Amenability for C*-dynamical systems

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Disclaimer: Locally compact groups G are assumed second countable if needed.

A LCG G is amenable $\overset{\text{def}}{\iff} \exists$ a G-invariant state $L^\infty(G) \to \mathbb{C}$.

(Reiter's condition) $\iff \exists\, \xi_i \in L^2(G)$ such that $\|\xi_i\| = 1$ and $\lim_i \|\xi_i - g\xi_i\| = 0$ for $\forall g \in G$ unif. on compacta

Zimmer 1977: amenability for measurable dyn. system $G \curvearrowright (X, \mu)$

- $G \curvearrowright (X, \mu)$ p.m.p. & amenable $\Rightarrow$ G amenable;
- $G \curvearrowright G$ is amenable

Anantharaman-Delaroche 1979&1987, AD–Renault 2000

For a von Neumann algebra M and a u-continuous action $G \curvearrowright M$

$G \curvearrowright M$ amenable $\overset{\text{def}}{\iff} \exists$ a G-equivariant cond. exp. $L^\infty(G) \bar{\otimes} M \to M$

$\iff G \curvearrowright \mathcal{Z}(M)$ amenable

If G is **discrete** $\iff \exists\, \xi_i \in L^2(G, \mathcal{Z}(M))$ s.t. $\langle \xi_i, \xi_i \rangle_{\mathcal{Z}(M)} = 1$ and $\lim_i \|\xi_i - g\xi_i\| = 0$ for $\forall g \in G$

If G is **discrete**, for a C*-algebra A and an action $G \curvearrowright A$,

$G \curvearrowright A$ amenable $\overset{\text{def}}{\iff} G \curvearrowright A^{**}$ is amenable (in the W*-sense)

$\iff G \curvearrowright X$ topologically amenable, if $A = C_0(X)$

The notion of amenability for measurable/topological dynamical systems has had numerous applications in ergodic group theory, study of exactness for C*-algebras, Baum–Connes conjecture, classification of von Neumann

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$G \curvearrowright M$ is amenable $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ a G-equivariant cond. exp. $L^\infty(G) \bar{\otimes} M \to M$

If G is discrete

$\Longleftrightarrow$ $G \curvearrowright \mathcal{Z}(M)$ amenable

$\Longleftrightarrow$ $\exists\ \xi_i \in L^2(G, \mathcal{Z}(M))$ s.t. $\langle \xi_i, \xi_i \rangle_{\mathcal{Z}(M)} = 1$ and $\lim_i \|\xi_i - g\xi_i\| = 0$ for $\forall g \in G$

If G is discrete, for a C*-algebra A and an action $G \curvearrowright A$,

$G \curvearrowright A$ amenable $\overset{\text{def}}{\Longleftrightarrow}$ $G \curvearrowright A^{**}$ is amenable

Problem 1: Is the discreteness assumption above necessary?

Bearden–Crann: Not necessary.

Problem 2: How do we define amenability for $G \curvearrowright A$ when G is not discrete? ⚠ $G \curvearrowright A^{**}$ may not be u-continuous.

Buss–Echterhoff–Willett, BC, OS: All reasonable definitions are equivalent.

Problem 3: Are there examples of amenable $G \curvearrowright A$ for interesting A?

Suzuki, O–Suzuki: Yes!

Let $\alpha\colon G \curvearrowright A$ be given. Equip $C_c(G, A)$ with the obvious (i.e., untwisted) A-bimodule structure, A-valued inner product

$$\langle \xi, \eta \rangle = \int_G \xi(x)^* \eta(x)\, dm(x) \in A,$$

and the diagonal G-action $G \curvearrowright C_c(G, A)$

$$(g\xi)(x) = \alpha_g(\xi(g^{-1}x)).$$

By completion, we obtain the (G, A)-C*-correspondence $L^2(G, A)$.
By adapting topological amenability of Anantharaman-Delaroche–Renault:

Definition (Exel–Ng 2002, Buss–Echterhoff–Willett 2019; modified)

$G \curvearrowright A$ has the QAP (quasi-central approx. property) if $\exists \xi_i \in L^2(G, A)$ s.t.

