Primitive Automorphic Forms*

Dedicated to Professor Tadao Oda
on the Occasion of His Sixtieth Birthday

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There seems to be a marvellous interaction taking place between mathematical physics and mathematics in the area of geometry, demanding a greater contribution from non-commutative structures and higher cohomologies. It may require a revolutional extension of the concept of spaces in order to explain the dualities there. I am not the proper person to talk about the whole subject, but will restrict myself to that part of the topic where I have been involved from the view point of complex geometry, namely periods of integrals over vanishing cycles. In order to attack a big mathematical problem, there are two approaches:

1) to generalize the problem and to develop a new general framework and language within which the problem finds a natural place, or
2) to attempt to examine a cross-section of the problem and to give a precise solution of that part of the problem.

The approach I adopt here in this paper is the second one, partly in the hope

* Copyedited by Chris Weeks.

that it will provide a prototype understanding of some aspect of the problem, but also for its inherent beauty since *the mathematical world consists not only of the general theories but also of interesting individuals* (C. L. Siegel).

Part I Primitive Forms

1. Motive

I have been attracted by mathematical objects where the arithmetic nature and the transcendental nature of the objects are closely combined. It is hard for me to define what I mean by this statement though I have an instincitive feel for it. So let me provide some typical examples.

The first examples are the transcendental numbers π and e, or, what amounts to the same thing, the trigonometric functions and the exponential function. They are certainly transcendental but also have arithmetic nature in a number of respects, one being the addition formula. Further examples are the functions related to the elliptic integral theory such as the elliptic modular function j, modular forms E_4, E_6, $\varDelta, \ldots$.

These examples are classical (some of them have a history of hundreds of years), but they will always retain their attraction and are certain to remain fresh in the future. Even more than that, experience tells us that it may not be absurd to expect these functions in the future to yield again a new branch of beautiful mathematics such as Moonshine from the Monster group to the elliptic modular function (Borcherds).

These examples of functions have their origins in the study of the integrals of arc-length of quadratic and cubic curves. This fact inspires the idea that integrals over algebraic objects leads to transcendental objects (but, then for instance, why the powers of π appear in the special values of the zeta-function, why the family of cubic curves are related to Moonshine? There remain still numerous unsolved questions).

A natural further development of the integral of arc-length for higher genus curves is the Abelian integral theory and the solution of Jacobi's inversion problem by theta-functions (already established in the 19th century). For higher dimensional varieties Hodge theory is a development. In fact, the study of Abelian varieties, (Siegel) modular forms and mixed Hodge structures have grown extensively and are attaining more and more importance in mathematics. However, I was annoyed, for instance, by the gap between the dimension $g(g+1)/2$ of the space of Abelian varieties and the dimension $3g-3$ of the space of Jacobian varieties of curves of genus $g>1$. The Siegel modular functions (or theta null-wert) are not functions on the space of Jacobians! This led me to study the integrals of the primitive forms, another generalization of the elliptic integral theory, explained in the Part I of this article.

2. Elliptic Integrals

We first review the classical elliptic integral theory in the Weierstrass form as the proto-type for what I explain later. Let $P(z,\underline{g}) := 4z^3 - g_2 z - g_3$ be a cubic polynomial in a complex variable z with two parameters g_2, g_3 and let $\Delta := 27g_3^2 - g_2^3$ be its discriminant. The elliptic integral of the first and the second kinds are:

$$I_1(g_1,g_2) = \oint P^{-1/2}dz \quad \text{and} \quad I_2(g_1,g_2) = \oint P^{-1/2}zdz, \tag{1}$$

where we integrate over closed paths (cycles) on the elliptic curve $E_{\underline{g}}$ defined by the equation $F(w,z,\underline{g}) := w^2 - P(z,\underline{g}) = 0$. As functions in $\underline{g} = (g_2, g_3)$, they satisfy the following total differential equation:

$$\begin{bmatrix} dI_1 \\ dI_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12}d\log\Delta & \omega \\ [6pt] - \frac{1}{12}g_2\omega & \frac{1}{12}d\log\Delta \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{2}$$

where $d\log\Delta$ and $\omega = \frac{-3g_2dg_3+(9/2)g_3dg_2}{\Delta}$ are the basis of logarithmic 1-forms along the discriminant. From (2), one deduces that the integral of the second kind is a logarithmic derivative of that of the first kind:

$$I_2 = \left(6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3}\right)I_1. \tag{3}$$

Eliminating I_2 in (2), one obtains the system of second order equation:

$$\left(\left(6g_3\frac{\partial}{\partial g_2}+\frac{1}{3}g_2^2\frac{\partial}{\partial g_3}\right)^2+\frac{g_2}{12}\right)u=0,\ \left(\frac{1}{3}g_2\frac{\partial}{\partial g_2}+\frac{1}{2}g_3\frac{\partial}{\partial g_3}+\frac{1}{12}\right)u=0, \tag{4}$$

satisfied by the first integral I_1. The equation is equivalent to the Gauss-Legendre hypergeometric differential equation.

