◎ Operator Algebra

ω ultrafilter $\quad \lim_\omega : l_\infty \mathbb{N} \to \mathbb{C}$ non-trivial character

- linear
- multiplicative
- $\lim_\omega a_n \in \overline{\mathrm{conv}}\{a_n : n \geq N\}$ for $\forall N$

R hyperfinite II_1 factor w/ a trace τ, $\tau(1) = 1$

$\prod R = \{(x_n)_{n=1}^\infty : x_n \in R, \sup \|x_n\| < +\infty\}$

$\tau_\omega : \prod R \to \mathbb{C}$, $\tau_\omega((x_n)_{n=1}^\infty) = \lim_\omega \tau(x_n)$

$N_{\tau_\omega} := \{(x_n)_{n=1}^\infty : \tau_\omega((x_n^* x_n)_n) = 0\}$

$R^\omega = \prod R / N_{\tau_\omega}$ von Neumann alg w/ a trace τ_ω.

① Connes '76 $\quad \forall$ sep II_1 factor $\hookrightarrow R^\omega$?
(or equivalently $\hookrightarrow \prod M_n/\omega$)

② Kirchberg $\quad C^*\mathbb{F}_\infty \otimes_{alg} C^*\mathbb{F}_\infty$ unique C^*-norm ?
(← universal C^*-alg generated by countably many unitary elements)
(companion results · $C^*\mathbb{F}_\infty \otimes_{alg} B(\ell_2)$ has unique C^*-norm
· $B(\ell_2) \otimes_{alg} B(\ell_2)$ has several (Junge-Pisier)

③ $\forall$ noncommutative L^1-space M_* is finitely representable in $S_1 = B(\ell_2)_*$

<u>Thm</u> (Kirchberg '93) These 3 conjectures are equivalent.

A C^*-alg $\quad \tau$ trace $\quad \tau(1) = 1$

$\forall a \quad \|a\|_2 := \tau(a^*a)^{1/2} \rightsquigarrow L^2(A, \tau) \ni \hat{a}$

$\|ax\|_2 = \tau(x^* a^* a x)^{1/2} \leq \tau(x^* \|a\|^2 x)^{1/2} = \|a\| \|x\|_2$

Similarly $\|xa\|_2 \leq \|x\|_2 \|a\|$ because τ is tracial

$\pi_\tau : A \to \mathbb{B}(L^2(A,\tau))$ "left action"

$\pi_\tau(a)\hat{x} = \widehat{ax}$

Also $\pi_\tau^{op} : A^{op} \to \mathbb{B}(L^2(A,\tau))$ right (or opposite) action

$\pi_\tau^{op}(a^{op})\hat{x} = \widehat{xa}$

$\pi_\tau \times \pi_\tau^{op} : A \underset{alg}{\otimes} A^{op} \to \mathbb{B}(L^2(A,\tau))$ $*$-hom

$\mu_\tau(z) := \langle \pi_\tau \times \pi_\tau^{op}(z)\,\hat{1}, \hat{1} \rangle$ for $z \in A \underset{alg}{\otimes} A^{op}$

$\mu_\tau(a \otimes b^{op}) = \langle \widehat{ab}, \hat{1} \rangle = \tau(ab)$

(If $A \subseteq \mathbb{B}(\mathcal{H})$, then $\underset{min}{\otimes}$-norm on $A \otimes A^{op}$ is induced from $A \otimes A^{op} \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{H}^{op})$. (Indep of choices of $A \subseteq \mathbb{B}(\mathcal{H})$))

Fact: $\pi_\tau \times \pi_\tau^{op}$ is continuous w.r.t. $\underset{min}{\otimes}$-norm

$\Leftrightarrow \mu_\tau$ ———— .

∵ $\hat{1}$ is a cyclic vector.

Thm (Connes, Kirchberg) For (A,τ) TFAE — τ is called amenable.

