Moduli of abelian surfaces, and regular polyhedral groups

Shigeru MUKAI *

May, 2003

Abstract : Let $\mathcal{A}_d$ be the moduli space of polarized abelian surfaces of type $(1, d)$. For $d = 2, 3$ and 4, the Satake compactification of $\mathcal{A}_d$ is isomorphic to the quotient of $\mathbf{P}^3$ by an action of $PSL(2, \mathbf{Z}/d) \times PSL(2, \mathbf{Z}/d)$. Let $PSL(2, \mathbf{Z}/5) =: G_5 \subset PGL(2)$ be the icosahedral group and $PGL(2) \subset \mathbf{P}^3$ the natural embedding into the projective space of 2×2 matrices. A small resolution of the Satake compactification of $\mathcal{A}_5$ (at the poit cusps) is isomorphic to the quotient of the blow-up $\tilde{\mathbf{P}}^3$ at the 60 points ($\simeq G_5$) by an action of $G_5 \times G_5$.

Let $X(d)$ be the moduli space of elliptic curves E with a full level d structure, *i.e.*, a symplectic isomorphism between the standard symplectic module

$$2[\mathbf{Z}/d)] := \left(\mathbf{Z}/d \oplus \mathbf{Z}/d, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

and the group E_d of d-torsion points with the Weil pairing. The modular curve $X(d)$ is rational if and only if $d \leq 5$. In particular, the finite group $PSL(2, \mathbf{Z}/d)$ is a regular polyhedral group $G_d \subset PGL(2)$ for $d = 2, 3, 4$ and 5. The compactified modular curve $\overline{X(d)}$ is identified with the circumscribing Riemann sphere of the regular polyhedron P_d with t vertices, where t is the number of cusps. The order of G_d is equal to dt.

d	2	3	4	5
P_d	triangle	tetrahedron	octahedron	icosahedron
t	3	4	6	12
G_d	S_3	A_4	S_4	A_5

Regular polyhedral groups are also closely related with Hilbert modular surfaces of small discriminant.

*Supported in part by the JSPS Grant-in-Aid for Exploratory Research ? and ? (Germann side) ?.

Example Let $\mathcal{O}_{\sqrt{5}}$ be the ring of integers of the quadratic field $\mathbf{Q}(\sqrt{5})$ and put

$$\Gamma = SL(2, \mathcal{O}_{\sqrt{5}} : \sqrt{5}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1_2 \bmod \sqrt{5},\ a, b, c, d, \in \mathcal{O}_{\sqrt{5}},\ ad - bc = 1 \right\}.$$

Then Γ acts on the product $H \times H$ of two copies of the upper half planes. Let $\tilde{\Gamma}$ be the group generated by Γ and the switch involution of $H \times H$. Then the Hilbert modular surface $Y_\Gamma := \overline{\tilde{\Gamma} \backslash H \times H}$ added with 6 point cusps is isomorphic to the projective plane $\mathbf{P}^2$. This Y_Γ has an action of the icosdahedral group G_5. Moreover, $X_\Gamma := \overline{\Gamma \backslash H \times H}$ is the double cover of Y_Γ with branch a G_5-invariant plane curve of degree 10.

In the 3-dimensional case the wreath product $2||G_d$ plays the role of G_d. Let $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ be the Segre embedding. The ambient space is the projectivization of the space of 2 by 2 matrices and the quadric $\mathbf{P}^1 \times \mathbf{P}^1$ parametrizes the rank one marices (modulo constant multiplication). Hence the complement is naturally identified with $PGL(2)$. This $\mathbf{P}^3$ is an equivariant compactification of the algebraic group $PGL(2)$ and the polyhedral group G_d acts on it from both sides. Let τ be the involution of this $\mathbf{P}^3$ interchanging $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its cofactor matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The fixed locus is the union of the constant matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and the traceless ones $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. τ interchanges the two factors of $\mathbf{P}^1 \times \mathbf{P}^1$. So the bipolyhedral group $2||G_d$ acts on $\mathbf{P}^3$.