The two linearly independent solutions $\omega_1,\ \omega_2$ of the equation (4) are obtained by the integrals $\oint_{\gamma_i}$ of the first kind over the two linearly independent 1-cycles γ_i $(i = 1,2)$ on the elliptic curve $E_{\underline{g}}$. For brevity, let us call γ_i *vanishing cycles* since they collapse to a singular point on the cuspidal cubic curve E_0 for $\underline{g} = 0$. The multivalued correspondence $\underline{g} \mapsto (\omega_1(\underline{g}), \omega_2(\underline{g}))$ from the complement $\{(g_2, g_3) \in \mathbb{C}^2 \mid \Delta(g_2, g_3) \neq 0\}$ of the discriminant locus $D := \{\Delta(\underline{g}) = 0\}$ in the parameter space to the period domain $\{(\omega_1, \omega_2) \in \mathbb{C}^2 \mid Im(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1) > 0\}$ is called a period map. The map is equi-2-dimensional and is locally bi-regular. Here one has a basic result (see, for instance, [Si, cha.1]): *the inverse to the period map is a univalent map. The coordinates* $\frac{1}{60}g_2$ *and* $\frac{1}{140}g_3$ *of the inversion are given by the Eisenstein series* $E_4 = \sum'_{m,n\in\mathbb{Z}} \frac{1}{(m\omega_1+n\omega_2)^4}$ *and* $E_6 = \sum'_{m,n\in\mathbb{Z}} \frac{1}{(m\omega_1+n\omega_2)^6}$*, respectively. The substitution* $\Delta(60E_4, 140E_6)$ *in the discriminant form is the generator of the ideal of cusp forms (the automorphic forms defined on the period domain which vanish at the cusps* $= \{(\omega_1,\omega_2) \mid \frac{\omega_2}{\omega_1} \in \mathbb{Q} \cup \{\infty\}\}$ *of the period domain).*

Thus, starting from such simple objects as cubic polynomials, one arrives at highly transcendental objects such as the elliptic modular forms and cusp forms. Let us give a summary of the above story:

1) All elliptic integrals are obtained by differentiating the integral of the first kind. This property may be called *primitivity*.
2) The dimension 2 of the parameters g_2, g_3 coincides with the rank 2 of the lattice of vanishing cycles generated by γ_1, γ_2. The period map is locally biregular. This property may be called *equi-dimensionality*.
3) The Eisenstein series E_4, E_6 combine globaly the two quantities: the pa-

rameters g_2, g_3 and the periods ω_1, ω_2, where the discriminant loci correspond to the cusps of the period domain.

3. Periods for Primitive Forms

We interpret the Weierstrass family $F(z, w, \underline{g}) = 0$ as a *universal unfolding*[1] *of the cusp* $F(z, w, 0) = w^2 - z^3$ *by the parameter* $\underline{g}$. Generally, let $f(\underline{x})$ be any polynomial in several complex variables $\underline{x}$, whose zeroloci defines a hypersurface with an isolated singular point at the origin and let $F(\underline{x}, \underline{g})$ be its universal unfolding[1] by the parameter $\underline{g} \in \mathbb{C}^\mu$, where the equation $F(\underline{x}, \underline{g}) = 0$ defines a flat family of affine varieties $X_{\underline{g}}$. There is a concept of the discriminant $\Delta(\underline{g})$ of $F(\underline{x}, \underline{g})$, whose zeroloci describe the parameter-values $\underline{g}$ at which $X_{\underline{g}}$ becomes singular. Remarkably, *the rank* μ *of the lattice of vanishing cycles of middle dimension in the homology group of* $X_{\underline{g}}$ *for* $\underline{g}$ *with* $\Delta(\underline{g}) \neq 0$ *(= the Milnor lattice, see* [Mi]*) coincides with the number* μ *of unfolding parameters* $(\underline{g}) = (g_1, \ldots, g_\mu)$. Thus, if one finds a family $\{\zeta_{\underline{g}}\}_{\underline{g}}$ of differential forms of degree $= \dim X_{\underline{g}}$ on $\{X_{\underline{g}}\}_{\underline{g}}$ whose covariant differentiations $\nabla_{g_i} \zeta_{\underline{g}}$ with respect to the parameters $g_1, \ldots, g_\mu$ form a basis of the de-Rham cohomology group dual to the lattice of vanishing cycles (at generic $\underline{g}$ with $\Delta(\underline{g}) \neq 0$), then the integrals $(\oint_{\gamma_j(g)} \zeta_g)_{j=1\ldots\mu}$ over a basis of the lattice of vanishing cycles should define a locally biregular period map since its Jacobian $(\frac{\partial}{\partial g_i} \oint_{\gamma_j(g)} \zeta_g)_{ij=1\ldots\mu} = (\oint_{\gamma_j(g)} \nabla_{g_i} \zeta_g)_{ij=1\ldots\mu}$ is, tautologically, of maximal rank.

