CURVES AND K3 SURFACES OF GENUS ELEVEN

Shigeru Mukai

Graduate School of Polymathematics, Nagoya University, Chikusa-ku, Nagoya, 464-01, Japan

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ABSTRACT. For a general curve C of genus 11, its embeddings into K3 surfaces are unique up to isomorphisms. Such embedding $\alpha : C \longrightarrow \hat{S}$ is constructed as a non-abelian analogue of the duality between the Picard and Albanese varieties. Let S be the moduli space of stable rank two vector bundles E of canonical determinant on C with $h^0(E) \geq 7$. Then S is a smooth K3 surface and the determinant line bundle h_{det} is a polarization of genus 11. The K3 surface $\hat{S}$ which contains C is the unique 2-dimensional component of the moduli space of stable rank two sheaves of determinant h_{det} on S.

In [11], we have begun our study of the Brill-Noether locus

$$M_C(2, K, n) = \{E | h^0(E) \geq n + 2\}/\text{isom}$$

in the moduli space $M_C(2, K)$ of stable rank two vector bundles E of canonical determinant over a curve, *i.e.*, a compact Riemann surface C. In the workshop at Sanda, we discussed $M_C(2, K, 3)$ for a curve C of genus seven, for which see the forthcoming article. Here, instead, we study another interesting case:

Theorem 1. *For a general curve C of genus eleven, the Brill-Noether locus $M_C(2, K, 5)$ is a smooth K3 surface and the restriction h_{det} of the determinant line bundle of $M_C(2, K)$ is a polarization of genus eleven,* i.e., $(h^2) = 20$.

Let (X, h) be a pair of a K3 surface and a line bundle h of degree 20 and $M_X(2, h, n)$ be the moduli space of stable sheaves $\mathcal{E}$ on X with $r(\mathcal{E}) = 2$, $\det \mathcal{E} \simeq h$ and $\chi(\mathcal{E}) = 2 + n$. By [6], $M_X(2, h, n)$ is smooth and of dimension $2(11 - 2n)$ if it is not empty. Hence $M_X(2, h, n)$ is a surface only if $n = 5$ and, in fact, $\hat{X} := M_X(2, h, 5)$ is a K3 surface if it is compact. Moreover, there is a line bundle $\hat{h}$ of degree 20 on $\hat{X}$ and $(\hat{\hat{X}}, \hat{\hat{h}})$ is isomorphic to (X, h) ([7], Theorem 1.4).

Let S be $M_C(2, K, 5)$ in the theorem and $\mathcal{U}$ the Poincaré bundle on $C \times S$. We normalize $\mathcal{U}$ so that $\det \mathcal{U} \simeq K_C \boxtimes h_{det}$ (see §4) and restrict it to fibres in the other direction. Namely we consider the family of vector bundles $\mathcal{U}_p := \mathcal{U}|_{p \times S}$ on S parametrized by C.

Theorem 2. *Let C, $S = M_C(2, K, 5)$, h_{det} and $\mathcal{U}$ be as above. Then the vector bundle $\mathcal{U}_p$ on S is stable with respect to h_{det} and belongs to $M_S(2, h_{det}, 5)$ for every $p \in C$. Moreover, the classification morphism $\alpha : C \longrightarrow M_S(2, h_{det}, 5), p \mapsto [\mathcal{U}_p]$, is an embedding.*

Let $\mathcal{M}_{11}$ and $\mathcal{F}_{11}$ be the moduli spaces of curves and polarized K3 surfaces (X, h) of genus 11, respectively. Let $\mathcal{P}_{11}$ be the $\mathbf{P}^{11}$-bundle $\coprod_{(X,h) \in \mathcal{F}_{11}} |h|$ over $\mathcal{F}_{11}$ and $\phi_{11} : \mathcal{P}_{11} \cdots \to \mathcal{M}_{11}$ the rational map which associates the isomorphism class for every $C \in |h|$ (cf. [5]). By the theorem, the rational map $\psi : \mathcal{M}_{11} \cdots \to \mathcal{P}_{11}$, $[C] \mapsto (M_S(2, h_{det}, 5), \alpha(C))$, satisfies $\phi_{11} \circ \psi = id$. Since $\mathcal{P}_{11}$ is irreducible and of the same dimension as $\mathcal{M}_{11}$, ϕ_{11} is birational and ψ is its inverse. Therefore, we have

Unité mixte de l'institute Henri Poincaré, CNRS Université Pierre et Marrie Currie, Paris.