(i) $\exists\, \varphi_n : A \to \mathbb{M}_{k(n)}$ unital completely positive s.t.
- $\mathrm{tr} \circ \varphi_n \to \tau$
- $\| \varphi_n(ab) - \varphi_n(a)\varphi_n(b) \|_2 \to 0$

(ii) $\mu_\tau : A \otimes A^{op} \to \mathbb{C}$ $\underset{min}{\otimes}$-conti

(iii) τ extends to an A-central state on $\forall$ superalg of A.

Pf of (i) $\Rightarrow$ (ii) $\mu_{tr} \circ (\varphi_n \otimes \varphi_n^{op})$ conti on $A \underset{min}{\otimes} A^{op}$ — contractive

$\downarrow$

μ_τ contractive

┘

Cor For (B,τ),

$$\pi_\tau(B)'' \underset{\text{(trace preserving)}}{\hookrightarrow} \prod \mathbb{M}_n/\omega \iff \tilde{\tau}: C^*\mathbb{F}_\infty \twoheadrightarrow B \xrightarrow{\tau} \mathbb{C} \text{ amenable}$$

Pf. $(\Rightarrow)$

$$C^*\mathbb{F}_\infty \twoheadrightarrow B \subseteq \prod \mathbb{M}_n/\omega \xrightarrow{\tau_\omega} \mathbb{C}, \quad C^*\mathbb{F}_\infty \dashrightarrow \prod \mathbb{M}_n \twoheadrightarrow \prod \mathbb{M}_n/\omega$$

By universality $\exists\, \varphi = (\varphi_n)_{n=1}^\infty : C^*\mathbb{F}_\infty \to \prod \mathbb{M}_n$ $*$-hom lift

$(\Leftarrow)$ Given $\varphi_n : C^*\mathbb{F}_\infty \to \mathbb{M}_{k(n)}$ approx multiplicative,

$$\varphi_\omega : C^*\mathbb{F}_\infty \to \prod \mathbb{M}_{k(n)}/\omega \quad *\text{-hom}$$

$$\varphi_\omega(C^*\mathbb{F}_\infty) \cong \pi_\tau(B). \quad ▥$$

This implies Kirchberg $\Rightarrow$ Connes.

⊙ Connes's Embedding Problem for group vN algs.

$$L\Gamma = (\mathbb{C}\Gamma)'' \subseteq B(\ell_2\Gamma) \quad \leftarrow \text{SOT closure}$$

$$\tau(x) := \langle x\delta_e, \delta_e\rangle \quad \text{trace}, \quad \langle x\delta_e,\delta_e\rangle = \hat{x}(1)$$

$\Gamma \hookrightarrow \mathcal{U}(R^\omega)$ algebraically (unitary group)

$\Updownarrow$ Fact

Def Γ (countable discrete) hyperlinear $\overset{\text{def}}{\iff} L\Gamma \hookrightarrow R^\omega$ (or $\prod \mathbb{M}_n/\omega$)

Lem Γ HL $\iff$ Γ has "microstates"

$\forall E \Subset \Gamma\ \forall \varepsilon > 0$
$\exists n\ \exists \pi : \Gamma \to \mathcal{U}(n)$ (unitary group of $\mathbb{M}_n$)

s.t.
- $\forall s,t \in E$
  $\|\pi(st) - \pi(s)\pi(t)\|_2 < \varepsilon$
- $\forall s \in E$
  $s \neq 1$: $|\tau(\pi(s))| < \varepsilon$

$\tau = \mathrm{tr}$ on $\mathbb{M}_n$
$\tau(1) = 1$
$\|x\|_2 = \tau(x^*x)^{1/2}$

Proof. Straightforward. ▥

"sofi" means finite in Hebrew.

HL

$\Uparrow$

Def Γ sofic $\overset{\text{def}}{\Longleftrightarrow}$ the same definition as HL but replacing $\mathcal{U}(n)$ with $\mathfrak{S}_n$:

$\forall E \Subset \Gamma \ \forall \varepsilon > 0$

$\exists n \ \exists \pi : \Gamma \to \mathfrak{S}_n$

s.t. • $\forall s, t \in E$

$\tau(\pi(st)^{-1}\pi(s)\pi(t)) > 1 - \varepsilon$.