For a polarized abelian surface (X, L) of type $(1, d)$, a (symplectic) isomorphism α between the standard symplectic module $2[\mathbf{Z}/d]$ and the group

$$K(L) := \{x \in X \mid T_x^* L \simeq L\}$$

with the Weil pairing is called a *canonical level structure.* By a canonical *colevel* structure, we mean a (symplectic) isomorphism between $2[\mathbf{Z}/d]$ and the quotient group $X_d/K(L)$, which has a symplectic structure as the complement of $K(L)$ in X_d. We denote the moduli space of polarized abelian surfaces with canonical level and colevel structure by $\mathcal{A}(1, d)$ and $\mathcal{A}(d, 1)$, respectively. The forgetful morphisms

$$\mathcal{A}(1, d) \longrightarrow \mathcal{A}_d \quad \text{and} \quad \mathcal{A}(d, 1) \longrightarrow \mathcal{A}_d$$

are both Galois coverings with Galois group $PSL(2, \mathbf{Z}/d)$. The fibre product

$$\mathcal{A}_d^{wbl} := \mathcal{A}(1, d) \times_{\mathcal{A}} \mathcal{A}(d, 1)$$

is called the moduli space of abelian surfaces with a *weak bilevel structure.*

For a polarized abelian surface (X, L) of type $(1, d)$, its dual $\hat{X}$ has a natural polarization $\hat{L}$ of the same type such that $\phi_{\hat{L}} \phi_L = d_X$. The colevel structure of (X, L) is equivalent to the level structure of its dual $(\hat{X}, \hat{L})$ and vice versa. Therefore, the moduli space $\mathcal{A}_d^{w\,bl}$ has an action of wreath product $2||PSL(2, \mathbf{Z}/d)$.

Remark 1 The moduli space $\mathcal{A}_d$ is the quotient of the Sigel upper half space of degree 2 by the full paramodular group

$$1_4 + \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & d\mathbf{Z} \\ d\mathbf{Z} & \mathbf{Z} & d\mathbf{Z} & d\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & d\mathbf{Z} \\ \mathbf{Z} & \frac{1}{d}\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{pmatrix} \cap Sp_4(\mathbf{Q}).$$

A pair (α, β) of isomorphisms $\alpha : 2[\mathbf{Z}/d] \xrightarrow{\sim} K(L)$ and $\beta : 2[\mathbf{Z}/d] \xrightarrow{\sim} K(\hat{L})$ is called a *canonical bilevel structure* of (X, L). The moduli space $\mathcal{A}_d^{bl}$ of polarized abelian surfaces of type $(1, d)$ with bilevel structure (X, L, α, β) is the quotient by the subgroup

$$1_4 + \begin{pmatrix} d\mathbf{Z} & & d\mathbf{Z} & d\mathbf{Z} \\ d\mathbf{Z} & d\mathbf{Z} & d\mathbf{Z} & d^2\mathbf{Z} \\ d\mathbf{Z} & & d\mathbf{Z} & d\mathbf{Z} \\ & & & d\mathbf{Z} \end{pmatrix} \cap Sp_4(\mathbf{Z}).$$

The moduli space $\mathcal{A}_d^{w\,bl}$ is the quotient of $\mathcal{A}_d^{bl}$ by the involution

$$(X, L, \alpha, \beta) \mapsto (X, L, \alpha, -\beta),$$

which corresponds to the element

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in Sp_4(\mathbf{Q}).$$

Theorem (1) *For $d = 2, 3$ and 4, the Satake compactification of $\mathcal{A}_d^{w\,bl}$ is $(2||G_d)$-equivariantly isomorphic to the projective 3-space $\mathbf{P}(M_{2\times 2}\ \mathbf{C})$.*

(2) *There exists a $(2||G_5)$-equivariant morphism*

$$\psi : \tilde{\mathbf{P}}^3 \longrightarrow \overline{\mathcal{A}}_d^{w\,bl}$$

onto the Satake compactificaiton and ψ contracts the strict transforms of the 72 special lines (see below) to the 72 point cusps, where $\tilde{\mathbf{P}}^3$ is the blow-up of $\mathbf{P}^3$ with center G_5. (The normal bundles of the strict transforms are isomorphic to $\mathcal{O}(-4) \oplus \mathcal{O}(-4)$.) ψ is an isomorphism elsewhere. Moreover, the exceptional divisors over the 60 points G_5 are the Hilbert modular surface Y_Γ in Example and parametrize the Comesatti surfaces, i.e., *abelian surfaces with real multiplication by $\mathcal{O}_{\sqrt{5}}$.*

(3) *In both cases (1) and (2),* $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}(M_{2\times 2}\ \mathbf{C})$ *parametrizes the products ow two elliptic curves (of degree* 1 *and* d*).*

Let $p_1, \ldots, p_t$ be the cusps of the elliptic modular curve $\overline{X(d)}$, $d = 2, 3, 4, 5$. Then by the theorem, the $2t$ lines $p_i \times \mathbf{P}^1$ and $\mathbf{P}^1 \times p_i$, $1 \le i \le t$, on $\mathbf{P}^1 \times \mathbf{P}^1$ are 1-dimensional (Satake) boundaries of $\mathcal{A}_d^{w\,bl}$. $\mathcal{A}_2^{w\,bl}$ and $\mathcal{A}_3^{w\,bl}$ are the complement of these $2t$ lines in $\mathbf{P}^3$. In order to describe $\mathcal{A}_4^{w\,bl}$ and $\mathcal{A}_5^{w\,bl}$, we need the following:

Definition A line in $\mathbf{P}^3$ joining two points $[g_1]$ and $[g_2]$ of $G_d \subset PGL(2)$ is *special* if $g_1 g_2^{-1} \in G_d$ is of order d.