- $(\langle \xi_i, \xi_i \rangle)_i$ is an approximate unit for A,
- $\|[\xi_i, a]\| \to 0$ for $a \in A$, and
- $\|\xi_i - g\xi_i\| \to 0$ for $g \in G$ uniformly on compacta.

A-D 1987: If G **discrete** and A **commutative**, then QAP $\Leftrightarrow$ amenability.
Is it true in general? Consider

$$\Phi\colon L^\infty(G) \bar{\otimes} A^{**} \ni f \mapsto \mathrm{LIM}_n \int_G \xi_n(x)^* f(x) \xi_n(x)\, dm(x) \in A^{**}.$$

The map Φ is u.c.p. and G-equivariant, but is it a conditional expectation?

Suzuki's example: $G \curvearrowright \mathcal{O}_2$

Let G be a countable discrete group that is *amenable at infinity* i.e., $\exists$ amenable $G \curvearrowright X$ with X **compact**. ⚠ $G \curvearrowright G$ always amenable.

Guentner–Kaminker 2000, O 2000 for discrete G, Brodzki–Cave–Li 2017

G amenable at infinity $\Longleftrightarrow$ G exact

$\exists \eta_n \colon X \to \mathrm{Prob}(G)$ cont's and $\lim_n \sup_x \|\eta_n(gx) - g\eta_n(x)\| = 0$ for $g \in G$.
Then $\xi_n = \eta_n^{1/2} \in L^2(G, C(X))$ satisfies $\langle \xi_n, \xi_n \rangle = 1$ and $\|\xi_n - g\xi_n\| \to 0$.
Consider $C(X) \rtimes G \subset \mathcal{O}_2$ and $A := \bigotimes_{\mathbb{N}} \mathcal{O}_2 \cong \mathcal{O}_2$ with the diag. G-action.
Then $\xi_n \in L^2(G, A)$ in the n-th tensor component witnesses the QAP:

$$\langle \xi_n, \xi_n \rangle = 1,\ \|[\xi_n, a]\| \to 0 \text{ for } a \in A, \text{ and } \|\xi_n - g\xi_n\| \to 0 \text{ for } g \in G.$$

Theorem (Szabo 2018 for amenable G, Suzuki 2020)

Let G be a countable discrete group that is amenable at infinity. Then, modulo strong cocycle conjugacy, there exists a **unique** action $\alpha \colon G \curvearrowright \mathcal{O}_2$ that is equivariantly $\mathcal{O}_2$-absorbing, pointwise outer, and with the QAP.
For any action $\beta \colon G \curvearrowright B$ on a unital simple separable nuclear C*-algebra, the diagonal action $\alpha \otimes \beta$ is strongly cocycle conjugate to α.

Let $\alpha\colon G \curvearrowright A$ be given. Recall that if G is discrete

$$G \curvearrowright A \text{ amenable} \overset{\text{def}}{\iff} G \curvearrowright A^{**} \text{ amenable (in the W*-sense)}$$

⚠ $G \curvearrowright A^{**}$ may not be u-continuous. For a fix, consider

$$A''_\alpha := \overline{A}^{\mathrm{w}^*} \subset (A \rtimes G)^{**},$$

the univ. enveloping vN algebra for covariant rep'ns. Note that

$$(A''_\alpha)_* = \{\phi \in A^* : g \mapsto g\phi \text{ is norm-continuous}\} = L^1(G) \cdot A^*.$$

Example: For a topological dynamical system $G \curvearrowright X$,

$$C_0(X)''_\alpha = \text{completion w.r.t. quasi-invariant probability measures.}$$

In particular, $C_0(G)''_\alpha = L^\infty(G)$. Generally very difficult to describe A''_α.