• $\forall s \in E$, $s \neq 1$, $\tau(\pi(s)) < \varepsilon$.

For $p \in \mathfrak{S}_n \subseteq \mathbb{M}_n$

$\tau(p) = \frac{1}{n}\#$ fixed pts of p

$\Longleftrightarrow \Gamma \hookrightarrow \prod \mathfrak{S}_n / N_\tau$

where $N_\tau = \{(p_n)_{n=1}^{\infty} \in \prod \mathfrak{S}_n : \lim_\omega \tau(p_n) = 1\} \trianglelefteq \prod \mathfrak{S}_n$.

◎ Examples & Permanence Properties

Example • Residually finite groups → In particular free groups are sofic.

$\Gamma \trianglerighteq N_n \quad N_1 \supseteq N_2 \supseteq \cdots \quad \bigcap N_n = \{1\}$

finite index

$\rightsquigarrow \pi_n : \Gamma \to \Gamma / N_n$

• Amenable groups

$\Gamma \quad \exists F_n \Subset \Gamma$ Følner sequence

$$\forall s \in \Gamma \quad \frac{|sF_n \,\Delta\, F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 0$$

$\rightsquigarrow \pi_n : \Gamma \to \mathfrak{S}_{F_n}$

$\pi_n(s)x = sx$ if $sx \in F_n$

OPEN PROBLEM: Is every group sofic?

(My feeling: Many counterexamples)

History Gottschalk's conjecture 1973
$\forall \Gamma$ is "surjunctive":
For $\varphi : \{1,\dots,n\}^{\Gamma} \to \{1,\dots,n\}^{\Gamma}$ continuous Γ-equivariant
surjectivity implies injectivity.
Thm (Gromov, Weiss) sofic $\Rightarrow$ surjunctive

Permanence Properties of sofic / HL groups
subgroups, directed union, direct product, free product, quotient by a finite normal subgroup, amenable extension, limit in the space of marked groups

$$\Big(\underset{\text{sofic}}{\Lambda} \trianglelefteq \Gamma, \ \underset{\text{amenable}}{\Gamma/\Lambda} \Rightarrow \underset{\text{sofic}}{\Gamma} \Big)$$

For $d \in \mathbb{N}$, $\mathcal{G}_d := \{(\Gamma; g_1, \dots, g_d) : \Gamma$ is a group generated by $g_1, \dots, g_d\}$

$$\begin{pmatrix} \mathbb{F}_d \twoheadrightarrow \Gamma \\ f_i \mapsto g_i \end{pmatrix} \longrightarrow \ \cong \{N : N \trianglelefteq \mathbb{F}_d\}$$

$\Gamma_n \to \Gamma$ in $\mathcal{G}_d \overset{\text{def}}{\Longleftrightarrow} \forall r \in \mathbb{N}$ the r-balls of the colored Cayley graphs of Γ_n eventually coincide with that of Γ

$\Longleftrightarrow \forall w \in \mathbb{F}_d \quad w = 1$ in Γ iff $w = 1$ in Γ_n eventually

Example ① $\Gamma \trianglerighteq N_n \quad N_1 \supseteq N_2 \supseteq \cdots \quad \bigcap N_n = \{1\}$
$\rightsquigarrow \Gamma_n = \Gamma/N_n \longrightarrow \Gamma$ in $\mathcal{G}_d$.

② $\Gamma = \langle S \mid r_1, r_2, \dots \rangle$ infinitely presented
$\Gamma_n = \langle S \mid r_1, \dots, r_n \rangle \longrightarrow \Gamma$ in $\mathcal{G}_d$

Q Consider the smallest class $\ne \emptyset$ of groups which is closed under the above permanence properties.
Is there $\forall$ group which does not belong to it?