A *primitive form* ζ_g ([O], [S2], [SaM]) is such a family of differential forms satisfying, further, a system of infinite bilinear differential equations with respect to $\underline{g}$. We shall not go into details of the bilinear equations defining the primitive form[2], but discuss some examples and consequences of the primi-

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[1] See [T] and also II.6 of the present paper. The description here is unprecise.

[2] The system of equations on a primitive form, satisfied by the elliptic integral of the first kind, is described in terms of higher residue pairings on the relative De-Rham cohomology groups. It

tive forms. The period map[3] for a primitive form from the complement of the discriminant loci $\Delta = 0$ to a suitable non-classical period domain recovers 1) *primitivity* and 2) *equi-dimensionality*, which we found in the elliptic integral theory.

We expect an analogy of 3) for suitable primitive forms. In the end, one needs to solve several problems, since the nature of 3) is global despite that of 1) and 2) are local. First of all, the existence of a primitive form is known only in a small neighborhood of the singular point except for a few good examples discussed later. So, these lead to the following *inversion problem* (in a rough and naive formulation).

"#problem *Under a suitable setting,*

1. *Give global existence and description of a primitive form.*
2. *Generalize the concepts of Eisenstein series. Using it, describe globally the inversion map to the primitive period map.*
3. *Does $\sqrt{\Delta}$ define the fundamental anti-invariant on the period domain? Is Δ the generator of cusp forms in a suitable sense?*

4. Examples

We discuss simple examples of primitive forms, where Problem 1 is answered by two expressions. They indicate the close connection of the primitive form with Lie theory and integrable systems.

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requires a 'purity of the Hodge component' of the covariant differentiation of the primitive form with respect to the parameter $\underline{g}$ (see [O], [S2], [SaM]). As a result, one obtains the analogue of the total differential equation (2) and Gauss-Legendre equation (4), satisfying the integrability conditions.

[3] i) If there is a monodromy invariant vanishing cycle, then the period map is better defined by the solutions of the generalized Gauss-Legendre equation. ii) There are examples where the primitive integral is carried out not only over local vanishing cycles but over global cycles: a) Seiberg-Witten integrals [Ta2], b) the period maps for the 14 exceptional unimodular singularities (see II.9, ii)).

There is a celebrated one-to-one correspondence, observed by McKay [Mc], between the conjugacy classes of finite subgroups of $SU(2)$ and the simply-laced Dynkin diagrams (see [B]) of types A_l, D_l and E_l for suitable l. So, let Γ be the Dynkin diagram assigned to a finite group $G \subset SU(2)$ (this means that the exceptional set of the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ is the Dynkin curve of type Γ (see [GSpV])). Then the universal unfolding of $\mathbb{C}^2/G$ is described by the coadjoint quotient map $\mathfrak{g}_\Gamma \to \mathfrak{h}_\Gamma/W_\Gamma$ where $\mathfrak{g}_\Gamma$, $\mathfrak{h}_\Gamma$ and W_Γ are the simple Lie algebra, its Cartan subalgebra and Weyl group of type Γ, respectively, and $\mathfrak{h}_\Gamma/W_\Gamma$ is the parameter space of the unfolding (Brieskorn [Br1], Kronheimer [Kr]). The primitive form for this case coincides with the Kostant-Killirov form, induced from the Killing form on $\mathfrak{g}_\Gamma$ and defining the symplectic structure on the coadjoint orbits (the first expression [Y1]). In these quotient singularity cases, the primitive form $\zeta_{\underline{g}}$ is given by $\mathrm{Res}[d\underline{x}/F(\underline{x},\underline{g})]$ in terms of the defining equation $F(\underline{x},\underline{g}) = 0$ of the unfolding (likewise the elliptic integral case) (the second expression, see II.7, [S2]).

Two different period maps are attached to this setting. The first one is straightforward: the lattice of vanishing cycles of $\mathbb{C}^2/G$ is identified with the root lattice $Q(\Gamma)(-1)$ of type Γ ([Br1]), and the primitive period map is identified with the inverse map $\mathfrak{h}_\Gamma/W_\Gamma \to \mathfrak{h}_\Gamma$. So, the inverse of the period map is the quotient map $\mathfrak{h}_\Gamma \to \mathfrak{h}_\Gamma/W_\Gamma$, where the discriminant Δ is the square of the fundamental W-anti-invariant polynomial. One easily deduces from classical results on finite reflection groups affirmative answers to the Problems 1–3 for this period map.

