F_1 maximizes the polynomial

$X(F_1(n))/r(F_1)$ among all subsheaves of E and then maximizes $r(F_1)$ among such subsheaves. The quotient $F_2 = E/F_1$ is torsion free and $\mathrm{Hom}_{\mathcal{O}_S}(F_1, F_2) = 0$ by our choice of F_1. Hence, by Corollary 2.8, we have

(*) $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_1, F_1) + \dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_2, F_2) \leq \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E)$.

Since $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) = (v(E)^2) + 2 \leq 2$,
we have $\dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_i, F_i) \leq 2$ for both $i = 1$ and 2. Hence $(v(F_i)^2) = \dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_i, F_i) - 2 \dim \mathrm{End}_{\mathcal{O}_S}(F_i) \leq 0$ for both $i = 1$ and 2. Since $r(F_i) < r(E)$, we have, by Corollary 2.17,

$$(v(F_1)^2) + (v(F_2)^2) \geq (v(E)^2).$$

Hence we have

$$\begin{aligned}
&\dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_1, F_1) + \dim \mathrm{Ext}^1_{\mathcal{O}_S}(F_2, F_2) \\
&\geq \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) + 2 \dim \mathrm{End}_{\mathcal{O}_S}(F_1) \\
&\quad + 2 \dim \mathrm{End}_{\mathcal{O}_S}(F_2) - 2 \\
&> \dim \mathrm{Ext}^1_{\mathcal{O}_S}(E, E),
\end{aligned}$$

which contradicts (*). q.e.d.

REMARK 3.15. If F is a rigid bundle of rank ≥ 2, then the semi-rigid sheaf E associated to F is not μ-stable. Hence, even if $\rho(S) = 1$, it is not always true that simple semi-rigid torsion free sheaf is μ-stable.

In the following two propositions, we consider the case where $c_1(E)$ is ample and study the stability of E with respect to $c_1(E)$.

PROPOSITION 3.16. *Let E be a semi-rigid sheaf with $v(E) = (r, \ell, s)$. Assume that ℓ is ample and E is stable with respect to ℓ. If s is divisible by r and $v(E)$ is primitive, then E is μ-stable with respect to ℓ.*

PROOF. Assume that E is not μ-stable. Then E has a proper quotient sheaf E_1 with $\mu(E_1) = \mu(E)$. We choose E_1 so that $r(E_1)$ is minimum among such quotients. Put $v(E_1) = (r_1, \ell_1, s_1)$. Since $\mu(E_1) = \mu(E)$, we have $(\ell. \ell_1 - r_1\ell/r) = 0$. Since E is semi-rigid, we have $\ell^2 = 2rs$. Therefore, we have

$$
\begin{aligned}
(v(E_1)^2) &= ((\ell_1 - r_1\ell/r) + r_1\ \ell/r)^2 - 2r_1 s_1 \\
&= (\ell_1 - r_1\ \ell/r)^2 + (r_1\ \ell/r)^2 - 2r_1 s_1 \\
&= (\ell_1 - r_1\ \ell/r)^2 + 2r_1\ (r_1 s/r - s_1).
\end{aligned}
$$

Since $v(E)$ is primitive, r and ℓ are coprime. Hence $\ell_1 - r_1\ell/r$ is not zero. Since $(\ell_1 - r_1\ \ell/r.\ell) = 0$ and ℓ is ample, $(\ell_1 - r_1\ \ell/r)^2$

is negative by the Hodge index theorem. On the other hand, since E is stable, the integer $r_1 s/r - s_1$ is negative. Therefore, we have $(v(E_1)^2) < -2r_1 \leq -2$, which contradicts Corollary 2.5 because E_1 is μ-stable and simple by our choice.

q.e.d.

PROPOSITION 3.17 *Let $v = (r, \ell, s)$ be a primitive isotropic vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ and E a sheaf with $v(E) = v$. Assume that ℓ is ample and E is semi-stable but not stable with respect to ℓ. Let*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E, \ n \geq 2$$

be a JHS-filtration of E. Then the successive quotients $F_i = E_i/E_{i-1}$ are rigid for every $i = 1, \ldots, n$.

PROOF. By Proposition 2.19 and Remark 2.20, we have $(v(F_i)^2) \leq 0$ for every i. Since v is primitive, equality is not attained for any i. Hence F_i is rigid by Proposition 3.2. q.e.d.

COROLLARY 3.18. *Let v be as above. Then the complement of $M_\ell(v)$ in the moduli space $\overline{M}_\ell(v)$ of semi-stable sheaves E with $v(E) = v$ is a 0-dimensional set.*

§4. Surface components of the moduli space

Let $v = (r, \ell, s)$ be an isotropic vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ and A an ample line bundle. Then each component of $M_A(v)$ has dimension 2. In this section, we study $M_A(v)$ in the case it is compact and we prove Theorem 1.4 and Theorem 1.5. By Langton's result [6] (see also [9] §5), the moduli space of semi-stable sheaves on S is compact. Hence we have

PROPOSITION 4.1. *$M_A(v)$ is compact if and only if every semi-stable sheaf E with $v(E) = v$ is stable. This is the case, e.g., if the greatest common divisor of r, $(\ell.A)$ and s is equal to* 1.

The above is true for every vector v. Using this proposition, we give some further sufficient conditions for $M_A(v)$ to be compact for given primitive isotropic vector v. Let c be the greatest common divisor of r, $(\ell.m)$ and s, where m runs over all divisor class of S. Then there exists an ample line bundle A such that the greatest common divisor of r, $(A.\ell)$ and s is equal to c. Hence if $c = 1$, then $M_A(v)$ is compact for such an ample line bundle A. For an application in §6, we consider the case $c \geqq 2$. We show that $M_A(v)$ is compact for an ample line bundle A in this case, too. Let N_S be the Néron-Severi group of S. N_S is a sublattice of $H^2(S, \mathbb{Z})$. Let N be the submodule generated by N_S and ℓ/c in $N_S \otimes \mathbb{Q}$. Since $(\ell^2) = 2rs$, (ℓ^2) is divisible by $2c^2$. By the definition of c, the bilinear form on $N_S \otimes \mathbb{Q}$ is integral and even on N. Hence N is an even lattice which contains N_S as a sublattice of index c.

PROPOSITION 4.2. *Let A be an ample line bundle on S such that G.C.D. $(r, (A.\ell), s) = c$. If there are no (-2)-vectors α in N with $(A.\alpha) = 0$, then $M_A(v)$ is compact.*

PROOF. Let E be a semi-stable sheaf with $v(E) = v$ and

$$E_*: 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

be a JHS-filtration of E We show that $n = 1$. Put $F_i = E_i/E_{i-1}$ and $v(F_i) = (r_i, \ell_i, s_i)$ for every $i = 1, \ldots, n$. Since E_* is a JHS-filtration, we have $r_i : (A.\ell_i) : s_i = r : (A.\ell) : s$ for every i. There exists an integer a_i such that $r_i = a_i r/c$, $(A.\ell)/c$ and $s_i = a_i s/c$. Put $m_i = \ell_i - a_i\ell/c \in N$. Then we have $(A.m_i) = 0$ and $(v(F_i)^2) = (m_i + a_i\ell/c)^2 - r_i s_i = (m_i^2) + 2a_i(m_i.\ell)/c$. Since $\sum_{i=1}^{m} m_i = 0$, there exists an i such that $(m_i.\ell) \leq 0$. For this i, we have $(m_i^2) \geq (v(F_i)^2) \geq -2$. Since $(A.m_i) = 0$, (m_i^2) is non-positive by the Hodge index theorem. Hence by our assumption, we have $(m_i^2) = 0$ and $m_i = 0$ by the Hodge index theorem.

Therefore, we have $v(F_i) = a_i v/c$. Since v is primitive, $v(F_i)$ is equal to v. Hence E is stable. q.e.d.

As an application of the above, we have the following proposition:

PROPOSITION 4.3. *Assume that there exists a semi-rigid sheaf E with $v(E) = v$ which is μ-stable with respect to an ample line bundle A'. Then there exists an ample line bundle A such that*

(1) *E is μ-stable with respect to A, and*

(2) *$M_A(v)$ is nonempty and compact.*

PROOF. There exists a neighbourhood U of A' in $\mathbb{P}(N_S \otimes \mathbb{R})$ such that E is μ-stable with respect to A for every ample line bundle $A \in U$. Let $\alpha_1, \ldots, \alpha_n$ be all the (-2)-vectors in N which are perpendicular to A'. If A_1 is an ample line bundle in $U - \bigcup_{i=1}^{n} \alpha_i^{\perp}$ and if A_1 is sufficiently near A', then $(A_1 . \alpha) \neq 0$ for any (-2)-vector α in N. Take such A_1 from $U - \bigcup_{i=1}^{n} \alpha_i^{\perp}$, and take an ample line bundle A_2 such that G.C.D. $(r, (A_2 . \ell), s) = c$. If n is sufficiently large, then $A = ncA_1 + A_2$ belongs to U and satisfies the last assumption of the preceding proposition. There are infinitely many n's such that G.C.D. $(r, (A . \ell), s)$

$= c$. Hence there exists an integer n such that $M_A(v)$ is compact and nonempty.

q.e.d.