Some exotic examples

<u>Thm</u> (Thom '08) $\exists$ sofic, (T), $\neg$RF

☹ ← limit of finite groups in $\mathcal{G}_d$

<u>Thm</u> (Cornulier '09) $\exists$ sofic, isolated in $\mathcal{G}_d$, non-amenable

☹ ← (locally-RF)-by-abelian

<u>Q.</u> Hyperbolic groups? One relator groups?

<u>Prop</u> (Elek-Szabo) $\Lambda \le \Gamma$ coamenable subgroup (i.e. $\exists F_n \Subset \Gamma/\Lambda$ approx Γ-invariant)

Λ sofic / HL $\Rightarrow$ Γ sofic / HL.

<u>Pf.</u>

$\sigma : \Gamma/\Lambda \to \Gamma$ lift $\qquad \alpha : \Gamma \times \Gamma/\Lambda \to \Lambda$

$\alpha(s, g) := \sigma(sg)^{-1} s\, \sigma(g)$

associated cocycle

$E \Subset \Gamma$, $\varepsilon > 0$ given. Choose a Følner set $F \Subset \Gamma/\Lambda$

For $\alpha(E, F) \Subset \Lambda$ & $\varepsilon > 0$, choose a sofic approx $\rho : \Lambda \to \mathfrak{S}_X$

Define $\pi : \Gamma \to \mathfrak{S}_{X \times F}$ by

$$\pi(t)(x, g) = (\rho(\alpha(t, g))x,\ tg)$$

whenever $tg \in F$.

Then for $s, t \in E$, one has

$$\pi(s)\pi(t)(x, g) = (\rho(\alpha(s, tg))\,\rho(\alpha(t, g))x,\ stg)$$

for most of $g \in F$

$$= (\rho(\alpha(s, tg)\alpha(t, g))x,\ stg)$$

for most of $x \in X$

$$= (\rho(\alpha(st, g))x,\ stg)$$

$$= \pi(st)(x, g).$$

Also $\tau(\pi(s)) \approx 0$ for $s \in E$. ▨

In particular, when N is central

Prop (Thom) $N \trianglelefteq \Gamma$ $\exists F_n \Subset N$ approx N-invariant & approx $\mathrm{Ad}\Gamma$-invariant

Γ HL $\Rightarrow \Gamma/N$ HL

(Rem: Not known for sofic case)

Pf. • Γ HL $\iff \lambda_\Gamma \times \rho_\Gamma : C^*\mathbb{F}_\infty \otimes_{alg} C^*\mathbb{F}_\infty \to \mathbb{B}(\ell_2\Gamma)$ $\otimes_{min}$ conti

• If N is as above, then $\lambda_{\Gamma/N} \times \rho_{\Gamma/N}$ is weakly contained in $\lambda_\Gamma \times \rho_\Gamma$. ▨

⊙ Application to Algebra

Def A unital ring R is direct finite if $ab=1$ implies $ba=$
It is stably direct finite if $\forall n$ $\mathbb{M}_n(R)$ is direct finite.

For a C^*-alg, direct finite $\iff$ no proper isometry (i.e. finite)

∵ Polar decomposition.

Thm $\mathbb{C}\Gamma$ is stably direct finite

Pf. $\mathbb{C}\Gamma \hookrightarrow L\Gamma$ finite vN alg. ▨

Kaplansky's Conjecture: $K\Gamma$ is (stably) direct finite for $\forall$ field K.

The ~~first~~ application of sofic groups: next to surjunctivity

Thm (Elek-Szabo) Sofic $\Rightarrow$ Kaplansky's Conj.