The second period map, which I explain below, is highly nontrivial. Solutions to the inversion problem for them are not known in general.

There is an operation: $*S^0$ (join with 0-dimensional sphere), which brings the lattices of even-dimensional vanishing cycles (with symmetric bilinear forms) to the lattices of odd-dimensional vanishing cycles (with skew-symmetric bilinear forms) and vice versa such that $(*S^0)^2 = -1$. Can we provide a description of the period map for the odd lattice $Q(\Gamma) * S^0$ for a root lattice $Q(\Gamma)$ of type Γ? In fact, $Q(A_1) * S^0$ and $Q(A_2) * S^0$ are identified with the first homology group of two-punctured conics and one punctured elliptic curves, respectively. The primitive forms for the types A_1 and A_2 are identified with the arcsine integral (Gauss) or the elliptic integral of the first kind, respectively. Therefore, as we saw already, their inversion maps are given by the exponential function and elliptic Eisenstein series, respectively. For any lattice $Q(\Gamma) * S^0$ of type Γ, one can also describe the primitive period map by a certain Abelian integral over affine curves, which maps the complement $\mathfrak{h}_\Gamma / W_\Gamma \setminus \{\Delta = 0\}$ of the discriminant loci to the period domain $\tilde{B}(\Gamma)$ locally biregulary, where the monodromy: a subgroup of the symplectic group $\mathrm{O}(\widetilde{Q(\Gamma)} * S^0)$ generated by transvections acts on $\tilde{B}(\Gamma)$ properly discontinuously. Even in such a classical setting, we know yet very little about the inversion map $\tilde{B}(\Gamma) \to \mathfrak{h}_\Gamma / W_\Gamma \setminus \{\Delta = 0\}$. Problems 2 and 3 are open except for the types A_1, A_2 (see [Mu], [S7]).

5. More Structures

Returning to the general setting of the period map for any primitive form, let us explain two more structures, which should help us to ask and formulate the inversion problems globally.

"#*Flat Structure* The unfolding parameter space $\{(\underline{g})\}$ posesses a global graded affine linear structure with respect to a flat metric[4]. Hence the inversion map to the primitive period map splits into graded linear coordinate components, each of which is an 'automorphic form' with respect to the monodromy group. For brevity, let us call each component a *primitive automorphic form* (see the examples below).

[4] More precisely, a primitive form identifies the tangent bundle of the parameter space with the Hodge bundle of the unfolding. Consequently, the tangent bundle obtains three structures: a *commutative algebra, Lie algebra* and a *flat metric* J with suitable compatibilities, which leads to a *potential to the flat metric*. This structure is called the *flat structure* [S2]. The structure is found also in 2-dimensional topological field theory and is called the *Frobenius structure* [Db], [Ma].

We illustrate the flat metric for the previous Examples by determining it elementarily only by a use of a *Coxeter element* [B], [S7]. Recall that the unfolding parameter space is given by $\mathfrak{h}/W = Spec(S(\mathfrak{h}^*)^W)$, where the ring $S(\mathfrak{h}^*)^W$ of W-invariant polynomials on $\mathfrak{h}$, according to Chevalley, is generated by algebraically independent homogeneous elements, say $P_1, \ldots, P_l$, of degrees $m_1 + 1 = 2 < \cdots < m_l + 1 = h$, and the discriminant $\Delta = (\prod l_\alpha)^2$ (l_α = the linear form defining the reflection hyperplane of W) has a form $\Delta = A_0 P_l^l + A_1 P_l^{l-1} + \cdots + A_l$ (A_i = a polynomial in $P_1, \ldots, P_{l-1}$ of total degree hi). A Coxeter element $c \in W$, defined as the product of simple reflections of W, has order h and its eigenvector, say ξ, for a primitive hth root of unity is regular (i.e. fixed by no reflections in W). Then the condition $\deg P_i < h$ implies $P_i(\xi) = 0$ ($1 \le i < l$) and the regularity of ξ implies $\Delta(\xi) \neq 0$, and therefore $A_0 \neq 0$. That is: *the discriminant Δ is a monic polynomial in P_l of degree l.* The Killing form $I(x)$ as an inner product on $\mathfrak{h}$ ($\simeq \mathfrak{h}^*$) induces also an inner product $I(dP_i, dP_j) = \sum_{m,n} \frac{\partial P_i}{\partial x_m} \frac{\partial P_j}{\partial x_n} I(x_m, x_n)$ on the cotangent vectors dP_i ($i = 1, \ldots, l$) on $\mathfrak{h}/W$. This inner product degenerates along the discriminant since $\det(I(dP_i, dP_j))_{ij=1}^l = \Delta$. By a use of the derivation $D := \frac{\partial}{\partial P_l}$ (which is unique up to a constant factor), we define a new inner product $J(dP_i, dP_j) := DI(dP_i, dP_j)$ on $\mathfrak{h}/W$. Then one can show *i)* $\det(J(dP_i, dP_j))_{ij=1}^l = A_0 \neq 0$ *i.e. the J is nondegenerate, ii) the form J is independent of the coordinates $P_1, \ldots, P_l$, iii) the Levi-Civita connection for J is flat, i.e. there is an affine linear coordinate, say $Q_1, \ldots, Q_l$*[5]*, on $\mathfrak{h}/W$ so that $J(dQ_i, dQ_j)$ are constants.* Such coordinates are called *flat*.