Corollary 1. *$M_S(2, h_{det}, 5)$, with $S = M_C(2, K, 5)$, is the unique K3 surface which contains C.*

Since $\mathcal{M}_{11}$ is unirational by [4], we also have

Corollary 2. *$\mathcal{F}_{11}$ is unirational.*

Thus $M_C(2, K, 5)$ and the morphism $\alpha : C \longrightarrow M_S(2, h_{det}, 5)$ are similar to the Picard variety and Albanese map $X \longrightarrow \mathrm{Alb}\, X = (\mathrm{Pic}^0 X)^\vee$, respectively. The morphism α is the K3 hull in the following sense:

Definition An embedding $\alpha : C \longrightarrow A$ of a curve C into a variety A is a *K3 hull* if

1) there exist a line bundle L on A and its global sections $s_1, \cdots, s_{n-1}$ such that $L|_C \simeq K_C$ and C is the complete intersection $s_1 = \cdots = s_{n-1} = 0$ in A,

2) every embedding of C into a K3 surface is isomorphic to the restriction of α to the common zero locus of a codimension one subspace of $< s_1, \cdots, s_{n-1} > \subset H^0(A, L)$, and

3) there is an exact sequence

$$0 \longrightarrow H^0(N_{C/A} \otimes K_C^{-1}) \longrightarrow H^1(K_C^{-2}) \longrightarrow \bigwedge^2 H^1(\mathcal{O}_C),$$

where the first map is the coboundary map of the long exact sequence associated to the natural exact sequence

$$[0 \longrightarrow T_C \longrightarrow T_A|_C \longrightarrow \underset{\displaystyle K_C^{\oplus(n-1)}}{\underset{\|}{N_{C/A}}} \longrightarrow 0] \otimes K_C^{-1}$$

and the second is the dual of the Wahl map $\bigwedge^2 H^0(K_C) \longrightarrow H^0(K_C^3)$ (see [13]).

The K3 hull exists for every general curve of genus $g \geq 7$. It is the symmetric space described in [9] for $g = 7, 8$ and 9, and C itself for $g = 10$ and $g \geq 12$ (cf. [3], [8], [5]). This will be discussed elsewhere.

Let G_d^r be the locus of curves with g_d^r in the moduli space $\mathcal{M}_{11}$ and C a general member of G_6^1. C is embedded into $\mathbf{P}^5$ by $|K_C \xi^{-1}|$ and the quadric hull S of its image $C_{14} \subset \mathbf{P}^5$ is a K3 surface of degree 8, where ξ is a g_6^1 of C. There exists a family $\{\mathcal{E}_x\}_{x \in S}$ of vector bundles

$$0 \longrightarrow \mathcal{O}_S(C - A) \longrightarrow \mathcal{E}_x \longrightarrow I_x(A) \longrightarrow 0$$

of Schwarzenberger type on S parametrized by S, where A is a hyperplane section of $S \subset \mathbf{P}^5$.

Theorem 3. *For a general member C of G_6^1, the Brill-Noether locus $M_C(2, K, 5)$ is smooth and isomorphic to S by the correspondence $S \ni x \mapsto \mathcal{E}_x|_C \in M_C(2, K, 5)$.*

The quadric hull S in the theorem is the unique K3 surface which contains the hexagonal curve C. In fact, if a K3 surface contains C, then the g_6^1 on C extends to a line bundle on it.

Another interesting divisor of $\mathcal{M}_{11}$ is the locus G_9^2. For a general member C of G_9^2, $S = M_C(2, K, 5)$ is still a K3 surface and contains a line D, *i.e.*, $\deg h_{det}|_D = 1$, which parametrizes the extensions $0 \longrightarrow \zeta \longrightarrow E \longrightarrow K_C\zeta^{-1} \longrightarrow 0$ with $h^0(E) = 7$, where ζ is a g_9^2 of C. The moduli space $M_S(2, h_{det}, 5)$ is the unique quartic surface which contains the image of $\Phi_{K_C\zeta^{-1}} : C \longrightarrow \mathbf{P}^3$.

We prove the theorem in §3 after some preparations in §§1 and 2, and Theorem 1 and 2 in the final section.