If $M_A(v)$ is compact, then $M_A(v)$ is irreducible. In fact, we hav

PROPOSITION 4.4. *Assume that $M_A(v)$ contains a connected component M which is compact and every member of M is locally free. Then we have*

(1) *$M_A(v)$ is irreducible, and*

(2) *every semi-stable sheaf E with $v(E) = v$ is stable.*

PROOF. Since $M_A(v)$ is smooth, M is irreducible. We snow that every semi-stable sheaf F with $v(F) = v$ belongs to M. Let $\mathscr{E}$ be the restriction to $S \times M$ of a quasi-universal family on $S \times M_A(v)$ (see Appendix 2). We consider the functor $\Phi^i(F) = R^i\pi_{M,*}(\mathscr{E}^\vee \otimes \pi_S^* F)$, $i = 0, 1$ and 2, of $\mathcal{O}_S$-module F into the category of $\mathcal{O}_M$-modules. If F is semi-stable, then, for every stable sheaf E with $v(E) = v(F)$, $H^i(S, E^\vee \otimes F) \neq 0$ is equivalent to $F \cong E$. Hence if F is semi-stable and $v(F) = v$, then $\Phi^i(F)$ is supported at most one point. Therefore, by Proposition 2.26, we have $\Phi^0(F) = \Phi^1(F) = 0$. Since $\dim S = 2$, $\Phi^2(F)$ is canonically isomorphic to $H^2(S, E^\vee \otimes F)$ at the poi[n]t $[E]$ of M, that is, $\Phi^2(F) \otimes k([E]) \cong H^2(S, E^\vee \otimes F)$. Hence $\Phi^2(F)$ is nonzero if and only if F is stable and belongs to M. On the other hand, the cohomology class $\alpha(F) = \mathrm{ch}(\Phi^0(F)) -$

$ch\ (\Phi^1\ (F)) + ch\ (\Phi^2\ (F)) \in H^*(M, \mathbb{Q})$ does not depend on F but depends only on $v(F)$ by the Grothendieck-Riemann-Roch theorem. If F belongs to M, the $\alpha\ (F)$ is nonzero. Hence $\alpha\ (F)$ is nonzero for every sheaf F with $v\ (F) = v$. Therefore every semi-stable sheaf F with $v(F) = v$ is stable and belongs to M, which proves (1) and (2). q.e.d.

REMARK 4.5. In the above proposition, the assumption that every member of M is locally free is superfluous. The proof works without this assumption, if one defines that functor Φ^i by $\Phi^i\ (F) = \pi_M \cdot \mathrm{Ext}\ (\mathscr{E}, \pi_S^* F)$, where π_M-Ext(*, *) is the sheaf associated to the presheaf assigning $\mathrm{Ext}_{\mathcal{O}_{S\times U}}\ (*|_{S\times U},\ *|_{S\times U})$ for every open subset U of M.

COROLLARY 4.6. *If every semi-stable sheaf E with $v(E) = v$ is stable, the $M_A\ (v)$ is compact and irreducible.*

We assume that the moduli space $M = M_A\ (v)$ is compact. Since the canonical bundle of M is trivial, ([12], Corollary 0.2), $M_A\ (v)$ is abelian or of type $K3$. We first consider the case where a universal family exists on $S \times M$.

LEMMA 4.7 *For every sheaf $\mathscr{E}$ on $S \times M$, the Chern character $ch(\mathscr{E})$ of E is integral, i.e., belongs to $H^*\ (S \times M,\ \mathbb{Z})$.*

PROOF. Put $ch(\mathscr{E}) = \sum_{i=0}^{4} ch^i\ (\mathscr{E}) \in \bigoplus_{i=0}^{4} H^{2i}(S \times M, \mathbb{Q})$. $ch^1\ (\mathscr{E})$

is the first Chern class $c_1(\mathscr{E})$ of $\mathscr{E}$ and is integral. Since $H^1(S) = 0$, $H^2(S \times M)$ is the direct sum of $H^2(S)$ and $H^2(M)$. Hence $c_1(\mathscr{E})$ is equal to $c_{1,S}(\mathscr{E}) + c_{1,M}(\mathscr{E}) \in H^2(S, \mathbb{Z}) \oplus H^2(M, \mathbb{Z})$. Since both S and M have trivial canonical bundles, both $c_{1,S}(\mathscr{E})^2$ and $c_{1,M}(\mathscr{E})^2$ are even. Hence $\mathrm{ch}^2(\mathscr{E}) = \frac{1}{2} c_1(\mathscr{E})^2 - c_2(\mathscr{E})$ is integral. By the Grothendieck-Riemann-Roch theorem, the $H^*(S) \otimes H^4(M)$-component of $\mathrm{ch}(\mathscr{E})\cdot td_M$ is equal to $(\sum_j (-1)^j \mathrm{ch}(R^j\pi_{S,*}\mathscr{E})) \otimes w$, where $w \in H^4(M)$ is the fundamental cocycle of M. Hence $ch^4(\mathscr{E})$ and the $H^2(S) \otimes H^4(M)$-component of $ch^3(\mathscr{E})$ are integral. Interchanging S and M, we have that the $H^4(S) \otimes H^2(M)$-component of $ch^3(\mathscr{E})$ is also integral. Since $H^6(S \times M)$ is the direct sum of $H^2(S) \oplus H^4(M)$, $ch^3(\mathscr{E})$ is integral. q.e.d.

Let $\mathscr{E}$ be a universal family on $S \times M$. Put $\mathbb{Z} = \pi_S^*\sqrt{td_S}\ ch(\mathscr{E})^* \cdot \pi_M^*\sqrt{td_M}$. By the lemma, $\mathbb{Z}$ belongs to $H^*(S \times M, \mathbb{Z})$. Z defines a homomorphism

$$\begin{array}{rcl} f : H^*(S, \mathbb{Z}) & \longrightarrow & H^*(M, \mathbb{Z}). \\ \cup\!\!\!| & & \cup\!\!\!| \\ \alpha & \longmapsto & \pi_{M,*}(\mathbb{Z}\cdot\pi_S^*\alpha) \end{array} \tag{4.8}$$

THEOREM 4.9. *Under the above situation, we have*

(1) *M is a $K3$ surface,*

(2) *f is an isometry from $\tilde{H}(S, \mathbb{Z})$ onto $\tilde{H}(M, \mathbb{Z})$ with respect to the quadratic forms defined in* (1.1), *and*

(3) *the inverse of f is equal to the homomorphism*

$$\begin{array}{ccc} f' : H^*(M, \mathbb{Z}) & \longrightarrow & H^*(S, \mathbb{Z}) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \beta & \longmapsto & \pi_{S,*}(Z' \cdot \pi_M^* \beta) \end{array}$$

defined by $Z' = \pi_S^* \sqrt{td_S} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td_M}$.

For the proof, the following is essential.

PROPOSITION 4.10. *Let $\mathscr{E}$ be a universal family on $S \times M$. Let π_{12} and π_{13} be the two projections of $S \times M \times M$ onto $S \times M$. Then $\pi_{M \times M} - \mathrm{Ext}^i(\pi_{12}^* \mathscr{E}, \pi_{13}^* \mathscr{E})$ is zero if $i \neq 2$ and $\pi_{M \times M} - \mathrm{Ext}^2(\pi_{12}^* \mathscr{E}, \pi_{13}^* \mathscr{E})$ is supported on the diagonal subscheme Δ of $M \times M$ and is a line bundle on Δ.*

PROOF. If $E, F \in M_A(v)$ and $E \not\cong F$, then $\mathrm{Ext}^i_{\mathcal{O}_S}(E, F) = 0$ for every i by Proposition 3.8. Hence the relative Ext-sheaf $\pi_{M \times M} - \mathrm{Ext}^i(\pi_{12}^* \mathscr{E}, \pi_{13}^* \mathscr{E})$ has a support on Δ. Since Δ is locally complete intersection, the relative Ext-sheaf is zero for both $i = 1$ and 2, by Proposition 2.26. By the base change theorem, $\pi_{M \times M} - \mathrm{Ext}^2(\pi_{12}^* \mathscr{E}, \pi_{13}^* \mathscr{E})$ is canonically isomorphic to the 1-dimensional vector space $\mathrm{Ext}^2_{\mathcal{O}_S}(E, E) \cong \mathrm{End}_{\mathcal{O}_S}(E)^\vee$ at the point $([E], [E]) \in \Delta$. Since M is a moduli space and $\mathscr{E}$ is a universal family, the sheaf $\pi_{M \times M} - \mathrm{Ext}^2(\pi_{12}^* \mathscr{E}, \pi_{13}^* \mathscr{E})$ is annihilated by the ideal $\mathscr{I}_\Delta$ of Δ.

Therefore, $\pi_{M\times M}$ — $\mathrm{Ext}^2(\pi^*_{12}E, \pi^*_{13}E)$ is a line bundle on Δ.

q.e.d.

PROOF OF THEOREM 4.9. : The following is the key to our proof.

CLAIM : The endomorphism $f \circ f'$ of $H^*(M, \mathbb{Z})$ is the identity.

The homomorphisms f and f' are given by cycles Z and Z' on $S \times M$. Using the projection formula, it can be easily shown that $f \circ f'$ is given by the cycle $\tilde{Z} = \pi_{M\times M,*}(\pi^*_{12}Z\cdot\pi^*_{13}Z')$, where π_{12} and π_{13} are same as in the above proposition. Precisely speaking, $(f \circ f')(\beta) = \pi_{1,*}(\tilde{Z}\cdot\pi^*_2\beta)$ for every $\beta \in H^*(M, \mathbb{Z})$, where π_1 and π_2 are two projections of $M \times M$ onto M. By the definition of Z and Z', we have $\tilde{Z} = (\pi_1^*\sqrt{td_M}).(\pi_2^*\sqrt{td_M})\cdot\pi_{M\times M,*}(U)$, where $U = (\pi^*_{12}\; ch(\mathscr{E})^*)/\pi_S^*td_S\cdot(\pi^*_{13}\, ch(\mathscr{E}))$. By the Grothendieck-Riemann-Roch theorem, the cycle $\pi_{M\times M,*}(U)$ is rationally equivalent to $\sum_i(-1)^i \quad ch(\pi_{M\times M}\text{-}\mathrm{Ext}^i(\pi^*_{12}\mathscr{E}, \pi_{13}^*\mathscr{E}))$. By the above proposition, $\tilde{Z}$ is rationally equivalent to $\pi_1^*\sqrt{td_M}.ch(\delta_*L).\pi_2^*\sqrt{td_M}$, where L is a line bundle M and $\delta : M \to M \times M$ is the diagonal embedding. Therefore, $f \circ f'$ is the multiplication by $ch(L) \in H^*(M)$, i.e., $(f \circ f')(\beta) = \beta\cdot ch(L)$ for every $\beta \in H^*(M, \mathbb{Z})$. Let ρ be the factor change of $M \times M$. Then $(1 \times \rho)^*U$ is equal to U^*. Hence, we have $\rho^*(\pi_{M\times M,*}U) = (\pi_{M\times M,*}U)^*$. On the other hand, since $\pi_{M\times M,*}U$ has a support on Δ, we have $\rho^*(\pi_{M\times M,*}U) = \pi_{M\times M,*}U$. Hence we have $ch(\delta_*L)^* = ch(\delta_*L)$. Since S is a $K3$ surface, the line bundle L is trivial. Therefore, $f \circ f'$ is the identity.