In particular, it turns out that $\underline{g} = (g_2, g_3)$ for the Weierstrass family $F(z, w, \underline{g})$ are the flat coordinates of type A_2, and so, the primitive automorphic forms

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[5] The iii) follows from the integrability of the Levi-Civita connection ∇ for the Killing form I ([S7]). The identification in footnote 4 induces that of ∇ and I with the Gauss-Manin connection and the intersection form on the lattice of vanishing cycles, respectively. The J is given by residue pairing on the de-Rham cohomology group, which corresponds to the Yukawa coupling in the topological field theory.

of type A_2 are the Eisenstein series E_4 and E_6.

"#*Infinite Dimensional Lie Algebra* One would like to generalize the McKay correspondence in such a manner that *one attaches to a singularity a Lie algebra which gives a global description of the unfolding as well as the primitive forms of the singularity.* This programme is in progress for some special cases ([S3], [S-T], [S-Y], [Sl 1,2], [H-S]). Here we describe an aproach generalizing the concept of a *root system*.

Root systems (historically, finite or affine) are quite simple combinatorial objects, where the data of the Lie algebras and groups as well as the Weyl groups are coded [B]. Let us reformulate the axioms for a root system in order to include a wider class of objects. A subset R of a real vector space $\mathfrak{h}_{\mathbb{R}}^*$ equipped with a symmetric bilinear form I consisting of elements of positive norms is a root system if 1) R generates a full lattice, 2) $2I(\alpha,\beta)/I(\alpha,\alpha) \in \mathbb{Z}$ for $\alpha, \beta \in R$, 3) the reflection for any $\alpha \in R$ preserves the set R, 4) R is irreducible. Actually, these axioms recover the classical case: *I is positive definite (resp. semidefinite with 1-dim. radical) if and only if R is a finite (resp. affine) root system.*

The set R of vanishing cycles of the unfolding of an even dimensional singularity[6] with the intersection form I satisfies the axioms of a root system. So, one has the correspondence: $\{\text{isolated singularities}\} \rightarrow \{\text{root systems}\}$. In the McKay case, as we saw, the lattice of vanishing cycles for the singularity $\mathbb{C}^2/G$ is $Q(\Gamma)(-1)$ and so, R is the finite root system of type Γ. But, McKay correspondence states something stronger: the diagram Γ gives a *simple basis* ([B]) of the root system R such that the product of reflections associated to Γ defines a Coxeter element (see II.7). This fact inspires *the second inversion problem*:

[6] The R is the collection of primitive elements α of the middle dimensional homology group of the general fibre X_t of the unfolding where the pararel translation of α vanishes as t tends to generic points of the discriminant loci (see [Br2, Appendix]).

"#problem

4 *Classify the generalized root system which admits a simple basis in a suitable sense such that the Coxeter element as the product of the reflections of the basis is of finite order h and that the eigenspace belonging to a primitive h-th root of unity is regular with respect to the Weyl group W generated by reflections of the roots*[7]. G

5 *Attached to such generalized root system R, construct the following i) invariant theory for the group W_R and the flat structure on $\tilde{B}\mathfrak{h}/W_R$, ii) a Lie algebra $\mathfrak{g}_R$ and co-adjoint quotient morphism $\tilde{B}\mathfrak{g}_R \to \tilde{B}\mathfrak{h}/W_R$, iii) the first expression of the primitive form as the Kostant Killirov form, primitive period map and primitive automorphic forms*[8].

6. What Is the Dual?

The primitive automorphic forms seem to me to be an arithmetic and transcendental way of describing the degeneration of spaces, likewise the exponential and elliptic functions can describe the degenerations of conics and cubics. From the view point of mirror symmetry, this is only one side. Then, what sort of global (Kählerian?) geometry do they describe in the dual model side i.e. in Gromov-Witten geometry? This question is not yet answered even for the elliptic integral, i.e. A_2-case, since the virtual dimension of the dual model is a fraction and the concept of spaces that we have at present is too narrow to allow such duality. A full understanding of the dual objects may, as was said

[7] If the Coxeter element (in this sense) is quasi-unipotent then the Witt-index of $I(=\mu_0+\mu_-)$ is even. Thus affine and hyperbolic root lattices are excluded as for the lattice of vanishing cycles (see II.7 and 8). For the study of root basis arising from singularities, one refers to a series of works by Ebeling ([Eb1]).