Notations $\mathbf{P}_*V$ and $\mathbf{P}^*V$ are the two projective spaces associated to a vector space V. The former parametrizes one-dimensional subspaces and the latter quotient spaces. S^nV is the n-th symmetric tensor product of a vector space or vector bundle V. For an $\mathcal{O}_X$-module

$\mathcal{E}$ and an $\mathcal{O}_Y$-module $\mathcal{F}$, the tensor product of their pull-backs on the product $X \times Y$ is denoted by $\mathcal{E} \boxtimes \mathcal{F}$.

1. Preliminary

Next two lemmas are useful to analyze and construct vector bundles in $M_C(2, K, n)$.

Lemma 1. *Let E be a rank two vector bundle of canonical determinant and ζ a line bundle on C. If ζ is generated by global sections, then we have*

$$\dim \mathrm{Hom}_{\mathcal{O}_C}(\zeta, E) \geq h^0(E) - \deg \zeta.$$

The proof is an easy exercise of the base-point-free pencil trick (see [10] Proposition 3.1).

Lemma 2. *Let ξ be a line bundle and consider non-trivial extensions*

$$0 \longrightarrow \xi \longrightarrow E \longrightarrow \eta \longrightarrow 0$$

of ξ by its Serre adjoint η.

1) *The extensions E with $h^0(E) = h^0(\xi) + h^0(\eta)$ are parametrized by the projective space $\mathbf{P}^* \mathrm{Coker}[S^2 H^0(\eta) \longrightarrow H^0(\eta^2)]$.*

2) *Assume that the multiplication map $S^2 H^0(\eta) \longrightarrow H^0(\eta^2)$ is surjective. Then $h^0(E) \leq h^0(\xi) + h^0(\eta) - 1$ for every non-trivial extension E. Moreover, the non-trivial extensions E with $h^0(E) = h^0(\xi) + h^0(\eta) - 1$ are parametrized by the quadric hull of the image of $\Phi_{|\eta|} : C \longrightarrow \mathbf{P}^* H^0(\eta)$. More precisely, for every point x of the quadric hull, there is the unique extension E such that the image of the linear map $H^0(E) \longrightarrow H^0(\eta)$ is the codimension one subspace corresponding to x.*

Proof. Let $e \in \mathrm{Ext}^1(\eta, \xi)$ be the extension class and $\delta_e : H^0(\eta) \longrightarrow H^1(\xi)$ the coboundary map. The condition that $h^0(E) = h^0(\xi) + h^0(\eta)$ is equivalent to $\delta_e = 0$, that is, e lies in the kernel of the linear map

$$\Delta : \mathrm{Ext}^1(\eta, \xi) \longrightarrow H^0(\eta)^\vee \otimes H^1(\xi), \quad e \mapsto \delta_e.$$

By the Serre duality, the linear map Δ is the dual of the multiplication map $H^0(\eta) \otimes H^0(\eta) \longrightarrow H^0(\eta^2)$. Hence $\mathrm{Ker}\, \Delta$ is the dual of $\mathrm{Coker}[S^2 H^0(\eta) \longrightarrow H^0(\eta^2)]$, which shows (1). The first assertion of (2) is a direct consequence of (1). The condition that $h^0(E) = h^0(\xi) + h^0(\eta) - 1$ is equivalent to $\mathrm{rank}\, \delta_e = 1$ by our assumption. There exists a nonzero linear map $\alpha : H^0(\eta) \longrightarrow \mathbf{C}$ such that δ_e is the composite of α and its dual $\alpha^\vee : \mathbf{C} \longrightarrow H^0(\eta)^\vee \simeq H^1(\xi)$. Let x be the point of $\mathbf{P}^* H^0(\eta)$ corresponding to α. Then $I_2 = \mathrm{Ker}[S^2 H^0(\eta) \longrightarrow H^0(\eta^2)]$, the degree 2 part of the homogeneous ideal of $\Phi_\eta(C)$, vanishes at x, since $S^2\alpha$ is the composite of $S^2 H^0(\eta) \longrightarrow H^0(\eta^2)$ and the linear map $H^0(\eta^2) \longrightarrow \mathbf{C}$ corresponding to e by construction. Thus E with $h^0(E) = h^0(\xi) + h^0(\eta) - 1$ determines a point in the quadric hull. Conversely, a point in the quadric hull determines e with $\mathrm{rank}\, \delta_e = 1$. Such e is unique since Δ is injective by our assumption. □