By the claim, $H^*(M, \mathbb{Z})$ is a direct summand of $H^*(S, \mathbb{Z})$. Since Z and Z' belong to $H^{ev}(S \times M, \mathbb{Z})$, f and f' preserve the decompositions $H^* = H^{ev} \oplus H^{odd}$ of the cohomology groups $H^*(M, \mathbb{Z})$ and $H^*(S, \mathbb{Z})$. Hence $H^{odd}(M, \mathbb{Z})$ is a direct summand of $H^{odd}(S, \mathbb{Z})$ which is zero, since S is a $K3$ surface. Since M has a trivial canonical bundle, we have, (1). By (1), $H^*(M, \mathbb{Z})$ and $H^*(S, \mathbb{Z})$ have the same rank (= 24). Therefore, f is an isomorphism, which shows (3). Let $\gamma = \gamma_S : S \to \operatorname{Spec} \mathbb{C}$ be the structure morphism of S. Then our inner product $(\alpha . \alpha')$ on $\tilde{H}(S, \mathbb{Z}) = H^*(S, \mathbb{Z})$ is equal to $\gamma_*(\alpha^* \cdot \alpha')$. Hence, by the projection formula, we have

$$(\alpha . f'(\beta)) = \gamma_{S,*}(\alpha^* \cdot \pi_{S,*}(\pi_S^* \sqrt{td_S} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td_M} \cdot \pi_M^* \beta))$$

$$= \gamma_{S,*} \pi_{S,*}(\pi_S^* \alpha^* \cdot \pi_S^* \sqrt{td_S} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td_M} \cdot \pi_M^* \beta)$$

$$= \gamma_{S \times M,*}(\pi_S^* \alpha^* \cdot \pi_M^* \beta \cdot ch(\mathscr{E}) \cdot \sqrt{td_{S \times M}}).$$

for every $\alpha \in H^*((S, Z)$ and $\beta \in H^*(M, Z)$. In a similar way, we have

$$(\beta . f(\alpha)) = \gamma_{S \times M,*}(\pi_M^* \beta^* \cdot \pi_S^* \alpha \cdot ch(\xi)^* \cdot \sqrt{td_{S \times M}}).$$

Therefore $(\alpha . f'(\beta)) = (f(\alpha) . \beta)$ for every $\alpha \in H^*(S. \mathbb{Z})$ and $H^*(M, \mathbb{Z})$, that is, f and f' are adjoint to each other with respect to the inner products (.) on $H^*(S, \mathbb{Z})$ and $H^*(M, \mathbb{Z})$. By (3), $f' \circ f$ is the identity. Hence we have $(f(\alpha) . f(\alpha')) = (\alpha . f'(f(\alpha))) = (\alpha . \alpha')$ for every $\alpha, \alpha' \in H^*(S, \mathbb{Z})$, which proves (2). q.e.d.

Now we assume only that $M = M_A(v)$ is compact and that $\mathscr{E}$ is a quasi-universal family on $S \times M$ and prove Theorem 1. 4 and 1. 5. Let $\sigma(\mathscr{E})$ be the similitude of $\mathscr{E}$ and put $Z = \pi^*_S\sqrt{td_S}\cdot ch(\mathscr{E}).\pi^*_M\sqrt{td_M}/\sigma(\mathscr{E}) \in H^{ev}(S \times M, \mathbb{Q})$. Z induces a homomorphism

$$\begin{array}{ccc} f: H^*(S, \mathbb{Q}) & \longrightarrow & H^{ev}(M, \mathbb{Q}) \\ \Psi & & \Psi \\ \alpha & \longmapsto & \pi_{M,*}(Z.\,\pi^*_S\alpha). \end{array}$$

The $H^0(M, \mathbb{Q})$-component of $f(\alpha)$ is equal to $(v.\alpha)$. Hence the orthogonal complement $v^\perp$ of v in $H^*(S, \mathbb{Q})$ is sent into $H^2(M, \mathbb{Q}) \oplus H^4(M, \mathbb{Q})$ by f.

LEMMA 4.11. *$f(v)$ is equal to the fundamental cocycle $w \in H^4(M, \mathbb{Z})$.*

PROOF. Let F be a member of $M = M_A(v)$ and let $\Phi^2(F)$ be same as in the proof of Proposition 4.4 and Remark 4.5. By the Grothendieck-Riemann-Roch theorem, we have $ch(\Phi^2(F)) = \pi_{M,*}(ch(\mathscr{E})^*.\pi^*_S(ch(F)\cdot td_S)) = \sigma(\mathscr{E})\sqrt{td_M}^{-1}\cdot f(ch(F).\sqrt{td_S}) = \sigma(\mathscr{E})\sqrt{td_M}^{-1} f(v)$. Now $\Phi^2(F)$ has a support at the point $x \in M$ corresponding to F and $\Phi^2(F) \otimes k(x)$ is canonically isomorphic to $\mathrm{Ext}^2_{\mathcal{O}_S}(\mathscr{E}|_{S\times x}, F)$. Since $\mathscr{E}$ is a quasi-universal family, $\mathscr{E}|_{S\times x}$ is isomorphic to $F^{\oplus\sigma(\mathscr{E})}$. Hence $\Phi^2(F) \otimes k(x)$ is a $\sigma(\mathscr{E})$-dimensional vector space. On the other hand, since M is the moduli space and $\mathscr{E}$ is a quasi-universal family, $\Phi^2(F)$ is annihilated by the maximal ideal at x. Hence $\Phi^2(F)$ is isomorphic to $k(x)^{\oplus\sigma(\mathscr{E})}$ and $ch(\Phi^2(F)) = \sigma(\mathscr{E})w$, which proves our lemma. q.e.d.

By this lemma, we see that f induces a homomorphism

$$\varphi_{\mathbb{Q}} : (v^{\perp} \text{ in } H^*(S, \mathbb{Q}))/\mathbb{Q}v \to H^2(M, \mathbb{Q}).$$

Proof of Theorem 1.4 and 1.5 : If $\mathscr{E}$ is a quasi-universal family on $S \times M$, then so is $\mathscr{E} \otimes \pi_M^* V$ for every vector bundle V on M. We first show that the two homomorphisms $\varphi_{\mathbb{Q}}$ and $\varphi_{\mathbb{Q},V}$ for $\mathscr{E}$ and $\mathscr{E} \otimes \pi_M^* V$ are same. The similitude $\sigma(\mathscr{E} \otimes \pi_M^* V)$ is equal to $\sigma(\mathscr{E})\, r(V)$. Hence $ch(\mathscr{E} \otimes \pi_M^* V)/\sigma(\mathscr{E} \otimes \pi_M^* V)$ is equal to $(ch(\mathscr{E})/\sigma(\mathscr{E}))\cdot\pi_M^*(ch(V)/r(V))$. Therefore, we have $f_{\mathbb{Q},V}(\alpha) = f_{\mathbb{Q}}(\alpha)(ch(V)/r(V))$ for every $\alpha \in H^*(S, \mathbb{Q})$. If $(v.\alpha) = 0$, then $H^0(M)$-component of $f_{\mathbb{Q}}(\alpha)$ is zero. Hence the $H^2(M)$ -component of $f_{\mathbb{Q},V}(\alpha)$ is same as that of $f_{\mathbb{Q}}(\alpha)$. Therefore, $\varphi_{\mathbb{Q},V}$ and $\varphi_{\mathbb{Q}}$ are same. If $\mathscr{E}$ and $\mathscr{F}$ are quasi-universal families on $S \times M$, then there exist vector bundles U and V on M such that $\mathscr{E} \otimes \pi_M^* U \cong \mathscr{F} \otimes \pi_M^* V$ (Definition A.4). Hence, by what we have shown, the two homomorphisms $\varphi_{\mathbb{Q}}$ s for $\mathscr{E}$ and $\mathscr{F}$ are same, which shows (1) of Theorem 1.5.

We prove (2) and (3) of Theorem 1.5 by a deformation argument. Both are reduced to the case where a universal family exists on $S \times M$. Let T be the moduli space of $K3$ surfaces S' with isometric markings $i' : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$. Let T_0 be the subspace of T consisting of (S', i')'s for which $i'(c_1(A))$ and $\ell = i'(\ell)$ lie in $H^{1,1}(S')$ and $i'(c_1(A))$ is positive. T_0 contains

(S, id) and has dimension 18 or 19 according as $c_1(A)$ and ℓ are linearly independent or not. Let A' be an ample divisor on S' such that $c_1(A') = i'(c_1(A))$ and put $v' = (r, \ell', s)$. The family of moduli spaces $M_{A'}(v')$ is smooth over an etale covering of T_0 ([12] Theorem 1.17). There exists a family of quasi-universal families $\mathcal{F}_t$ on $S_t \times M_{A_t}(v_t)$, $t \in T_0$, which is flat over an etale covering of T_0. By Proposition 4.1, the compactness of $M_{A'}(V')$ is an open condition : There exists an open neighbourhood U of (S, id) such that $M_{A'}(v')$ is compact for every $(S', i') \in U$. On the other hand the set of (S', i') which satisfy

(*) there exists a divisor class $m \in H^{1,1}(S; \mathbb{Z})$ such that G.C.D. $(r, (\ell.m), s) = 1$

is dense in T_0. By Theorem A. 6 and Remark A. 7, for such S', there exists a universal family on $S \times M_{A'}(v')$. Hence there exists a pair (S', i') for which $M' = M_{A'}(v')$ is compact and a universal family $\mathcal{E}'$ exists on $S \times M'$. By Theorem 4.9, M' is a K3 surface and (2) and (3) of Theorem 1.5 are true for this S' and $\mathcal{E}$. Hence M is a $K3$ surface and (2) and (3) of Theorem 1.5 are true for this S' and for every quasi-universal family $\mathcal{F}$ on $S \times M'$. Since (S, id) and $\mathcal{F}$ is a flat deformation of (S', i') and $\mathcal{F}'$, (2) and (3) are also true for S. The second half of Theorem 1.4 follows from (2) and (3) of Theorem 1.5. q.e.d.

