A generalization of Mumford's example (joint work with H. Nasu)

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Let $\mathrm{Hilb}^{sc} V$ be the Hilbert scheme parametrizing smooth curves in a smooth projective variety V. In [3], Mumford showed that $\mathrm{Hilb}^{sc}\mathbb{P}^3$ has a generically non-reduced component. More precisely the following is proved:

Example A Let S be a smooth cubic surface in $\mathbb{P}^3$, E a (-1)-$\mathbb{P}^1$ in S and $C \subset S$ a smooth member of the linear system $|4h + 2E| \simeq \mathbb{P}^{37}$ on S. (C is of degree 14 and genus 24.) Such space curves C are parametrized by $W^{56} \subset \mathrm{Hilb}^{sc}\mathbb{P}^3$, an open subset of a $\mathbb{P}^{37}$-bundle over $|3H| \simeq \mathbb{P}^{19}$. Here H is a plane in $\mathbb{P}^3$ and h is its restriction to S. Then W^{56} is an irreducible component of $(\mathrm{Hilb}^{sc}\mathbb{P}^3)_{red}$ and $\mathrm{Hilb}^{sc}\mathbb{P}^3$ is nowhere reduced along W^{56}.

It is well known that every infinitesimal (embedded) deformation of $C \subset V$ is unobstructed if $H^1(N_{C/V}) = 0$. Conversely we find a sufficient condition for a first order infinitesimal deformation of a curve C in a 3-fold V to be obstructed, abstracting an essence from the arguments in [1] and [4]. As application we construct generically non-reduced components of the Hilbert schemes of uniruled 3-folds V including Examples A and B as special cases:

Example B ([2]) *Let V_3 be a smooth cubic 3-fold in $\mathbb{P}^4$, S its general hyperplane section, E a (-1)-$\mathbb{P}^1$ in S and $C \subset S$ a smooth member of $|2h+2E| \simeq \mathbb{P}^{12}$. ($C$ is of degree 8 and genus 5.) Such curves C in V_3 are parametrized by $W^{16} \subset \mathrm{Hilb}^{sc}\ V$, an open subset of $\mathbb{P}^{12}$-bundle over the dual projective space $\mathbb{P}^{4,\vee}$. Then W^{16} is an irreducible component of $(\mathrm{Hilb}^{sc} V_3)_{red}$ and $\mathrm{Hilb}^{sc} V_3$ is nowhere reduced along W^{16}.*

The curves C of genus 24 in Example A are not (moduli-theoretically) general but the curves C of genus 5 in Example B are general. Hence, with the help of Sylvester's pentahedral theorem ([5]), Example B gives a counterexample to the following problem:

Problem 1 Is every component of the Hom scheme $\mathrm{Hom}(X, V')$ generically smooth for a smooth curve X with general modulus and for a general member V' in the Kuranishi family of V?

Let $\mathrm{Hom}_8(X_5, V_3)$ be the Hom scheme of morphisms of degree 8 from a curve X_5 of genus 5 with general modulus to a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$.

Theorem ([2]) *If V_3 is also moduli-theoretically general, then $\mathrm{Hom}_8(X_5, V_3)$ has a generically non-reduced component of expected dimension ($= 4$).*

The following seems still open:

Problem 2 Let G/P be a projective homogeneous space, *e.g.*, a Grassmann variety and X a curve with general modulus. Is every component of $\mathrm{Hom}(X, G/P)$ generically smooth?

The answer is affirmative for the projective space $\mathbb{P}^n$ by virtue of Gieseker's theorem (= Petri's conjecture).

References

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