See [11] §4 for the following criterion of smoothness:

Proposition 1. *Let E be a member of $M_C(2, K)$ with $h^0(E) = n + 2$ and put $\sigma = 3g(C) - 3 - (n+2)(n+3)/2$. Then we have*

1) *$\dim_{[E]} M_C(2, K, n) \geq \sigma$, and*

2) *$M_C(2, K, n)$ is smooth and of dimension σ at $[E]$ if and only if the multiplication map $S^2 H^0(E) \longrightarrow H^0(S^2 E)$ is injective.*

2. Hexagonal curve of genus 11

Let $S \subset \mathbf{P}^5$ be a smooth complete intersection of three quadrics. S is a K3 surface by the adjunction formula and the Lefschetz theorem. Throughout this and next sections we assume that S contains a normal elliptic curve B of degree 6 but no bisecant lines of B. Such a surface exists by the following:

Lemma 3. *Let $B \subset \mathbf{P}^5$ be a normal elliptic curve of degree 6. Then the intersection of three general quadrics passing through B is a smooth surface and does not contain a bisecant line of $B \subset \mathbf{P}^5$.*

Proof. The first assertion follows from Bertini's theorem, since $B \subset \mathbf{P}^5$ is an intersection of quadrics. All S's which contain B are parametrized by a non-empty open subset of the Grassmannian $G(3, H^0(I_B(2)))$, which is of dimension $3(9-3) = 18$. Let ℓ be a bisecant line. All S's which contain both B and ℓ are parametrized by an open subset of the Grassmannian $G(3, H^0(I_{B\cup\ell}(2)))$, which is of dimension 15. Since the bisecant lines are parametrized by a surface, we have the second assertion. □

Let $C \subset S$ be a smooth member of the complete linear system $|A + B|$, where A is a hyperplane section of $S \subset \mathbf{P}^5$. Such C exists since $|A + B|$ is free from base points. Since

$$(A^2) = 8, (A.B) = 6, (B^2) = 0 \text{ and } (A + B)^2 = 20,$$

C is of genus 11. We denote the restriction of $\mathcal{O}_S(B)$ and $\mathcal{O}_S(A)$ to C by ξ and η, respectively. By the adjunction formula, the canonical line bundle K_C of C is the tensor product of ξ and η. By the exact sequences

$$0 \longrightarrow \mathcal{O}_S(-A) \longrightarrow \mathcal{O}_S(B) \longrightarrow \xi \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S(-B) \longrightarrow \mathcal{O}_S(A) \longrightarrow \eta \longrightarrow 0,$$

and by the lemma below, the restriction maps $H^0(\mathcal{O}_S(B)) \longrightarrow H^0(\xi)$ and $H^0(\mathcal{O}_S(A)) \longrightarrow H^0(\eta)$ are isomorphisms. In particular, $|\xi|$ is a g^1_6 and its Serre adjoint η is a g^5_{14}. The embedding $C \hookrightarrow S \hookrightarrow \mathbf{P}^5$ is given by the complete linear system $|\eta|$.

Lemma 4. $H^i(\mathcal{O}_S(A - B)) = H^i(\mathcal{O}_S(B - A)) = 0$ *for every i.*

Proof. Since $(B - A.A) < 0$, we have $H^0(\mathcal{O}_S(B - A)) = 0$ and hence $H^2(\mathcal{O}_S(A - B)) = 0$ by the Serre duality. Since $(A - B)^2 = -4$, we have $\chi(\mathcal{O}_S(B - A)) = 0$. Hence it suffices to show $H^0(\mathcal{O}_S(A - B)) = 0$. Assume the contrary and let D be a member of $|A - B|$. Since $(D^2) = -4$ and $(D.A) = 2$, D is a union of two disjoint lines. Since $(D.B) = 6$, one of them meets B at at least three points. This is a contradiction. □

Tensoring $\mathcal{O}_S(A)$ with the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-B) \longrightarrow \mathcal{O}_S^{\oplus 2} \longrightarrow \mathcal{O}_S(B) \longrightarrow 0,$$

we have

Lemma 5. *The multiplication map*

$$H^0(\mathcal{O}_S(B)) \otimes H^0(\mathcal{O}_S(A)) \longrightarrow H^0(\mathcal{O}_S(A + B))$$

is an isomorphism.