§5. Existence of simple μ-semi-stable semi-rigid sheaves

In this section, we show the existence of simple μ-semi-stables sheaves E with $v(E) = v$ for primitive isotropic vectors v of $\tilde{H}^{1,1}(S, \mathbb{Z})$.

THEOREM 5.1. *Let $v = (r, \ell, s)$ be a primitive isotropic vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ of rank $r \geq 1$ and A an arbitrary ample divisor. Then there exists a simple μ-semi-stable sheaf E with $v(E) = v$, i.e. $SM_A(v)$ is nonempty.*

By virtue of Theorem A.1, this theorem is equivalent to the following stronger version :

THEOREM 5.2. *Let m be a divisor class of S. Then the simple μ-semi-stable sheaf E can be chosen so that E satisfies the following condition :*

(*) $(c_1(F).m)/r(F) \geq (c_1(E).m)/r(E)$ holds for every non-torsion quotient sheaf F of E with $\mu(F) = \mu(E)$.

In fact, if $n >> 0$, then $nA + m$ is ample. By Theorem 5.1, there exists a simple sheaf E_n with $v(E_n) = v$ and which is μ-semi-stable with respect to $A + \frac{1}{n}m$. By Theorem A.1, there exists a simple sheaf E which is μ-semi-stable with respect to infinitely many $A + \frac{1}{n}m$. It is easy to see that this E satisfies (*) in Theorem 5.2. We prove these theorems by induction on r. In the case $r = 1$, $E = \mathcal{O}_S(\ell) \otimes \mathfrak{m}$ satisfies our requirement for a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$. In fact, $v(E) = v$ and E is μ-stable with respect to any ample line bundle. Assume that Theorem 5.2 is true in the case of rank $< r$. Under this assumption, we shall show that Theorem 5.1 is true for every v of rank r.

Step I. Assume that $-r < s < 0$ and $(\ell.A) = 0$. Then there exists a simple μ-semi-stable sheaf E with $v(E) = v$.

PROOF. By the induction hypothesis, there exists a simple μ-semi-stable sheaf F with $v(F) = (-s, \ell, -r)$. Since $\mu(F) = 0$, the canonical homomorphism $f : H^0(S, F) \otimes \mathcal{O}_S \to F$ is injective and for every nonzero homomorphism $g : F \to \mathcal{O}_S$, the cokernel of g is of finite length. Here we apply Theorem 5.2, putting $m = -\ell$. Then we can take F so that

$$-(c_1(G).\ell)/r(G) \geq -(\ell^2)/r(F)$$

holds for every nontorsion quotient G of F with $\mu(G) = \mu(F)$. Since $(\ell^2) = 2rs < 0$, $(c_1(G).\ell)$ is negative. Hence, for this F, we have $\mathrm{Hom}_{\mathcal{O}_S}(F, \mathcal{O}_S) = 0$. Therefore, by the Serre duality, $H^2(S, F) = 0$ and F satisfies (2.21). Let E be the reflection of F (see §2). Then $v(E) = v$ and there is an exact sequence

$$0 \to H^0(S, F) \otimes \mathcal{O}_S \hookrightarrow F \to E \to H^1(S, F) \otimes \mathcal{O}_S \to 0.$$

Since F is μ-semi-stable and $\mu(F) = \mu(\mathcal{O}_S)$, the cokernel of f is torsion free and μ-semi-stable. Hence E is torsion free and μ-semi-stable. By Proposition 2.25, E is simple.

q.e.d.

We do not use the full strength of the above step but only the existence of simple torsion free sheaves on monogonal $K3$ surfaces. A quasi-polarized $K3$ surface (S,A) is called *monogonal* if there exists a smooth elliptic curve C on S with $(A.C) = 1$. Put $g = \frac{1}{2}(A^2) + 1$. Then $(A - gC)^2 = -2$ and $(C. A. - gC) = 1$. Hence there exists an effective divisor D such that $D \sim A - gC$.

If $\rho(S) = 2$, then Pic S is generated by C and D and D is a smooth rational curve. S is a double cover of the $\mathbb{P}^1$-bundle $\mathbb{F}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbb{P}^1$. A divisor $aC + b(C + D)$ on S is ample if and only if $a > b > 0$.

Step. II. *Assume that S is monogonal and $\rho(S) = 2$. Then there exists a simple torsion free sheaf E on S with $v(E) = v$.*

PROOF. ℓ is equal to $aC + b(C + D)$ for some integers a and b. Take an integer b' so that $b' \equiv b \bmod r$ and $|b'| \leqq r/2$. Then take an integer a' congruent to a modulo r so that $r/2 < |a'| \leqq 3r/2$ and $a'b' < 0$ if $b' \neq 0$ and so that $-r < a' \leqq 0$ if $b' = 0$. Put $\ell' = a'C + b'(C + D)$. ℓ' is congruent to ℓ modulo r and $s' = (\ell'^2)/2r$ is an integer. We show the existence of a simple torsion free sheaf E' on S with $v(E') = (r, \ell', s')$. Then $E = E' \otimes \mathcal{O}_S((\ell - \ell')/r)$ is a simple torsion free sheaf and satisfies $v(E) = v$. If $b' \neq 0$, then $-3r^2/4 \leqq a'b' < 0$ by our choice of a' and b'. Since $(\ell'^2) = 2a'b'$, we have $-3r/4 \leqq s' < 0$. Put $H = a'C - b'(C + D)$. If $b' \neq 0$, then H or $-H$ is ample. Since $(H.\ell') = 0$, there exists a simple torsion free sheaf E' with $v(E') = (r, \ell', s')$ by Step I. If $b' = 0$, then $s' = 0$. Since v' is primitive, r and a' are coprime. Hence there exists a simple vector bundle ξ on the elliptic curve C of rank $-a'$ and degree r by [1] (see also §2 [18]). ξ is generated by global sections and $H^1(C, \xi) = 0$ (see Lemma 5.3 below). We regard ξ as a sheaf on S supported by C. Let E' be the kernel of the natural homomorphism $\varphi : H^0(S, \xi) \otimes \mathcal{O}_S \to \xi$. Then φ is surjective and E' is a vector bundle. Since $\dim H^0(S, \xi) = \dim H^0(C, \xi) = r$, the rank of E' is equal to r. Since ξ is a simple sheaf and since

$H^1(S, \xi) = 0$, E' is simple. (Every endomorphism of E comes from that of ξ.) q.e.d.

LEMMA 5.3. *Let E be an indecomposable vector bundle of rank r and degree d on an elliptic curve C. If $d > r$, then E is generated by global sections and $H^1(C, E) = 0$.*

PROOF. Let h be the greatest common divisor of r and d. Then E has a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_h = E$$

such that E_i/E_{i-1} is indecomposable and has rank r/h and degree d/h for every $i = 1, 2, \ldots, h$. Hence we may assume that r and d are coprime. Then, by Lemma 2.2 [1], E is simple. Let d'/r' be the greatest irreducible fraction with $d'/r' < d/r$ and $0 < r' < r$. There exists a simple vector bundle E' on C with rank r' and degree d'. Since $r'd - rd' = 1$, we have $\chi(E', E) = 1$ by the Riemann-Roch theorem. Applying Part II [1] for $E'^{\vee} \otimes E$, we have $\mathrm{Ext}^1_{\mathcal{O}_C}(E', E) = 0$ and $\dim \mathrm{Hom}_{\mathcal{O}_C}(E', E) = 1$.

Since E' and E are stable, the canonical homomorphism $\varphi: E' \otimes \mathrm{Hom}_{\mathcal{O}_C}(E', E) \to E$ is injective and the cokernel E'' has no torsion. Since $\mathrm{Hom}_{\mathcal{O}_C}(E'', E') = 0$, we have $\mathrm{Ext}^1_{\mathcal{O}_C}(E', E'') = 0$ by the Serre duality. Hence every endomorphism of E'' is induced by that of E. Therefore, E'' is simple. So we have obtained an exact sequence of simple vector bundles

$$0 \to E' \to E \to E'' \to 0$$

Case for which $d'/r' > 1$: By the induction hypothesis, our assertion is true for E' and E''. Hence so is for E.

Case for which $d'/r' = 1$: By our choice of d'/r', we have $r' = d' = 1$ and $d = r + 1$. E' is a line bundle of degree 1 and isomorphic to $\mathcal{O}_C(p)$ for a point p on C. By the induction hypothesis, E'' is generated by global sections. Hence E is generated by global sections except at p. Let L be the kernel of the canonical homomorphism $\psi : H^0(C, E) \otimes \mathcal{O}_C \to E$. Since ψ is generically surjective and since $h^0(C, E) = r(E) + 1$, L is a line bundle. If dim $\mathrm{Hom}_{\mathcal{O}_C}(L, \mathcal{O}_C) < h^0(C, E)$, then E would be decomposable. Hence we have $h^0(C, L^{-1}) \geqq d = r + 1$. By the Riemann-Roch theorem we have deg $L \leqq -d$ and deg (Image ψ) $\geqq d$. Hence ψ is surjective.

q.e.d.

Next we study the case where A is primitive and ℓ is a multiple of A, say $\ell = kA$ for an integer k. In this case, the moduli space $M_{S,A}(v)$ is defined for every polarized $K3$ surface (S, A) of a fixed degree, say $d = (A^2)$. Let F_d (resp. $\overline{F}_d$) be the moduli space of polarized (resp. quasi-polarized) $K3$ surfaces (S, A) of degree d. By the Torelli theorem([7], [20]), F_d and F_d are irreducible.