<u>Pf.</u> $\pi_n : \Gamma \to \mathfrak{S}_n$ sofic approximation

$$K\Gamma \to K\mathfrak{S}_n \subseteq \mathbb{M}_n(K)$$

<u>Lem.</u> $\operatorname{rank}(1-ab) = \operatorname{rank}(1-ba)$ in $\mathbb{M}_n(K)$

$\because$ Let $\xi_1, \cdots, \xi_d$ be a basis of $\ker(1-ab)$.
Then $b\xi_i$ linearly indep ($\because$ $ab\xi_i = \xi_i$),
and $(1-ba)b\xi_i = 0$.
$\rightsquigarrow$ $\dim_K \ker(1-ab) \leq \dim_K \ker(1-ba)$. ▨

$$\rho(a) := \frac{1}{n}\operatorname{rank}(a) \quad \text{for } a \in \mathbb{M}_n(K)$$

$$\rho_\omega : \prod \mathbb{M}_n(K) \longrightarrow [0,1]$$
$$(a_n) \longmapsto \lim_\omega \rho(a_n)$$

$\ker \rho_\omega$ is an ideal of $\prod \mathbb{M}_n(K)$.

$\pi_\omega : K\Gamma \longrightarrow \prod \mathbb{M}_n(K)/\ker\rho_\omega$ homomorphism

$\prod \mathbb{M}_n(K)/\ker\rho_\omega$ is stably direct finite

$\because$ $a = (a_n)_{n\in\omega}$, $b = (b_n)_{n\in\omega}$, $ab = 1$

$$\Rightarrow \rho_\omega(1-ba) = \lim_\omega \rho(1-b_n a_n)$$
$$= \lim_\omega \rho(1-a_n b_n)$$
$$= \rho_\omega(1-ab) = 0. \ ▨$$

Claim ρ_ω is injective.

$\because$ Let $f \underset{\neq 0}{\in} K\Gamma$, $E := \operatorname{supp} f$

n fixed.

Take a maximal subset $X \subseteq \{1, \cdots, n\}$

s.t. for $\forall s, t \in E$ and $\forall x, y \in X$

either $s \neq t$ or $x \neq y$ $\Rightarrow \pi_n(s)x \neq \pi_n(t)y$.

$\rightsquigarrow \ker \pi_n(f) \cap KX = \{0\}$

$\rightsquigarrow \rho(\pi_n(f)) \geq |X|/n$.

Estimate of $|X|$.

Pick $x \notin X$. By maximality of X,

either $\exists s, t \in E$ $s \neq t$ and $\pi_n(s)x = \pi_n(t)x$,

$\rightsquigarrow$ rare event.

or

$\exists s, t \in E$ $\exists y \in X$ s.t. $\pi_n(s)x = \pi_n(t)y$.

$$\rightsquigarrow x = \pi_n(s)^{-1}\pi_n(t)y \in \pi_n(E)^{-1}\pi_n(E)X$$

$\rightsquigarrow |X^c| \lesssim |E|^2 |X|$

$\rightsquigarrow |X| \gtrsim \frac{1}{1+|E|^2} n$

$\rightsquigarrow \rho_\omega(f) = \lim_\omega \rho(\pi_n(f)) \geq \frac{1}{1+|E|^2} > 0$ ▨

Consequently $K\Gamma \hookrightarrow \underbrace{\prod M_n(K)/\ker \rho_\omega}$

$\in$ stably direct finite ▨

⓪ L^2-Theory $\quad \mathbb{Z}\Gamma \subseteq L\Gamma$

X free finite Γ-CW complex (e.g. universal cover of M, $\Gamma = \pi_1(M)$)

$C_*(X)$ cellular $\mathbb{Z}\Gamma$-chain $\cdots \to \mathbb{Z}\Gamma^{\oplus k_p} \xrightarrow{d_p} \mathbb{Z}\Gamma^{\oplus k_{p-1}} \to \cdots$

$C_*^{(2)}(X) = C_*(X) \otimes_{\mathbb{Z}\Gamma} \ell_2\Gamma \quad \cdots \to \ell_2\Gamma^{\oplus k_p} \xrightarrow{d_p} \ell_2\Gamma^{\oplus k_{p-1}} \to \cdots$