[8] Here $\tilde{B}$ is an unexplained operation to define the period domain (cf. II.9). For an elliptic singularity, Problems 4, 5 i) are solved in [S3], [Sat 1,2]. The Lie algebra $\mathfrak{g}(R)$ can be constructed by a use of vertex operators (see [S-Y]) which have close similarities with the simple Lie algebras because of the existence of the Coxeter element, but are beyond Kac-Moody algebras (see [S-Y], [Sl2]).

at the beginning, require a new concept of spaces allowing for tensor products, for which we shall have to wait for the mathematics of the 21st century.

Part II Regular Systems of Weights

We proposed the Inversion Problems 1–5 for primitive period maps in Part I. We saw that even for the finite root system cases, the description of the primitive automorphic forms for the lattice of odd dimensional vanishing cycles remains as an attractive open problem. Nevertheless, the purpose of this present Part II is to propose a wider class of primitive period maps than these classical cases, where I suspect that all Inversion Problems should have reasonable answers. Roughly speaking, the classical cases treated only finite root systems with positive definite forms I. We extend the study to infinite root systems with semi-definite or indefinite forms I. How are we to find them?

The new objects are introduced by a use of *regular weight systems* (see II.1 below. References on the subject are [S4-6]). Attached to a regular weight system, one considers several mathematical structures such as the singularity and its vanishing cycles, the flat structure, the primitive form and its period map and, finally, the primimitive automorphic forms. Their descriptions are fragmentary, which altogether may form a puzzle, since we don't know yet the whole picture and many things are yet to be worked out. Nevertheless, altogether they seem to form a quite rich mathematical world worthwhile studying, whose goal as mentioned in the title of the present paper should be the study of the primitive automorphic forms for the regular weight systems.

1. Regular Weight System

Let us start with elementary arithmetics. We call a system $W = (a, b, c; h)$ of 4 integers with $0 < a, b, c < h$ and $\gcd(a, b, c) = 1$ a *regular weight system* if the rational function:

$$\chi_W(T) := T^{-h}\frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}$$

does not have a pole except at $T = 0$. Then, χ_W has a development

$$\chi_W(T) = \sum_{i=1}^{\mu} T^{m_i}$$

in a finite sum of Laurent monomials, where $m_i \in \mathbf{Z}$ are called *exponents*. The smallest exponent $= a + b + c - h$ is denoted by $\varepsilon(W)$. The number $\mu = \mu_W := \chi_W(1) = (h-a)(h-b)(h-c)/abc$ of exponents shall be called the *rank* and $h = h_W$ shall be called the *Coxeter number*.

2. Classification

There is a one to one correspondence: $\{$regular weight systems having only positive exponents$\} \simeq \{$finite root systems of type A_l, D_l and $E_l\}$ such that their exponents coincide with each other. Let us call such a weight system *classical*. $D_l : (2, l-2, l-1; 2(l-1))$, $E_6 : (3, 4, 6; 12)$, $E_7 : (4, 6, 9; 18)$, $E_8 : (6, 10, 15; 30)$, where $\varepsilon(W) = 1$ for all.

There are three regular weight systems: $\tilde{E}_6$: $(1, 1, 1; 3)$, $\tilde{E}_7$: $(1, 1, 2; 4)$, $\tilde{E}_8$: $(1, 2, 3; 6)$, having only non-negative but 0 exponents. They are called *elliptic*. These are the cases when $\varepsilon(W) = 0$.

There are $14 + 8 + 6 + 3$ regular weight systems when $\varepsilon(W) = -1$. The first 14 are the cases when there is no exponent equal to 0 or 1 and we shall call them, tentatively, e-hyperbolic (an abbreviation for elliptic hyperbolic. See II.7). For any $\varepsilon \in \mathbb{Z}_{<0}$ there are a finite number of regular weight systems with $\varepsilon(W) = \varepsilon$. Each of them has its own arithmetics and beauty (e.g. II.7), but we shall not go into details here.