The multiplication map $S^2H^0(\mathcal{O}_S(A)) \longrightarrow H^0(\mathcal{O}_S(2A))$ is surjective and its kernel is of dimension 3. Since S is smooth, every quadric passing through S is smooth at $x \in S$. Therefore, we have

Lemma 6. *The multiplication map $S^2H^0(I_x(A)) \longrightarrow H^0(\mathcal{O}_S(2A))$ is injective for every $x \in S$, where I_x is the maximal ideal of $\mathcal{O}_S$ at x.*

The non-existence of bisecant lines in S implies the following:

Lemma 7. *For an effective divisor D, we have*
1) $h^0(\xi(D)) = 2$ if D is of degree ≤ 3, and
2) $h^0(\xi(D)) + h^0(\eta(-D)) \leq max\ \{8-d, d+2\}$, where d is the degree of D.

Proof. If $\deg D \leq 2$, then $h^0(\eta(-D)) = h^0(\eta) - \deg D$, since $|\eta|$ is very ample. Since $\xi\eta = K_C$, we have $h^0(\xi(D)) = h^0(\xi) = 2$ by the Riemann-Roch theorem. Assume that $\deg D = 3$ and $h^0(\xi(D)) > 2$. Then D is contained in a trisecant line ℓ of $C \subset \mathbf{P}^5$. This is a contradiction since ℓ is contained in S and $(\ell.B) = (\ell.C - A) = 2$. So we have proved (1). By the Riemann-Roch theorem, we have

$$h^0(\xi(D)) + h^0(\eta(-D)) = 2h^0(\xi(D)) + 4 - d.$$

Hence (2) immediately follows from (1) if $d = 2$ or 3. If $d \geq 4$, then $h^0(\xi(D)) \leq d - 1$ by (1). Hence we have (2). □

Now we are ready to prove the following:

Proposition 2. *Let E be a semi-stable rank two vector bundle of canonical determinant on C. Then we have*
1) $h^0(E) \leq 7$, and
2) if $h^0(E) = 7$, then E is stable and contains a line subbundle isomorphic to ξ or $\xi(p)$ for a point $p \in C$.

Proof. If $h^0(E) < 7$, then there is nothing to prove. Hence we may assume that $h^0(E) \geq 7$. By Lemma 1, E contains a subsheaf isomorphic to ξ and we have an exact sequence

$$0 \longrightarrow \xi(D) \longrightarrow E \longrightarrow \eta(-D) \longrightarrow 0$$

for an effective divisor D of degree ≤ 4. Hence we have $h^0(E) \leq h^0(\xi(D)) + h^0(\eta(-D))) \leq 7$ and $\deg D = 0$ or 1 by Lemma 7, which shows (1) and the second assertion of (2). Assume that E is not stable and let $0 \longrightarrow \alpha \longrightarrow E \longrightarrow \beta \longrightarrow 0$ be an exact sequence with $\deg\alpha = \deg\beta = 10$. Then either α or β is isomorphic to $\xi(D)$ for an effective divisor D of degree 4. Hence we have $h^0(E) \leq h^0(\xi(D)) + h^0(\eta(-D)) \leq 6$ by Lemma 7. This is a contradiction and E is stable. □

3. Proof of Theorem 3

Let A, B and $C \subset S \subset \mathbf{P}^5$ be as in the previous section. $(S, \mathcal{O}_S(A+B))$ is a polarized K3 surface of genus 11. We study $M_S(2, A+B, 5)$ and their restrictions to C, using vector bundles of Schwarzenberger type. Let

$$H^j(S, \mathcal{E}xt^i_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B))) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B)),$$

be the local-global spectral sequence of extension groups, where I_x is the ideal of a point x of S. By Lemma 4, the natural map

$$(1) \qquad \mathrm{Ext}^1_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B)) \longrightarrow H^0(S, \mathcal{E}xt^1_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B))) \simeq \mathbf{C}$$

is an isomorphism. Let

$$(2) \qquad 0 \longrightarrow \mathcal{O}_S(B) \longrightarrow \mathcal{E}_x \longrightarrow I_x(A) \longrightarrow 0.$$

be the unique non-trivial extension of $I_x(A)$ by $\mathcal{O}_S(B)$. Since $H^1(\mathcal{O}_S(B)) = 0$, we have the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_S(B)) \longrightarrow H^0(\mathcal{E}_x) \longrightarrow H^0(I_x(A)) \longrightarrow 0.$$