Step III. There is a nonempty open subset U of F_d such that $M_{S,A}$ is nonempty for every polarized $K3$ surface $(S, A) \in U$.

PROOF. If (S, A) is monogonal and $\rho(S) = 2$, then there exists simple torsion free sheaf E with $v(E) = v$. Since $\{Spl_S(v)\}_{(S,A)\in F_d}$ is a smooth family over an etale covering of F_d (Theorem 1.17 [12]), there exists a simple torsion free sheaf E' on S' with $v(E') = (r, kA', s)$ for every small deformation (S', A') of (S, A). The polarized $K3$ surfaces (S', A') with $\rho(S') = 1$ form a dense subset in F_d. Hence there exists a polarized $K3$ surface (S', A') with $\rho(S') = 1$ and a simple torsion free sheaf E' on S' with $v(E') = (r, kA', s)$. Since $(v(E')^2) = 0$ and $\rho(S') = 1$, E' is stable, by virtue of Proposition 3.14. Since $\{M_{S,A}(v)\}_{(S,A)\in F_d}$ is a smooth family over an etale covering of F_d, there exists an open neighbourhood U of (S', A') which satisfies our requirement. q.e.d.

Step IV. If ℓ is a multiple of A, then there exists a sheaf E with $v(E) = v$ and which is stable with respect to A, *i.e.*, $M_{S,A}(v)$ is nonempty for every (S, A).

PROOF. By Langton's theorem ([6] see also [9] §5), the family $\{\overline{M}_{S,A}(v)\}_{(S,A)\in F_d}$ of the moduli spaces of semi-stable sheaves is proper over F_d. By Step III, $\overline{M}_{S,A}(v)$ is nonempty over a dense open subset of F_d. Therefore $\overline{M}_{S,A}(v)$ is nonempty for every $(S, A) \in F_d$. Let $\pi : \mathscr{S} \to F$ be a family of polarized K3 surfaces. Then, by Maruyama [9] § 4, the (coarse) moduli space $\Pi : \overline{M}_{\mathscr{S}/F} \to F$ of semi-stable sheaves on $\mathscr{S}/F$ exists and each fibre of Π is canonically isomorphic to the moduli space of semi-stable sheaves on the corresponding fibre of π. In particular, the function $F_d \ni (S, A) \mapsto \dim \overline{M}_{S,A}(v)$ is upper semi-continuous. Since $\dim \overline{M}_{S,A}(v) \geq \dim \overline{M}_{S,A}(v) = 2$ for every member (S, A) of U in Step II, we have $\dim \overline{M}_{S,A}(v) \geq 2$ for every polarized K3 surface (S, A). By Proposition 3.14, the complement of $M_{S,A}(v)$ in $\overline{M}_{S,A}(v)$ is discrete. Hence $M_{S,A}(v)$ is nonempty for every $(S, A) \in F_d$. q.e.d.

Now we return to the general case.

Step V. *There exists a simple sheaf E with $v(E) = v$ and which is μ-semi-stable with respect to A.*

PROOF. If a sheaf E is stable with respect to A, then $E \otimes L$ is simple and μ-stable with respect to A for every line bundle L. Hence, by Step IV, our assertion is true if $\ell \equiv kA \mod r$ for an integer k. In particular, $SM^{\mu}_{rnA+\ell}(r, \ell, s)$ is nonempty for every $n >> 0$. Since the sequence $\{A + \ell/rn\}$ of $\mathbb{Q}$-divisors converges to A, we have, by Theorem A. 1, $SM^{\mu}_{A}(r, \ell, s)$ is nonempty. q.e.d.

We have completed the proof of Theorem 5.1 and Theorem 5.2. By Step IV, we have also proved the following.

THEOREM 5.4. *Let $v = (r, \ell, s)$ be a primitive isotropic vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ and assume that ℓ is ample. Then there exists a sheaf E with $v(E) = v$ and stable with respect to ℓ, i.e., $M_{\ell}(r, \ell, s) \neq \phi$.*

§6. Application to the Hodge conjecture

In this section, we apply the results in § § 4 and 5 to show that certain Hodge cycles Z on a product $S \times S'$ of two algebraic $K3$ surfaces S and S' are algebraic (Theorem 1.9). We first consider the special case for which $T_{S'} \cong \varphi(T_S)$, where $\varphi = f^{\tau}_{Z}$ as in Theorem 1.9.

Step I. *Let $\varphi : T_S \overset{\sim}{\to} T_{S'}$ be a Hodge isometry between the transcendental lattices of S and S'. Then there exists an algebraic cycle $W \in H^4(S \times S', \mathbb{Q})$ on $S \times S'$ such that $\varphi(\alpha) = \pi_{S',*}(W \cdot \pi_S^* \alpha)$ for every $\alpha \in T_S$.*

We remark that there exists an isomorphism $f : S' \to S$ such that $f^* = \varphi$ on T_S if $\rho(S) > 11$ (proposition 6.2). But this is not true in general if $\rho(S) \leq 11$. In fact, there is a pair of $K3$ surfaces S and S' such that $T_S \cong T_{S'}$ but $N_S \not\cong N_{S'}$ as lattices. We note that two lattices $\tilde{H}^{1,1}(S, \mathbb{Z})$ and $\tilde{H}^{1,1}(S', \mathbb{Z})$ are isomorphic to each other, which is the key of our proof of Step I. More strongly, by Theorem 1.14.2 and 1.14.4 in [17], we have

PROPOSITION 6.1 *Let $\varphi_1, \varphi_2 : T \to H$ be two primitive embeddings of a lattice T into an even unimodular lattice H. Assume that the orthogonal complement N of $\varphi_1(T)$ in H satisfies one of the following :*

(1) *N contains the hyperbolic lattice $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as a sublattice or*

(2) *N is indefinite and* rank $N \geq$ rank $T + 2$.

Then φ_1 and φ_2 are equivalent, i.e., there exists an isometry γ of H such that $\varphi_1 = \gamma \circ \varphi_2$.

We give a proof of the fact remarked above, which is a prototype of our proof of Step I.

PROPOSITION 6.2. *Let S and S' be algebraic K3 surfaces and $\varphi : T_S \to T_{S'}$ be a Hodge isometry. If $\rho(S) > 11$, then there exists an isomorphism $f : S \to S$ such that $f^* = \varphi$ on T_S.*

For the proof, we need a version of Torelli theorem of K3 surfaces :

PROPOSITION 6.3. *Let S and S' be K3 surfaces and $\psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ be a Hodge isometry. Then there exists an isomorphism $f : S' \to S$ such that $f^* = \psi$ on T_S.*

PROOF. By the strong Torelli theorem ([7]), there exists an isomorphism $f : S' \to S$ and reflections r_i $(i = 1, \ldots, n)$ by (-2)- curves $C_i \cong \mathbb{P}^1$ on S such that $\psi = f^* \circ r_1 \circ \cdots \circ r_n$. Since $[C_i]$ is perpendicular to T_S, r_i is identity on T_S for every $i = 1, \ldots, n$. Hence we have our proposition. q.e.d.

Proof of Proposition 6.2 : Apply (2) of Proposition 6.1 to two primitive embeddings $T_S \hookrightarrow H^2(S, \mathbb{Z})$ and $T_{S'} \hookrightarrow H^2(S \;\; \mathbb{Z})$. Since $H^2(S, \mathbb{Z})$ and $H^2(S', \mathbb{Z})$ are isomorphic to each other as lattices, we obtain an isometry $\tilde{\varphi} : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$

such that $\tilde{\varphi} | T_S = \varphi$. By the above proposition, there exists an isomorphism $f : S' \to S$ such that $f^* = \tilde{\varphi}$ which proves our proposition. q.e.d.

Proof of Step I : The orthogonal complement of T_S in the extended $K3$ lattice $\tilde{H}(S, \mathbb{Z})$ is isomorphic to $N_S \perp U$. Applying (1) of Proposition 6.1 to the embedding of T_S and $T_{S'}$ into $\tilde{H}(S, \mathbb{Z})$ and $\tilde{H}(S', \mathbb{Z})$, we see that there exists an isometry $\Phi : \tilde{H}(S', \mathbb{Z}) \to \tilde{H}(S, \mathbb{Z})$ such that $\Phi| T_S = \varphi$. Put $v = \Phi(0, 0, 1) = (r, \ell, s)$ and $u = \Phi(1, 0, 0) = (p, k, q)$. Φ maps $\tilde{H}^{1,1}(S')$ onto $\tilde{H}^{1,1}(S)$. Hence both ℓ and k are divisor classes on S. Let m be a divisor class on S. The Chern character e^m of the line bundle $\mathcal{O}_S(m)$ is a unit of the cohomology ring $H^*(S, \mathbb{Z})$. Hence the multiplication by e^m induces a Hodges isometry Φ_m, $\Phi_m(r, \ell, s) = (r, \ell+rm, s+(m.\ell)+\frac{r}{2}(m^2))$ of the extended $K3$ lattice $\tilde{H}(S, \mathbb{Z}) \cong H^*(S, \mathbb{Z})$. Replacing Φ by $\Phi_m \circ \Phi$ for a sufficiently ample divisor m, we can choose Φ so that s is positive. Since the change of r and s is an Hodge isometry, we choose Φ so that r is positive. Since $(u.v) = -1$, the greatest common divisor r, $(\ell.k)$ and s is equal to 1.

CLAIM: There exists an integer n such that r and $s + n(\ell.k)$ are coprime.

Let d be the greatest common divisor of s and $(\ell.k)$. Since s/d and $(\ell.k)/d$ are coprime, there exists an integer n such that r and $s/d + n(\ell.k)/d$ are coprime. Since r and d are coprime, so are r and $s + n(\ell.k)$.