$b_p^{(2)}(X) = \dim_{L\Gamma}(\ker d_p \ominus \overline{\operatorname{ran}}\, d_{p+1})$

$d_p \in \mathbb{M}_{k_{p-1}, k_p}(\mathbb{Z}\Gamma)$

$\ker d_p \ominus \overline{\operatorname{ran}}\, d_{p+1} = \ker(\underbrace{d_p^* d_p + d_{p+1} d_{p+1}^*}_{\Delta_p \in \mathbb{M}_{k_p}(\mathbb{Z}\Gamma)_+})$

$(M, \tilde{\tau}) = (\mathbb{M}_k(L\Gamma), \operatorname{Tr} \otimes \tau)$

$\tilde{\tau}([x_{ij}]) = \sum \tau(x_{ii})$

For $\Delta \in \mathbb{M}_k(L\Gamma)_+$,

$F_\Delta(\lambda) := \tilde{\tau}(\chi_{[0,\lambda]}(\Delta))$ spectral density function

$\rightsquigarrow b_p^{(2)}(X) = F_{\Delta_p}(0)$.

(Δ_p depends on the choice of a basis of $C_*(X)$, but F_{Δ_p} does not.)

Fuglede - Kadison determinant

$\det \Delta := \exp\left(\int_{0+}^{\infty} \log \lambda \, dF_\Delta(\lambda)\right)$ or 0 if not integrable

integration on $[0, \|\Delta\|]$.

Example $(M, \tilde{\tau}) = (\mathbb{M}_n, c\operatorname{Tr})$, $A \in (\mathbb{M}_n)_+$

$\det A = (\prod \text{non-zero eigenvalues})^c$

$\det 0 = 1$

$A \in \mathbb{M}_n(\mathbb{Z})_+ \Rightarrow \det A = (\text{positive integer})^c \geq 1$

coeff of the char polyn

Determinant Conjecture (Lück)

$\det \Delta \geq 1$ for $\forall \Delta \in \underline{M_k(\mathbb{Z}\Gamma)_+}$

$\overset{\text{def}}{=} \{d^*d : d \in M_{\ell,k}(\mathbb{Z}\Gamma)\}$

How to compute $b_p^{(2)} = F_\Delta(0)$?

e.g. $\begin{bmatrix} d_p^* & d_{p+1} \end{bmatrix} \begin{bmatrix} d_p \\ d_{p+1}^* \end{bmatrix}$

Approximation of $\Delta \in M_k(\mathbb{Z}\Gamma)_+$

① $\Gamma \trianglerighteq N_n$, $N_1 \geq N_2 \geq \cdots$, $\bigcap N_n = \{1\}$, $\pi_n : \Gamma \to \Gamma/N_n$

$\Delta_n = \pi_n(\Delta) \in M_k(\mathbb{Z}\,\Gamma/N_n)$

② $\pi_n : \Gamma \to \mathfrak{S}_n$ sofic approximation, $\Delta = d^*d$

$\Delta_n = \pi_n(d)^* \pi_n(d) \in M_k(\mathbb{Z}\mathfrak{S}_n)$

$\hookrightarrow \det \Delta_n \geq 1$

In both cases, $\Delta_n \to \Delta$ in moments & $\sup \|\Delta_n\| < +\infty$.

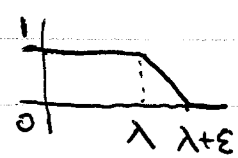
Approximation Conjecture : $F_{\Delta_n}(0) \to F_\Delta(0)$.

Lem $\limsup F_{\Delta_n}(\lambda) \leq F_\Delta(\lambda)$

$\liminf F_{\Delta_n}(\lambda - 0) \geq F_\Delta(\lambda - 0)$

In particular, $F_{\Delta_n} \to F_\Delta$ a.e.