The following is obvious:

Lemma 8. 1) *$\dim H^0(\mathcal{E}_x) = 7$ and $\mathcal{E}_x$ is generated by global sections.*
2) *$H^i(\mathcal{E}_x) = 0$ for $i = 1, 2$.*

Since the three linear maps
i) $S^2H^0(\mathcal{O}_S(B)) \longrightarrow H^0(\mathcal{O}_S(2B))$,
ii) $H^0(\mathcal{O}_S(B)) \otimes H^0(I_x(A)) \longrightarrow H^0(I_x(A + B))$ and
iii) $S^2H^0(I_x(A)) \longrightarrow H^0(I_x^2(2A))$.
are injective by Lemma 5 and 6, we have

Lemma 9. *The natural linear map $S^2H^0(\mathcal{E}_x) \longrightarrow H^0(S^2\mathcal{E}_x)$ is injective.*

Let E_x be the restriction of the vector bundle $\mathcal{E}_x$ to the curve C. Since $\det \mathcal{E}_x \simeq \mathcal{O}_S(C)$, we have the exact sequence

$$0 \longrightarrow \mathcal{E}_x^\vee \longrightarrow \mathcal{E}_x \longrightarrow E_x \longrightarrow 0.$$

By Lemma 8 and the Serre duality, we have

Lemma 10. *The restriction map $H^0(\mathcal{E}_x) \longrightarrow H^0(E_x)$ is an isomorphism.*

In particular, $\dim H^0(E_x) = 7$ and E_x is generated by global sections. Restricting the above (2) to the curve C, we have the exact sequence

$$(3) \qquad 0 \longrightarrow \xi \longrightarrow E_x \longrightarrow \eta \longrightarrow 0$$

and

$$(4) \qquad 0 \longrightarrow H^0(\xi) \longrightarrow H^0(E_x) \longrightarrow H^0(I_x(A)) \longrightarrow 0$$

if $x \notin C$. If $x \in C$, then $I_x(A) \otimes \mathcal{O}_C$ contains the sky-scraper sheaf $k(x)$ as a subsheaf and the torsion-free quotient is isomorphic to $\eta(-x)$. Hence we have the exact sequence

$$(5) \qquad 0 \longrightarrow \xi(x) \longrightarrow E_x \longrightarrow \eta(-x) \longrightarrow 0.$$

Proposition 3. *E_x is stable for every $x \in S$.*

Proof. Assume that $x \notin C$. Since $h^0(E_x) < h^0(\xi) + h^0(\eta)$ by (4), the exact sequence (3) does not split. Let ζ be a line subbundle of E_x. If the composite $\zeta \longrightarrow E_x \longrightarrow \eta$ is zero, then $\deg \zeta = \deg \xi < 10$. Otherwise ζ is isomorphic to $\eta(-D)$ for a nonzero effective divisor D. If $\deg D = 1$ and $D = (p)$ for a point $p \in C$, then the extension class e of (3) lies in the kernel of $\mathrm{Ext}^1_{\mathcal{O}_C}(\eta, \xi) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\eta(-p), \xi(p))$. Hence the point corresponding to e in the way of Lemma 2 is p. This is a contradiction. Hence we have $\deg D \geq 2$. Since $h^0(E) = 7$, we have $\deg D \geq 5$ by Lemma 7. Therefore E_x is stable.

Claim: The exact sequence (5) does not split.

Assume the contrary and let $s : E_x \longrightarrow \xi(x)$ be a splitting. Let $\mathcal{F}$ be the kernel of the composite $\mathcal{E}_x \longrightarrow E_x \longrightarrow \xi(x)$. $H^0(\mathcal{F})$ is a 5-dimensional subspace of $H^0(\mathcal{E}_x)$ and mapped onto $H^0(I_x(A))$. Since $c_1(\mathcal{F}) = 0$, the image of the evaluation homomorphism $H^0(\mathcal{F}) \otimes \mathcal{O}_S \longrightarrow \mathcal{F} \subset \mathcal{E}$ is of rank one. Hence the image is isomorphic to $I_x(A)$, which is a contradiction.