Take n as in the claim and replace Φ by $\Phi_{nk} \circ \Phi$. Then, by the claim, r and s are coprime. Replace Φ by $\Phi_{rA} \circ \Phi$ again for a sufficiently ample divisor A. Then r and s are still coprime and ℓ become ample. Let M be the moduli space $M_\ell(v)$ of sheaves E with $v(E) = v$ which is stable with respect to ℓ. By Theorem 5.4. M is nonempty. Since r and s are coprime, every semi-stable sheaf is stable. Hence M is compact and hence irreducible by Corollary 4.6. By Theorem A.6 and Remark A.7, there exists a universal family $\mathscr{E}$ on $S \times M$. By Theorem 4.9, the cycle $Z = \pi_S^* \sqrt{td_S} \cdot ch(\mathscr{E}) \quad \pi_M^* \sqrt{td_M}$ induces a Hodge isometry $\psi : \tilde{H}(M, \mathbb{Z}) \to \tilde{H}(S, \mathbb{Z})$, with $\Psi(\delta) = v$, where $\delta = (0, 0, 1)$. $\Phi^{-1} \circ \Psi$ is an isometry and sends δ to δ. Hence $\Phi^{-1} \circ \Psi$ induces a Hodge isometry from $H^2(M, \mathbb{Z}) = \Psi^{-1}(v^\perp / \mathbb{Z}v)$ onto $H^2(S', \mathbb{Z}) = \Phi^{-1}(v^\perp / \mathbb{Z}v)$.

By Proposition 6.3, there exists an isomorphism $f : S' \to M$ such that $f^* : H^2(M, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ coincides with $\Phi^{-1} \circ \Psi$ on T_M. Then the Chern character $ch((1 \times f)^* \mathscr{E}) \in H^*(S \times S', \mathbb{Z})$ of $(1 \times f)^* E$ induces a Hodge isometry $\Psi' : \tilde{H}(S', \mathbb{Z}) \to \tilde{H}(S, \mathbb{Z})$ which coincides with Φ (or equivalently φ) on $T_{S'}$. The $H^4(S \times S')$-component W of Z induces a homomorphism τ of the Hodge structure $H^2(S', \mathbb{Z})$ to $H^2(S' \mathbb{Z})$. τ maps $T_{S'}$ onto T_S and coincides with φ on $T_{S'}$. q.e.d.

Let $v = (r, \ell, s)$ be a primitive isotropic vector of $\tilde{H}^{1,1}(S, \mathbb{Z})$ and assume that the moduli space $M = M_A(v)$ of stable sheaves E with $v(E) = v$ is nonempty and compact. Then, by Theorem 1.5, there exists an algebraic cycle Z on $S \times M$ defined by using the Chern character of a quasi-universal family and Z induces a Hodge isometry $\varphi : v^{\perp}/\mathbb{Z}v \to H^2(M, \mathbb{Z})$. The transcendental lattice T_S (regarded as a sublattice of $\tilde{H}(S, \mathbb{Z})$) is perpendicular to v and $T_S \cap \mathbb{Z}v = 0$. Hence $v^{\perp}/\mathbb{Z}v$ contains a sublattice isomorphic to T_S and φ induces a Hodge isometry $\varphi : T_S \to T_M$. φ is injective but not surjective in general.

PROPOSITION 6.4. *Let $v = (r, \ell, s)$, M and φ be as above. Let $n = n(v)$ be the minimum of $|(u.v)|$, where u runs over all vectors of $\tilde{H}^{1,1}(S, \mathbb{Z})$ with $(u.v) \neq 0$. Then we have*

(1) *the cokernel of φ is a cyclic group of order n,*

(2) *there exists a transcendental cycle $\lambda \in T_S$ such that $\ell + \lambda \in H^2(S, \mathbb{Z})$ is divisible by n, and*

(3) *if λ satisfies (2), then $\varphi(\lambda) \in T_M$ is divisible by n and $\varphi(\lambda)/n$ generates the cokernel of φ.*

PROOF. For every $v \in H^{1,1}(S, \mathbb{Z})$, $(u.v)/n$ is an integer. Since $\tilde{H}(S, \mathbb{Z})$ is unimodular and $\tilde{H}^{1,1}(S, \mathbb{Z})$ is a primitive sublattice, there exists $w \in \tilde{H}(S, \mathbb{Z})$ such that $(u.v)/n = (w.v)$ for every $v \in \tilde{H}^{1,1}(S, \mathbb{Z})$. $\lambda = nw - u \in \tilde{H}(S, \mathbb{Z})$ is perpendicular to $\tilde{H}^{1,1}(S, \mathbb{Z})$ and hence lies in T_S. It is clear that λ satisfies (2). Assume that λ satisfies (2). Then $w = (\lambda+v)/n$ lies in $v^{\perp}$ and nw is congruent to λ modulo $\mathbb{Z}v$. Hence $\varphi(\lambda)/n$ lies in T_M. We show that $\varphi(\lambda)/n$ generates the cokernel of φ. The transcendental lattice T_M is isomorphic to $(v^{\perp} \cap \tilde{H}^{1,1}(S, \mathbb{Z}))^{\perp}/\mathbb{Z}v \cong (\mathbb{Q}v \oplus T_S \otimes \mathbb{Q}) \cap \tilde{H}(S, \mathbb{Z})/\mathbb{Z}v$. Let α be a vector of $(\mathbb{Q}v \oplus T_S \otimes \mathbb{Q}) \cap \tilde{H}(S, \mathbb{Z})$. Then $\alpha = av + \nu$ for $a \in \mathbb{Q}$

and $v \in T_S \otimes \mathbb{Q}$. Take a vector $u \in \tilde{H}^{1,1}(S, \mathbb{Z})$ such that $(u.v) = n$. Then we have $an = a(u.v) = (\alpha.v) \in \mathbb{Z}$. Since $v = nw - \lambda$, we have $\alpha = (an)w + (v - a\lambda)$. Since an is an integer, $v - a\lambda$ lies in T_S and α is congruent to $(an)w$ modulo T_S. Hence $\varphi(\lambda)/n$ generates the cokernel of φ, which shows (3). If $m\varphi(\lambda)/n$ lies in T_S, then mw lies in $T_S + \mathbb{Z}v$ and is equal to $\lambda' + bv$ for $\lambda' \in T_S$ and $b \in \mathbb{Z}$. We have $m(\lambda+v) = n(\lambda'+bv)$. Since $T_S \cap \mathbb{Z}v = 0$, m is equal to nb and divisible by n. Hence $\varphi(\lambda)/n$ has order n in Coke φ, which shows (1). q.e.d.

We have thus proved the following

COROLLARY 6.5. *Let M be a compact surface compoenet of the moduli space of stable sheaves on S. Then there exists an algebraic cycle on $S \times M$ which induces a homomorphism φ : $T_S \to T_M$* such that *$\varphi \otimes \mathbb{Q}$ is an isometry and the cokernel of φ is a finite cyclic group.*

Conversely, we have

PROPOSITION 6.6. *Let S be an algebraic $K3$ surface and $\Psi : T_S \to T$* be *an embedding of the transcendental lattice T_S of S into an even lattice T. Assume that the cokernel of Ψ is a cyclic group of order $r < \infty$. Then there exists a compact component M of the moduli space of stable sheaves of rank r on S which satisfies the following :*

(1) *there is an isometry $i : T \xrightarrow{\sim} T_M$, and*

(2) *there is an algebraic cycle on* $S \times M$ *which induces* $i \circ \psi$.

PROOF. Take a transcendental cycle $\tau \in T_S \otimes \mathbb{Q}$ so that $(\psi \otimes \mathbb{Q})(\tau)$ belongs to T and generates T modulo $\psi(T_S)$. By our assumption, $\lambda = r\tau$ belongs to T_S. Since $\psi \otimes \mathbb{Q}$ is an isometry, $(\tau . \beta)$ is equal to $((\psi \otimes \mathbb{Q})(\tau) \cdot \psi(\beta))$ and is an integer for every $\beta \in T_S$. Since $H^2(S, \mathbb{Z})$ is a unimodular lattice and since T_S is a primitive sublattice of $H^2(S, \mathbb{Z})$, there exists a cycle $\alpha \in H^2(S, \mathbb{Z})$ such that $(\alpha . \beta) = (\tau . \beta)$ for every transcendental cycle $\beta \in T_S$. Then, the cycle $\ell = r(\alpha - \tau)$ belongs to $H^2(S, \mathbb{Z})$ and perpendicular to T_S. Hence ℓ is a divisor class of S. Moreover, $\ell + \lambda$ is equal to $r\alpha$ and divisible by r in $H^2(S, \mathbb{Z})$. Replacing α by α + (a sufficiently ample divisor), we can choose α so that ℓ becomes an ample divisor class. We put $s = (\ell^2)/2r = r(\alpha - \tau)^2/2$ and $v = (r, \ell, s) \in \tilde{H}^{1,1}(S, \mathbb{Z})$ and consider the moduli space $M = M_A(v)$ of stable sheaves E with $v(E) = v$. Since (τ^2) is an even integer, so is $(\alpha - \tau)^2$. Hence s is divisible by r. Since τ is transcendental, $(\ell . m)$ is equal to $r(\alpha . m)$ and hence divisible by r for every divisor class m of S. Hence the number $n(v)$ (see Proposition 6.4) is equal to r. $M_\ell(v)$ is nonempty, by Theorem 5.4 and $M_\ell^\mu(v)$ is nonempty, by Proposition 3.16. Hence by Proposition 4.3, there exists an ample line bundle A such that $M = M_A(v)$ is nonempty and compact and irreducible. By Proposition 6.4, there exists an isometry $i : T \to T_M$ such that $\varphi = i \circ \Psi$ and φ is induced by an algebraic cycle on $S \times M$. q.e.d

Step II. Let $\varphi : T_S \to T_{S'}$ be a homomorphism of Hodge structures and assume that $\varphi \otimes \mathbb{Q}$ is an isometry. Then there exists an algebraic cycle $W \in H^4(S \times S', \mathbb{Q}|)$ on $S \times S'$ which induces φ.