Pf.

h_ε 

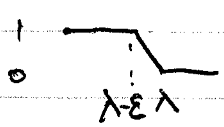
continuous

$\exists$ Weierstrass approximation by polynomials

$F_{\Delta_n}(\lambda) \leq \tilde{\tau}(h_\varepsilon(\Delta_n))$

$\to \tilde{\tau}(h_\varepsilon(\Delta)) \approx F_\Delta(\lambda)$.

Hence $\limsup F_{\Delta_n}(\lambda) \leq F_\Delta(\lambda)$.



Similarly $\liminf F_{\Delta_n}(\lambda - 0) \geq F_\Delta(\lambda - 0)$.

Increasing functions have at most countably many discontinuous points. ∎

Thm (Lück, Schick, Elek–Szabo)

$\forall n \ \det \Delta_n \geq 1 \Rightarrow \det \Delta \geq 1$ & Approx Conj holds.

Cor $\forall n \ \Gamma_n = \Gamma / N_n$ sofic $\Rightarrow$ Approx Conj holds.

$f(\lambda) = F_\Delta(\lambda) - F_\Delta(0)$

$\int g \, df = gf - \int f \, dg = gf - \int f g'$

Proof of Thm $C \geq \|\Delta_n\|, \|\Delta\|$.

$$\log \det \Delta = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon^+}^{C} \log \lambda \, dF_\Delta(\lambda)$$

$$= \lim_{\varepsilon \downarrow 0} \left[(\log C)(F_\Delta(C) - F_\Delta(0)) - (\log \varepsilon)(F_\Delta(\varepsilon) - F_\Delta(0)) - \int_{\varepsilon^+}^{C} \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda} d\lambda \right.$$

$$= (\log C)(F_\Delta(C) - F_\Delta(0)) - \int_{0^+}^{C} \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda} d\lambda$$

possibly $-\infty = -\infty$.

(If either $\int_{0^+}^{C} \log \lambda \, dF_\Delta(\lambda)$ or $\int_{0^+}^{C} \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda} d\lambda$ converges, then $\lim_{\varepsilon \to 0} (\log \varepsilon)(F_\Delta(\varepsilon) - F_\Delta(0)) = 0$.)

$\alpha := \limsup F_{\Delta_n}(0) \leq F_\Delta(0)$

$$\int_{0^+}^{C} \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda} d\lambda = \int_{0^+}^{C} \frac{F_\Delta(\lambda) - \cancel{\alpha}}{\lambda} - \frac{F_\Delta(0) - \alpha}{\lambda} d\lambda$$

$F_{\Delta_n} \to F$ a.e. $\leq \int_{0^+}^{C} \liminf \frac{F_{\Delta_n}(\lambda) - F_{\Delta_n}(0)}{\lambda} d\lambda$

Fatou $\leq \liminf \int_{0^+}^{C} \frac{F_{\Delta_n}(\lambda) - F_{\Delta_n}(0)}{\lambda} d\lambda$

$\leq \lim (\log C)(F_{\Delta_n}(C) - F_{\Delta_n}(0)) = (\star)$

$< +\infty$

$\rightsquigarrow \int_{0^+}^{C} \frac{F_\Delta(0) - \alpha}{\lambda} d\lambda < +\infty \rightsquigarrow F_\Delta(0) = \alpha \xrightarrow{\text{subsubseq argument}} F_{\Delta_n}(0) \to F_\Delta(0)$

$(\star) = (\log C)(F_\Delta(C) - F_\Delta(0))$

$\rightsquigarrow \log \det \Delta \geq 0$. ▨

◎ Ergodic Theory $\longrightarrow$ Bowen's lecture

◎ Orbit Equivalence Relations

$\Gamma \curvearrowright (X,\mu)$ p.m.p. $\rightsquigarrow \mathcal{R}_{\Gamma \curvearrowright X} = \{(sx, x) : \substack{s\in\Gamma \\ x\in X}\}$

$(\mathrm{dom}\, \varphi\psi = \psi^{-1}\mathrm{dom}\,\varphi)$ orbit equiv rel

$[\![\mathcal{R}]\!]$ = the pseudo group of partial isomorphisms on X whose graphs are contained in $\mathcal{R}$.