The number of special lines is iqual to 9, 16, 18 and 72 for $d = 2, 3, 4$ and 5. In the case $d = 2, 3$, the special lines parametrize the polarized abeian surfaces (X, L) which have symplectic automorphism of order d.

Proposition (1) *The moduli space* $\mathcal{A}_4^{w\,bl}$ *is the complement of* 12 *lines* $p_i \times \mathbf{P}^1$, $\mathbf{P}^1 \times p_i$ *and the* 18 *special lines in* $\mathbf{P}^3$*.*

(2) *The moduli space* $\mathcal{A}_5^{w\,bl}$ *is the complement of the strict trransform of* 12 *lines* $p_i \times \mathbf{P}^1$, $\mathbf{P}^1 \times p_i$ *and the* 72 *special lines in the blow-up* $\tilde{\mathbf{P}}^3$*.*

Remark 2 Let K_4 be the klein's subgroup of the octahedral group G_4. The action of $K_4 \times K_4$ on $\mathbf{P}^3$ is the projectivization of the Schrödinger representation of the Heisenberg group. Each of the 15 involutions in $K_4 \times K_4$ has the union of two skew lines as fixed point locus. The 18 and 12 lines in (1) of the proposition coincide with these 30 fixed lines.

Acknowledgement This note is essentially [?]. But a confusion among the bilevel structure and the weak one is corrected. This was found in the discussion of the author with Prof. Klaus Hulek in his stay at Universität Hannover in 2003. He is very grateful for its hospitality.

References

[1] Aluffi, P. and Faber, C.:Linear orbits of d-tuples of points in $\mathbf{P}^1$, J. f. Reine Angew. Math., **445** (1993), 205–220.

[2] Freitag, E.: Zur theorie der Modulformen zweiten Grades, Göttingen Nachrichten, **11**(1965).

[3] — : Modulformen zweiten Grades zum rarionalen und Gaussen Zhalkörper, Sitz. der Heidelberger Akad. Wiss. (1967), 3-49.

[4] Geer, G. van der : *Hilbert modular surface*, Springer Verlag.

[5] Gritsenko, V.A. and Nikulin, V.V. : Automorphic forms and Lorentzian Kac-Moody Algebras, Part II, Internat. J. Math. **9**(1998), 201-275.

[6] Gundlach, K.B.: Funktionen zur Modulgruppe von $\mathbf{Q}(\sqrt{5})$, Math. Ann. **152**(1963), 226-256.

[7] Horrocks, G. and Mumford, D.: A rank 2 vector bundles on $\mathbf{P}_4$ with $15,000$ symmetries, Topology, **12** (1973), 63-81.

[8] Hulek, K., Kahn, C. and Weitraub, S.H. : *Moduli Spaces of Abelian Surfaces: Compactification, degenerations, and Theta Functions*, de Gruyter, Berlin, 1993.

[9] Ibukiyama, T. and Onodera, F.: On the graded ring of modular forms of the Siegel paramodular group of level 2, Abh. Math. Sem. Univ. Hamburg **67**(1977), 297-305.

[10] Igusa, J.: On Siegel modular forms of genus two, Amer. J. Math. **84**(1962), 175-199.

[11] Mukai, S.: New developments in the theory of Fano 3-folds: Vector bundle method and moduli problem, Sugaku, **47**(1995), 125–144.: English translation, Sūgaku Expositions, Amer. Math. Soc., ?(2003), ?–?.

[12] —: Moduli of abelian surfaces, and regular polyhedral groups, in '*Moduli of Algebraic Varieties and The Monster*', Proceedings, Sapporo, January 1999, ed. I. Nakamura, Hokkaido Univ., 1999, pp. 68-74.

[13] — and Umemura, H.: Minimal rational threefolds, in *Algebraic Geometry*, Proceedings, Tokyo/Kyoto 1982, Lecture Notes in Math. **1016**, Springer-Verlag, 1983, pp. 490–518.

[14] Sarti, A.: Pencils of symmetric surfaces in $\mathbf{P}^3$, J. Algebra, **246**(2001), 429–452.

Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa Oiwake-chō, Sakyō-ku
Kyoto 606-8502, Japan
e-mail address : mukai@kurims.kyoto-u.ac.jp