Assume that $x \in C$ and let ζ be a line subbundle of E_x. We may assume that $\zeta \simeq \eta(-x-D)$ for an effective divisor D. D is nonzero by the claim and E_x is stable by Lemma 7. □

By the proposition, $\mathcal{E}_x$ is also stable (with respect to $\mathcal{O}_S(C)$) and every endomorphism is a constant multiplication. Hence, by the exact sequence

$$0 \longrightarrow sl\ \mathcal{E}_x \longrightarrow S^2\mathcal{E}_x \longrightarrow S^2E_x \longrightarrow 0,$$

the restriction map $H^0(S^2\mathcal{E}_x) \longrightarrow H^0(S^2E_x)$ is injective. By Lemma 9 and 10, we have

Lemma 11. *The natural linear map $S^2H^0(E_x) \longrightarrow H^0(S^2E_x)$ is injective.*

Now we prove Theorem 3. Let T be a copy of S and Δ be the diagonal of $S \times T$. By the Leray spectral sequence, we have an isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_S}(I_\Delta(\sigma^*A), \mathcal{O}_{S\times T}(\sigma^*B))) \longrightarrow \tau_*\mathcal{E}xt^1_{\mathcal{O}_{S\times T}}(I_\Delta(\sigma^*A), \mathcal{O}_{S\times T}(\sigma^*B))) \simeq \mathcal{O}_T(B-A)$$

whose fibres are (1), where σ and τ are the projections of $S \times T$ onto S and T, respectively. Hence we have the universal extension

$$(6) \qquad 0 \longrightarrow \mathcal{O}_{S\times T}(\sigma^*B) \longrightarrow \mathcal{F} \longrightarrow I_\Delta(\sigma^*A + \tau^*(B-A)) \longrightarrow 0,$$

whose restriction to $S \times x$ is (2) for every $x \in T$. The restriction of $\mathcal{F}$ to $C \times T$ is a Poincaré bundle of the family $\{E_x\}_{x\in T}$. By Lemma 8, 10 and Proposition 3, we obtain the classification morphism $B_C : S \longrightarrow M_C(2, K, 5)$. B_C is injective by the following:

Lemma 12. $\dim \mathrm{Hom}(\xi, E_x) = 1$ *for every $x \in S$.*

Proof. Assume the contrary. Then E_x contains two distinct subsheaves isomorphic to ξ. Let L be the subsheaf generated by them. If L is of rank one, then L is isomorphic to $\xi(D)$ for an effective divisor D and $h^0(L) \geq 3$, which contradicts Lemma 7. If L is of rank two, then L is isomorphic to $\xi \oplus \xi$, which contradicts Lemma 11. □

Let E be a member of $M_C(2, K, 5)$. By Proposition 2, there is an exact sequence

$$0 \longrightarrow \xi \longrightarrow E \longrightarrow \eta \longrightarrow 0$$

or

$$0 \longrightarrow \xi(p) \longrightarrow E \longrightarrow \eta(-p) \longrightarrow 0$$

for a point $p \in C$. In the latter case, E is isomorphic to E_p by (1) of Lemma 2 and the claim in the proof of Proposition 3. In the former case, there exists a point $x \in S$ such that the image of $H^0(E) \longrightarrow H^0(\eta)$ is $H^0(I_x(A))$ by (2) of Lemma 2. By the stability of E, x does not belong to C. Hence E is isomorphic to E_x. Therefore, B_C is surjective. $M_C(2, K, 5)$ is smooth and of dimension 2 by Proposition 1 and Lemma 11.

4. Proof of Theorem 1 and 2

Let C be a general curve of genus 11. C does not have a g^3_{10}. Hence we have $h^0(E) \leq 6$ for every strictly semi-stable rank two vector bundle E of canonical determinant on C. Since $M_C(2, K, 5)$'s form a proper family when C varies, they are smooth K3 surfaces by Theorem 3.