PROOF. We prove our assertion by induction on the length ℓ of the cokernel of φ. In the case $\ell = 1$, our assertion was proved in Step I. Hence we assume that $\ell > 1$. Take a sublattice T of $T_{S'}$ such that $\varphi(T_S) \subsetneqq T$ and $T/\varphi(T_S)$ is a cyclic group. Then, by Proposition 6.6, there exists a $K3$ surface M which is a compact component of the moduli space of stable sheaves such that $T_M \cong T$ and there exists an algebraic cycle W_1 on $S \times M$ which induces $T_S \to T \cong T_M$. By induction hypothesis, there exists an algebraic cycle W_2 on $M \times S'$ which induces $T_M \cong T \to T_{S'}$. Then, the cycle $Z = \pi_{S\times S'},{}_*(\pi^*_{S\times M} W_1 \cdot \pi_{M\times S'}{}^* W_2)$ on $S \times S'$ is algebraic and induces φ on T_S. q. e. d.

PROOF OF THEOREM 1.9 : By our assumption, there exists a primitive embedding $T \hookrightarrow \Lambda$ of T into a $K3$ lattice Λ. Since $T \otimes \mathbb{Q} \cong T_S \otimes \mathbb{Q}$ the Hodge decomposition of $T_S \otimes \mathbb{C}$ induces that of $T \otimes \mathbb{C}$ We regard T as a polarized Hodge structure by this Hodge decomposition. The orthogonal complement of T in Λ is a hyperbolic lattice, *i.e.*, has signature $(1, *)$. By virtue of the surjectivity theorem of the period map for $K3$ surfaces [23], there exists a $K3$ surface S'' and an isometry $i : \Lambda \to H^2(S'', \mathbb{Z})$ such that $i(T) = T_{S''}$ and $i|_T$ is a homomorphism of Hodge structures. Both T_S and $T_{S'}$ contain $T_{S''}$ as a sublattice of finite index. By Step II, there exist algebraic cycles on $S'' \times S$ and on $S'' \times S'$ which induce the isometries $T_{S''} \hookrightarrow T_S$ and

$T_{S''} \hookrightarrow T_{S'}$, respectively. Therefore, the composition of the two algebraic cycles induces the Hodge isometry between $T_S \otimes \mathbb{Q}$ and $T_{S'} \otimes \mathbb{Q}$. q. e. d.

APPENDIX 1. Boundedness and existence of μ-semi-stable sheaves.

In this section, S is an arbitrary complete algebraic surface over $\mathbb{C}$. We study the behaviour of moduli spaces of μ-semi-stable sheaves with respect to A_n, $n = 1, 2, 3, \ldots$, when ample $\mathbb{Q}$-divisors A_n converge to an ample divisor A.

THEOREM A. 1. *Let $\{A_n\}$ be a sequence of ample $\mathbb{Q}$-divisors which converges to an ample divisor A. Let c_1 be a numerical equivalence class of divisors and c_2 an integer. Assume that, for every n, there exists a sheaf E_n on S with Chern classes c_1 and c_2 (modulo numerical equivalence) and which is μ-semi-stable with respect to A_n. Then there exists a sheaf E on S which satisfies the following :*

(1) *there exists an infinite subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that E is μ-semi-stable with respect to every A_{n_k}, and*

(2) *E is μ-semi-stable with respect to A.*

Let P be a Zariski-open condition for sheaves on S which is independent of A_n, e.g., simpleness or local freeness. If the open condition P holds for every E_n, then E can be chosen so that E satisfies P.

For the proof of the above theorem, a certain boundedness of μ-semi-stable sheaves is essential. Let $\mathscr{A}$ be the ample cone in $H^{1,1}(S, \mathbb{R})$ and $\bar{\mathscr{A}}$ its closure.

THEOREM A. 2. *Let H be an ample divisor and B a bounded subset of $\bar{\mathscr{A}} \cap H^2(S, \mathbb{Q})$. Let $S^r_A(c_1, c_2)$ denote the set of isomorphic classes of rank r sheaves with Chern classes c_1 and c_2 modulo numerical equivalence and which are μ-stable with respect to an ample $\mathbb{Q}$-divisor A. Then the union $\bigcup_{b \in B} S^r_{H+b}(c_1, c_2)$ is bounded.*

In the case $B = \{0\}$, this was proved by Maruyama in [8] and our proof of Theorem A.2 is quite parallel to his proof in §2 [8]. Let $\alpha_1, \ldots, \alpha_{r-1}$ be a sequence of $r-1$ rational numbers and let $S^r_B(\alpha_1, \ldots, \alpha_{r-1} : c_1, c_2)$ be the set of isomorphism classes of rank r torsion free sheaves of type $\alpha_1, \ldots, \alpha_{r-1}$ with respect to $H + b$ for some $b \in B$ (see p. 28 [8]) and with Chern classes c_1 and c_2 modulo numerical equivalence. Our Theorem A. 2 is a special case of the boundedness of $S^r_B(\alpha_1, \ldots, \alpha_{r-1} : c_1, c_2)$ which follows from Theorem A. 3 below and Theorem 1.14 in [8].

THEOREM A.3. *Let a be an integer and let $S^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$ be the union of $S^r_B(\alpha_1, \ldots, \alpha_{r-1} : c_1, c_2)$ for all $c_2 \leq a$. Then there are two constants b_0 and b_1 (independent of each c_2) such that for any member E of $S^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$, $\dim H^0(S, E) \leq b_0$ and $\dim H^0(C, E \otimes \mathcal{O}_C) \leq b_1$ for any curve C in an open set $U(E)$ of $|H|$, where $U(E)$ may depend on E.*

PROOF. Our proof is quite similar to that of Theorem 2.5 in [8]. We only indicate the parts to be modified. It suffices to show the theorem for the subset $VS'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$ of $S'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$ consisting of vector bundles in $S_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$. We prove our theorem by induction on r. Assume that the theorem is true in the case rank $r-1$. Under this assumption, we shall show that our theorem holds for $VS'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$. Since B is bounded, there exists an integer n such that $H^0(S, E(n)) \neq 0$ for every member E of $VS'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$ (cf. Lemma 2.1 in [8]), where $E(n)$ is the abbreviation of $E \otimes H^{\otimes n}$. Hence, for every member E of $VS'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$, there exists an exact sequence

$$0 \to \mathcal{O}_S(D) \otimes H^{\otimes(-n)} \to E \to F \to 0$$

where D is an effective divisor and F is a torsion free sheaf of rank $r-1$. Let L be the set of effective divisors D such that $\mathcal{O}_S(D) \otimes H^{\otimes(-n)}$ is contained in some member E of $VS'^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$.

CLAIM: L is bounded.

$\mathcal{O}_S(D)$ is a subsheaf of $E(n)$ and $E(n)$ is of type $\alpha_1, \ldots, \alpha_{r-1}$ with respect to $H+b$ for some $b \in B$. Hence we have

$$\begin{aligned}(D \cdot H+b) &\leq \mu_{H+b}(E(n)) + \alpha_{r-1}/(r-1) \\ &= (c_1 \cdot H+b)/r + n(H \cdot H+b) + \alpha_{r-1}/(r-1)\end{aligned}$$

$$= (c_1 \,.\, H)/r + n(H^2) + (c_1/r + nH \,.\, b) + \alpha_{r-1}/(r-1).$$

Since B is bounded, $R = \sup_{b \in B} (c_1/r + nH \,.\, b) < \infty$. Since b belongs to $\mathscr{A}$ and D is effective, $(b \,.\, D)$ is nonnegative. Hence, we have

$$(D \,.\, H) \leq (D \,.\, H+b) \leq (c_1 \,.\, H)/r + n(H^2) + R + \alpha_{r-1}/(r-1).$$

Therefore, L is bounded.

Let G be a rank s quotient sheaf of F. Since G is a quotient of E and since E is of type $\alpha_1, \ldots, \alpha_{r-1}$ with respect to $H + b$, we have

$$\mu_{H+b}(E) - \alpha_s \leq \mu_{H+b}(G).$$

Put $\alpha_{s,D,b} = \alpha_s + \{n(H \cdot H+b) + (c_1/r - D \cdot H+b)\}/(r-1)$. Then we have $\mu_{H+b}(E) - \alpha_s = \mu_{H+b}(F) - \alpha_{s,D,b}$. Put $\alpha'_s = \sup_{D \in L,\, b \in B} \alpha_{s,D,b}$. Then we obtain $\mu_{H+b}(F) - \alpha_s \leq \mu_{H+b}(G)$. Hence F is of type $\alpha_1, \ldots, \alpha_{r-2}$ with respect to $H + b$. Let Q be the set of isomorphic classes of F's which are obtained from some E in $VS^r_{B,a}(\alpha_1, \ldots, \alpha_{r-1} : c_1)$ as above. Then, by the above result, Q is a subset of

$$\coprod_{\lambda \in \Lambda} \coprod_{c_2 \leq a + \beta} S^{r-1}_B(\alpha'_1, \ldots, \alpha'_{r-2} : c_1 - \lambda + nc_1(H), c_2)$$

where $\Lambda = L/$(numerical equivalence) and $\beta = \max_{D \in L} \{-(c_1(D+nH \cdot D-nH)\}$. By induction hypothesis, our theorem is true for any member F of Q and our proof can be completed in the same way as Theorem 2.5 in [8]. q.e.d.

PROOF OF THEOREM A.1: Take an integer N so that $NA_n - A$ is ample for every n. Applying Theorem A.2 for $H = A$ and $B = \{NA_n - A\}$, we see that the set $\mathscr{E}$ of isomorphic classes of sheaves on S which are μ-semi-stable with respect to A_n for some n is bounded. All E_n s belong to $\mathscr{E}$ and hence there exists a subfamily $\{F_t : t \in V\}$ of $\mathscr{E}$ parametrized by a variety V which contains E_n for infinitely many n, say, for $n = n_1, n_2, \ldots$. Since μ-semi-stability is an open condition, for each n_k, there exists a Zariski open set U_k of V such that F_u is μ-semi-stable with respect to A_{n_k} (and satisfies the property P) for every $u \in U_k$. V is a variety over $\mathbb{C}$ and is a Baire space. Hence the intersection of all U_k s is nonempty. Therefore, we have (1). (2) follows immediately from (1), because $\mu_A(F) = \lim_{k\to\infty} \mu_{A_{n_k}}(F)$ for every sheaf F on S. q.e.d.