3. Singularity

For a given system of weights $W = (a, b, c; h)$, consider the graded polynomial ring $\mathbb{C}[x, y, z]$ with $\deg x = a, \deg y = b, \deg z = c$. Consider a weighted homogeneous polynomial of degree h $f_W(x,y,z) = \sum_{la+mb+nc=h} c_{l,m,n} x^l y^m z^c$ whose coefficients are generic. Then, *the weight system W is regular $\Leftrightarrow$ the hypersurface $X_0 := \{f_W(x,y,z) = 0\}$ has an isolated singular point at the origin.* This is equivalent to saying that the Jacobi ring $J_W := \mathbb{C}[x,y,z]/(\frac{\partial f_W}{\partial x}, \frac{\partial f_W}{\partial y}, \frac{\partial f_W}{\partial z})$ is of finite rank, and then the Poincaré polynomial $P_{J_W}(T) := \sum \dim(J_{W,d}) T^d$ is equal to $T^{-\varepsilon(W)} \chi_W(T)$.

If W is classical, i.e. $\varepsilon(W) = 1$, then there is a finite subgroup $G \subset SU(2)$ so that $X_{0,\mathbb{C}} \simeq \mathbb{C}^2/G$. This means that *the classical regular weight systems give the inverse of the McKay correspondence* in a sense.

If W is elliptic, i.e. $\varepsilon(W) = 0$, then depending on f_W, there is $\tau \in \mathbb{H}$ and an action on $\mathbb{C}^2$ of an infinite cyclic central extension H of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ so that $X_{0,\mathbb{C}} \setminus \{0\} = \mathbb{C}^2/H$ is a principal $\mathbb{C}^\times$-bundle over an elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. The singularity is called simply elliptic and named $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ according to the weight system [S1].

If $\varepsilon(W) = -d < 0$, then there is a co-compact Fuchsian group $\Gamma \subset PSL(2,\mathbb{R})$ and its lifting $\tilde{\Gamma} \subset \tilde{P}SL(2,\mathbb{R})$ (=the central extension by $\mathbb{Z}/d$) so that $X_{0,\mathbb{C}} \setminus \{0\} = \tilde{\mathbb{H}}_d/\tilde{\Gamma}$ (Arnold[Ar], Dolgachev[Dl1], Pinkham).

4. Duality

For any regular weight system W, define a polynomial:

$$\varphi_W(\lambda) := \prod_{i=1}^{\mu} (\lambda - exp(2\pi\sqrt{-1} m_i/h))$$

which is a cyclotomic polynomial [9] and hence decomposes in a form:

$$\varphi_W(\lambda) := \prod_{i|h} (\lambda^i - 1)^{e_W(i)}$$

for suitable $e_W(i) \in \mathbb{Z}$. We call two weight systems W and W^* *dual* to each other if their Coxeter number h coincides and $e_W(i) + e_{W^*}(h/i) = 0$ for all

[9] The φ_W shall be naturally identified with the characteristic polynomial of the Coxeter element acting on the root lattice Q_W (cf. II.8).

$i|h$ (with some exceptions, see [S6], [Ta1]).

All the classical weight systems of type A_l, D_l, E_l are selfdual. There is an involution $*$ on the set of 14 e-hyperbolic weight systems such that W and W^* are dual to each other. In fact, this purely arithmetically proven duality on the 14 weight systems induces the *strange duality of Arnold* on the 14 exceptional unimodular singularities [Ar]. It suggests a duality between the lattices of algebraic and transcendental cycles.

It is interesting to observe that i) *All the regular weight systems of rank 24 (there are 12 such weight systems having negative exponent) are selfdual,* ii) *Except for 4 cases, all selfdual characteristic polynomials of the Conway group* $\cdot 0$ *are described as either that of direct sum of classical weight system (Niemeier case), that of dual pair of e-hyperbolic weight systems or that of the 12 rank 24 weight systems* ([S6], [E2]).

5. Eta-Product

Before we go further, let us look at some arithmetics. Let $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind eta-function for $q = \exp(2\pi\sqrt{-1}\tau)$ and $\tau \in \mathbb{H}$. For W, define an *eta-product* [10] by

$$\eta_W(\tau) := \prod_{i|h} \eta(i\tau)^{e_W(i)}.$$

By a use of the eta-product, the duality between two weight systems W and W^* is characterized by the relation $\eta_W(-1/h\tau) \cdot \eta_{W^*}(\tau) \cdot \sqrt{d_W} = 1$, where $d_W = d_{W^*}$ is a constant called the *discriminant* of W.

The eta-product η_W is holomorphic (resp. cuspidal) if and only if the dual-rank $\nu_W := -\sum_{i|h} i \cdot e_W(h/i)$ is non-positive (resp. negative). Let $\eta_W = \sum_n c(n)q^n$ be the Fourier expansion at ∞. Then, we ask:

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[10] The eta-product can be more naturally introduced as the Poincare series for the action of the Coxeter element on the symmetric tensor algebra of the root lattice Q_W, which is the building block for the infinite dimensional Lie algebra attached to the singularity (cf. I.5 and II.7, 8).