(• $\varphi \in [\![\mathcal{R}_{\Gamma\curvearrowright X}]\!] \iff \mathrm{dom}\,\varphi = \coprod E_n \subseteq X$, $\varphi|_{E_n} = s_n|_{E_n}$ $s_n \in \Gamma$

• $1_E \in [\![\mathcal{R}]\!]$)

$\tau(\varphi) := \mu(\{x : \varphi(x) = x\})$

$\tau : \mathbb{C}[\![\mathcal{R}]\!] \to \mathbb{C}$ trace

$\underset{\text{GNS}}{\rightsquigarrow}$ $vN(\mathcal{R})$ $(\cong L^\infty(X) \rtimes \Gamma$ if essentially free$)$

<u>Def</u> $\mathcal{R}$ sofic $\overset{\text{def}}{\iff}$ $[\![\mathcal{R}]\!]$ has a sofic approximation :

$\forall E \Subset [\![\mathcal{R}]\!]$ $\forall \varepsilon > 0$

$\exists n$ $\exists \pi : [\![\mathcal{R}]\!] \to [\![\mathfrak{S}_n]\!]$

s.t. $\|\pi(\varphi)\pi(\psi) - \pi(\varphi\psi)\|_2 < \varepsilon$

$|\tau(\pi(\varphi)) - \tau(\varphi)| < \varepsilon$

for $\forall \varphi, \psi \in E$.

Thm $\mathcal{R}$ sofic $\Rightarrow$ $vN(\mathcal{R}) \hookrightarrow \prod \mathbb{M}_n/\omega$

Elek–Lippner

EL

Thm Γ sofic $\Rightarrow$ Bernoulli shift $\mathcal{R}_{\Gamma \curvearrowright (X,\mu)^\Gamma}$ sofic

Q. $\mathbb{Z}/2 \wr \Gamma$ sofic ? It is HL by this theorem.

Proof of Thm We may assume (X,μ) is (finite, uniform).

$E \Subset \Gamma$ given. Take $\rho : \Gamma \to \mathfrak{S}_F$ sofic approximation

Define $\pi : \Gamma \to \mathfrak{S}_{F \times X^F}$ by

$$\pi(s)(\sigma, \xi) = (\rho(s)\sigma, \xi).$$

cylinder set

For $\alpha : D_\alpha \to X$, $D_\alpha \Subset \Gamma$, let $e_\alpha := 1_{\{g \in X^\Gamma : g|_{D_\alpha} = \alpha\}}$

$$\pi(e_\alpha) := 1_{\{(\sigma,\xi) : \forall t \in D_\alpha \ \xi(\rho(t)^{-1}\sigma) = \alpha(t)\}} \in [\![\mathfrak{S}_{F\times X^F}]\!]$$

$$\rightsquigarrow \tau(\pi(e_\alpha)) = \tau(e_\alpha)$$

$s e_\alpha s^{-1} = e_{s\alpha}$ where $s\alpha : sD_\alpha \to X$

$$(s\alpha)(st) = \alpha(t)$$

$$\pi(e_{s\alpha}) = 1_{\{(\sigma,\xi) : \forall t \in D_\alpha \ \xi(\rho(st)^{-1}\sigma) = \alpha(t)\}}$$

On the other hand,

$$\pi(s)\pi(e_\alpha)\pi(s)^{-1} = 1_{\{(\sigma,\xi) : \forall t \in D_\alpha \ \xi(\rho(t)^{-1}\rho(s)^{-1}\sigma) = \alpha(t)\}}.$$

Hence $\| \pi(e_{s\alpha}) - \pi(s)\pi(e_\alpha)\pi(s)^{-1} \|_2 \approx 0$.

This is sufficient for $R_{\Gamma \curvearrowright X^\Gamma}$ to be sofic. ▨