APPENDIX 2. Existence of a (quasi-) universal family

Let X be a scheme and $\mathscr{M}$ a connected component of the moduli functor $\mathscr{S}pl_X$ of simple sheaves on S.

DEFINITION A.4. (1) Let T be a scheme. A sheaf $\mathscr{E}$ on $X \times T$ is a *quasi-family* of sheaves in $\mathscr{M}$ if $\mathscr{E}$ is T-flat and if, for every $t \in T$, there exist an integer σ and a member E of $\mathscr{M}$ such that

$\mathscr{E}|_{X \times T} \cong E^{\oplus \sigma}$. If T is connected, then the positive integer σ does not depend on $t \in T$ and called the similitude of $\mathscr{E}$.

(2) Two quasi-families $\mathscr{E}$, $\mathscr{E}'$ of sheaves in $\mathscr{M}$ on $X \times T$ are *equivalent* if there exist vector bundles V and V' on T such that $\mathscr{E} \otimes \pi_T^* V \cong \mathscr{E}' \otimes \pi_T^* V'$.

(3) A sheaf $\mathscr{E}$ on $X \times M$ is a *quasi-universal family* of sheaves in $\mathscr{M}$ if $\mathscr{E}$ is a quasi-family and, for every scheme T and quasi-family $\mathscr{F}$ on $X \times T$, there exists a unique morphism $f : T \to M$ such that $f^*\mathscr{E}$ and $\mathscr{F}$ are equivalent.

By definition, if $\mathscr{E}$ on $X \times M$ and $\mathscr{E}'$ on $X \times M'$ are quasi-universal families, then M and M' are isomorphic to each other and $\mathscr{E}$ and $\mathscr{E}'$ are equivalent.

THEOREM A.5. *Assume that $\mathscr{M}$ is representable by a scheme M of finite type in the usual topology (if $k = \mathbb{C}$) or in the etale topology. Then there exists a quasi-universal family on $X \times M$.*

PROOF. For simplicity, we assume that $k = \mathbb{C}$ and M is representable in the usual topology. There exists an open covering $M = \bigcup_i U_i$ (in the usual topology) and a universal family $\mathscr{E}_i$ on $U_i \times X$ for every i. Take a sufficiently ample line bundle L such that all higher cohomology groups $H^i(X, E \otimes L)$ vanish for every member E of M. By the base change theorem, the direct image $V_i = \pi_{i,*}(\mathscr{E}_i \otimes L)$ is a vector bundle on U_i, where π_i is the projection of $X \times U_i$ onto U_i. Shrink the covering

$\bigcup_i U_i$ so that $\mathrm{Pic}\,(U_i \cap U_j) = 0$ for every $i \neq j$. Then there exists an isomorphism $f_{ij} : \mathscr{E}_i|_{X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_j|_{X\times(U_i\cap U_j)}$. f_{ij} induces an isomorphism $\bar{f}_{ij} = \pi_{ij,*}(f_{ij} \otimes L) : V_i \xrightarrow{\sim} V_j$, on $U_i \cap U_j$ where π_{ij} is the projection of $X \times (U_i \cap U_j)$ onto $U_i \cap U_j$. We put $\Phi(f_{ij}) = f_{ij} \otimes \pi_{ij}^*(\bar{f}_{ij}^{-1})^{\vee}: \mathscr{E}_i \otimes \pi_i^* V_i^{\vee}|_{X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_j \otimes \pi_j^* V_j^{\vee}|_{X\times(U_i\cap U_j)}$.

CLAIM: $\Phi(f_{ij}) \circ \Phi(f_{jk}) \circ \Phi(f_{ki})$ is identity over $X \times (U_i \cap U_j \cap U_k)$ for every i, j and k.

By the functoriality of Φ, $\Phi(f_{ij}) \circ \Phi(f_{jk}) \circ \Phi(f_{ki})$ is equal to $\Phi(g_{ijk})$, where $g_{ijk} = f_{ij} \circ f_{jk} \circ f_{ki}$. g_{ijk} is an automorphism of $\mathscr{E}_i|_{X\times(U_i\cap U_j\cap U_k)}$ over $U_i \cap U_j \cap U_k$. Since $\mathscr{E}_i$ is simple over U_i, the automorphism g_{ijk} of $\mathscr{E}_i$ over $U_i \cap U_j \cap U_k$ is multiplication by an invertible element of $\mathscr{O}_{U_i \cap U_j \cap U_k}$. Hence $\Phi(g_{ijk})$ is identity.

By the claim, $\mathscr{E}_i \otimes \pi_i^* V_i^{\vee}$ can be glued together by $\Phi(f_{ij})$'s. We obtain a sheaf $\mathscr{E}$ on $X \times M$ whose restriction to $U_i \times X$ is isomorphic to $\mathscr{E}_i \otimes \pi_i^* V_i^{\vee}$ for every i. We show that $\mathscr{E}$ is a quasi-universal family. Let $\mathscr{F}$ be a quasi-family of sheaves in M on $X \times T$. Since $\mathscr{E}_i$ are universal families, there exist a unique morphism $f : T \to M$, a vector bundle F_i on $f^{-1}(U_i)$ and an isomorphism $h_i : \mathscr{F}|_{X\times f^{-1}(U_i)} \xrightarrow{\sim} ((1\times f)^*\mathscr{E}_i) \otimes_{\mathscr{O}_T} F_i$ for every i. We show that two quasi-families $\mathscr{F}$ and $\mathscr{G} = (1 \times f)^*\mathscr{E}$ on $X \times T$ are equivalent. Define the homomorphism $\varphi : \mathscr{G} \otimes \pi^*\pi_* \mathscr{H}om_{\mathscr{O}_{X\times T}}(\mathscr{G}, \mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_{X\times T}}(\pi^*\pi_* \mathscr{E}nd_{\mathscr{O}_{X\times T}}(\mathscr{G}), \mathscr{F})$ by

$\varphi(g \otimes f)(e) = f(e(g))$ for every $g \in \mathscr{G}$, $f \in \pi^*\pi_*\mathscr{H}om_{\mathcal{O}_{X\times T}}(\mathscr{G},\mathscr{F})$ and $e \in \pi^*\pi_*\mathscr{E}nd\,\mathcal{O}_{X\times T}(\mathscr{G})$, where π is the projection of $X \times T$ onto T. By using the isomorphisms h_i, it can be easily checked that this φ is an isomorphism. Since $\pi_*\mathscr{H}om_{\mathcal{O}_{X\times T}}(\mathscr{G},\mathscr{F})$ and $\pi_*\mathscr{E}nd_{\mathcal{O}_{X\times T}}(\mathscr{G})$ are vector bundles on T, two quasi-families F and G are equivalent. q.e.d.

A quasi-universal family of similitudes 1 is nothing but a universal family. On the existence of a universal family, we have the following by an argument similar to the above and by an idea in [16] (and its improvement Theorem 6.11 in [9]).

THEOREM A.6. *Let the assumption be same as in above theorem. Let μ be the greatest common divisor of $\chi(E \otimes N)$, where E is a member of $\mathscr{M}$ and N runs over all vector bundles on X. If $\mu = 1$, then there exists a universal on $X \times M$.*

PROOF. Let μ_0 be the greatest common divisor of $\chi(E \otimes N)$, where N runs over all vector bundles on X which satisfy

(*) all higher cohomology groups $H^i(X, E \otimes N)$ vanish for every member E of M.

We show that $\mu_0 = 1$. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then there exists an integer m_0 such that $N(m)$ satisfies (*) for every $m \geqq m_0$. $\chi(E \otimes N(m))$ is divisible by μ_0 for every $m \geqq m_0$ by definition. Since $\chi(E \otimes N(m))$ is a numerical polynomial on m, $\chi(E \otimes N)$ is divisible by μ_0. Since N is an arbitrary vector bundle, μ_0 divides μ and hence $\mu_0 = 1$ by our assumption. Therefore, there exist vector bundles N_j with the property (*) and integers a_ν ($1 \leq \nu \leq n$) such that $\sum_{\nu=1} a_\nu \chi(E \otimes N_\nu) = -1$. Let $M = \cup_i U_i$, $\mathscr{E}_i$ and $f_{ij} : \mathscr{E}_i|_{X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_j|_{X\times(U_i\cap U_j)}$ be same as in the proof of Theorem A.5. By the property (*), $\pi_{i,*}(\mathscr{E}_i \otimes \pi_X{}^*N_\nu)$ is a vector bundle of rank $\chi(E \otimes N_\nu)$ on U_i for every i and ν. Put $L_i = \bigotimes_{\nu=1}^{n} \det(\pi_{i,*}(\mathscr{E}_i \otimes \pi_X^* N_\nu))^{\otimes a_\nu}$, where det denotes the highest nonzero exterior power of a vector bundle. The isomorphism f_{ij} induces the isomorphism $p_{ij} : L_i|_{U_i\cap U_j} \xrightarrow{\sim} L_j|_{U_i\cap U_j}$ for every i,j. By the same argument as in Theorem A.5, we can show that $\mathscr{E}_i \otimes \pi_i^* L_i$ on $X \times U_i$ can be glued together by the isomorphisms $f_{ij} \otimes p_{ij}$ and we obtain a sheaf $\mathscr{E}$ on $X \times M$ whose restriction to $X \times U_i$ is isomorphic to $\mathscr{E}_i \otimes \pi_i^* L_i$ for every i. Since $\mathscr{E}_i$ are universal families, $\mathscr{E}$ is a universal family.

q.e.d.

REMARK A.7. If X is smooth, then every sheaf on X has a resolution by a locally free sheaves. Hence μ in the theorem is equal to the greatest common divisor of $\chi(E \otimes N')$ where $E \in \mathscr{M}$ and N' runs over all sheaves on X. If X is smooth and dim $X = 2$, then μ is equal to the greatest common divisor of $r(E)$, $(c_1(E).D)$ and $\chi(E)$, where D runs over all divisors of X.

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