Definition/Theorem (BEW, BC, SO 2020)

$G \curvearrowright A$ is *amenable* if it satisfies one of the following equivalent conditions

(i) $G \curvearrowright A''_\alpha$ is amenable: $\exists$ a G-equiv. cond. exp. $L^\infty(G) \bar{\otimes} A''_\alpha \to A''_\alpha$.

(ii) $G \curvearrowright A$ has the QAP.

(iii) $\exists$ a G-equivariant cond. exp. $L^\infty(G) \bar{\otimes} \mathcal{Z}(A^{**}) \to \mathcal{Z}(A^{**})$.

(iv-x) AP by positive type functions, central sequence algebras, ...

Furnishing examples to the ongoing classification program for amenable actions on Kirchberg algebras by Izumi–Matui, Szabo, ...

Theorem (Pimsner 1995, Meyer 2019, Suzuki–O 2020)

For $\forall G \curvearrowright A$ amenable, $\exists G \curvearrowright B$ such that $A \subset B$ such that

- B is simple, purely infinite, and nuclear (provided that A is);
- $G \curvearrowright B$ is pointwise outer and amenable;
- $A \subset B$ induces KK^G-equivalence.

Pimsner–Meyer constr'n: $G \curvearrowright A \rightsquigarrow (G, A)$-C*-corresp. $\mathcal{E} \rightsquigarrow B := \mathcal{T}(\mathcal{E})$
Want to show amenability of $G \curvearrowright B$.
Strategy: Look at the fixed-pt subalgebra $B^{\mathbb{T}}$ of the gauge action $\mathbb{T} \curvearrowright B$.
$G \curvearrowright B^{\mathbb{T}}$ is built up by $G \curvearrowright \mathbb{K}(\mathcal{E}^{\otimes n})$, which are amenable in this situation.

Theorem (Suzuki–O 2020)

For $G \times K \curvearrowright A$, where K is compact, TFAE

(i) $G \curvearrowright A$ amenable,
(ii) $G \curvearrowright A^K$ amenable,
(iii) $G \curvearrowright A \rtimes K$ amenable,
(iv) $G \times K \curvearrowright A$ amenable.

Applying the previous construction to the Higson–Kasparov proper G-C*-algebra $\mathcal{A}(H)$, one obtains

Corollary

If G has Haagerup prop'ty, $\exists$ amenable $G \curvearrowright \mathcal{O}_\infty \otimes \mathbb{K}$ which is KK^G-trivial.

It is unclear for which G $\exists$ an amenable $G \curvearrowright \mathcal{O}_\infty$ which is KK^G-trivial.

Theorem (Suzuki–O 2020)

For $G \times K \curvearrowright A$, where K is compact, TFAE

(i) $G \curvearrowright A$ amenable,

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(iii) $G \curvearrowright A \rtimes K$ amenable,

(iv) $G \times K \curvearrowright A$ amenable.

(i) $\Rightarrow$ (ii): $\exists$ G-equivariant cond. exp. $A \to A^K$.

(ii) $\Rightarrow$ (iii): $A \rtimes K = (A \otimes \mathbb{K}(L^2(K)))^K = \varinjlim (A \otimes \mathbb{K}(p_i L^2(K)))^K$ by P.–W.
Hence it suffices to deal with a finite-index inclusion $A \subset B$.

(iii) $\Rightarrow$ (iv): $L^\infty(G \times K) \mathbin{\bar{\otimes}} A''_{\alpha\times\beta} \to L^\infty(G) \mathbin{\bar{\otimes}} (A''_{\alpha\times\beta} \bar{\rtimes} K) \to A''_{\alpha\times\beta} \bar{\rtimes} K \to A''_{\alpha\times}$

(iv) $\Rightarrow$ (i): $L^\infty(G) \mathbin{\bar{\otimes}} \mathcal{Z}(A^{**}) \subset L^\infty(G \times K) \mathbin{\bar{\otimes}} \mathcal{Z}(A^{**}) \to \mathcal{Z}(A^{**})$. □