Proposition 4. *There exists a vector bundle $\mathcal{U}$ on $C \times M_C(2,K,5)$ such that $\mathcal{U}|_{C\times[E]} \simeq E$ for every $E \in M_C(2,K,5)$ and $\det \mathcal{U} \simeq K_C \boxtimes h_{det}$. Such $\mathcal{U}$, called the normalized Poincaré bundle, is unique up to isomorphism.*

Proof. The moduli space $M_C(2,K)$ is the quotient of an open subset R of a Quot scheme by an action of $PGL(\nu)$. Let R' be the inverse image of $M_C(2,K,5)$ by the quotient morphism $R \longrightarrow M_C(2,K)$ and $\tilde{\mathcal{U}}$ the restriction of the universal quotient bundle. By Proposition 2, $h^0(E) = 7$ for every $E \in M_C(2,K,5)$. Hence the direct image $\pi_{R'*}\tilde{\mathcal{U}}$ is a vector bundle of rank 7. The direct image $\pi_{R'*}(\tilde{\mathcal{U}} \otimes_{\mathcal{O}_C} K_C)$ is of rank 40. Following [12], we consider the vector bundle

$$\tilde{\mathcal{U}} \otimes \pi_{R'}^*((\det \pi_{R'*}\tilde{\mathcal{U}})^{17} \otimes \det(\pi_{R'*}(\tilde{\mathcal{U}} \otimes_{\mathcal{O}_C} K_C))^{-3}).$$

The action of a central element $t \in \mathbf{G}_m \subset GL(\nu)$ on the three factors are t, $t^{7\cdot 17}$ and $t^{-40\cdot 3}$. Hence this tensor product has an action of $PGL(\nu)$ and descends to a Poincaré bundle $\mathcal{U}$ on $C \times M_C(2,K,5)$.

Since $M_C(2,K,5)$ is regular, there exists a line bundle L on $M_C(2,K,5)$ such that $\det \mathcal{U} \simeq K_C \boxtimes L$. Since $\mathcal{U}^\vee \otimes_{\mathcal{O}_C} K_C \simeq \mathcal{U} \otimes_{\mathcal{O}_M} L^{-1}$, we have

$$(R^1\pi_{M*}\mathcal{U})^\vee \simeq \pi_{M*}(\mathcal{U}^\vee \otimes K_C) \simeq (\pi_{M*}\mathcal{U}) \otimes L^{-1}$$

by the (relative) Serre duality. Hence we have

$$h_{det} = (\det R^1\pi_{M*}\mathcal{U}) \otimes (\det \pi_{M*}\mathcal{U})^{-1} \simeq L^7 \otimes N^{-2}, \tag{7}$$

where we put $N = \det \pi_{M*}\mathcal{U}$. Therefore, the universal bundle $\mathcal{U} \otimes_{\mathcal{O}_M} (L^3 \otimes N^{-1})$ satisfies our requirement (cf. [1] p. 582). The normalized Poincaré bundle is unique since universal bundles differ only by $\operatorname{Pic} M(2,K,5)$ and the Picard group has no 2-torsion. □

The determinant line bundle h_{det} is ample by [2]. We compute its degree. In the hexagonal case, $\mathcal{F}|_{C\times T}$ is a universal family. Since $\det(\mathcal{F}|_{C\times T}) \simeq K_C \boxtimes \mathcal{O}(B-A)$ by the exact sequence (6), we have $h_{det} \simeq \mathcal{O}_T(7(B-A)) \otimes (\det \tau_*(\mathcal{F}|_{C\times T}))^{-2}$ by (7). By Lemma 10 and (6), we have the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_S(B)) \otimes \mathcal{O}_T \longrightarrow \tau_*(\mathcal{F}|_{C\times T}) \longrightarrow H^0(\mathcal{O}_S(A)) \otimes \mathcal{O}_T(B-A) \longrightarrow \mathcal{O}_T(B) \longrightarrow 0$$

and $\det(\tau_*\mathcal{F}|_{C\times T}) \simeq \mathcal{O}_T(5B-6A)$. Hence h_{det} is isomorphic to $\mathcal{O}_T(5A-3B)$, which is a line bundle of degree 20. So we have proved Theorem 1.

For a hexagonal curve C, $\mathcal{U} = (\mathcal{F}|_{C\times T}) \otimes \mathcal{O}_T(3A-2B)$ is the normalized Poincaré bundle. The restriction $\mathcal{U}_p$ is stable for every $p \in C$ by Proposition 3 and α is an embedding by Theorem 3. Since the properties to be shown are stable by small deformations, we have Theorem 2.

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