"#**conjecture** *The Fourier coefficients $c(n)$ at ∞ of the eta-product η_W attached to W are non-negative if and only if η_W is not a cusp form.*

One direction 'only if' is trivial. The case $\nu_W < 0$ is also trivial. So the remaining case is when $\nu_W = 0$. The eta-products for the elliptic weight systems $\tilde{E}_l$ for $l = 6, 7$ and 8 are the first non-trivial non-cuspidal holomorphic automorphic forms with $\nu_W = 0$, where the conjecture is proven by a use of the following Euler product expansions of their Mellin transforms $L_W(s) := \sum_{n=1}^{\infty} c(n) n^{-s}$ of the eta-product.

$$L_{\tilde{E}_6}(s) = \prod_{p \neq 3} \frac{1}{(1 - \left(\frac{p}{3}\right) p^{-s})(1 - p^{-s})} = \prod_{p \equiv 1\,(3)} \frac{1}{(1-p^{-s})^2} \prod_{p \equiv 2\,(3)} \frac{1}{1-p^{-2s}}$$

$$L_{\tilde{E}_7}(s) = \frac{1}{4} \prod_{p \equiv 1\,(8)} \frac{1}{(1-p^{-s})^2} \prod_{p \equiv 5,7\,(8)} \frac{1}{1-p^{-2s}} \left(\prod_{p \equiv 3\,(8)} \frac{1}{(1-p^{-s})^2} - \prod_{p \equiv 3\,(8)} \frac{1}{(1+p^{-s})^2} \right)$$

$$L_{\tilde{E}_8}(s) = \frac{1}{4} \prod_{p \equiv 1\,(12)} \frac{1}{(1-p^{-s})^2} \prod_{p \equiv 7,11\,(12)} \frac{1}{1-p^{-2s}} \left(\prod_{p \equiv 5\,(12)} \frac{1}{(1-p^{-s})^2} - \prod_{p \equiv 5\,(12)} \frac{1}{(1+p^{-s})^2} \right)$$

6. Universal Unfolding, Discriminant and Primitive Forms

We define a universal unfolding (Thom [T]) of the polynomial f_W as a weighted homogeneous polynomial $F_W(x, y, z, \underline{g})$ where $\underline{g} = (g_1, \ldots, g_\mu)$ is a set of unfolding parameters with $\deg g_i = m_i + \varepsilon(W)$ $(1 \leq i \leq \mu)$ such that i) $F_W(x, y, z, 0) = f_W(x, y, z)$, ii) $\frac{\partial}{\partial g_\mu} F_W(x, y, z, \underline{g}) = 1$, iii) $\frac{\partial}{\partial g_i} F_W(x, y, z, 0)$ $(i = 1, \ldots, \mu)$ span the Jacobi ring J_W. [11]

[11] The F_W is not uniquely determined by these conditions, since the universal unfolding is a local analytic property near the origin [T]. In fact, two universal unfoldings differ globally by a finite correspondence of the unfoldings-parameter spaces. This fact should cause later a subtle problem of levels of primitive automorphic forms with respect to the modular groups Γ_0 and Γ_- coming from the discriminants $D_{W,0}$ and $D_{W,-}$. But, we do not go into them at this early stage.

The projection $\varphi_W : X_W := \{(x,y,z,\underline{g}) \mid F_W(x,y,z,\underline{g}) = 0\} \to S_W =$ the unfoling parameter space $= \{\underline{g}\}$ induces a $\mathbb{A}^{\times}$-equivariant fibration, where the 2-dimensional fibre over a point $\underline{g}$ is denoted by $X_{\underline{g}}$. Note that S_W obtains negatively weighted coordinate components if and only if $\varepsilon(W) < 0$. The discriminant loci $D_W = \{\Delta = 0\}$ of the fibration φ_W (= the set of points $\underline{g}$ of S_W where the fibre $X_{\underline{g}}$ 'degenerates' in a suitable sense) consists of three different type components:

$D_{W,+} = \{\underline{g} \in S_W \mid X_{\underline{g}}$ obtains an isolated singular point whose vanishing cycles also vanish at $0 \in X_0\}$,

$D_{W,0} = \{\underline{g} \in S_W \mid X_{\underline{g}}$ obtains non-isolated singular loci$\}$.

$D_{W,-} = \{\underline{g} \in S_W \mid X_{\underline{g}}$ obtains an isolated singular point whose vanishing cycles do not vanish at $0 \in X_0\}$,

where $D_{W,0} \cup D_{W,-}$ depends heavily on F_W[11]. The discriminant form Δ (= the equation for the discriminant loci as a divisor in S_W) should play a basic role in a description of the flat structure on S_W, but here we use the discriminant forms only to describe a conjecture on primitive forms.

Primitive forms are holomorphic 2-forms defined in a neighbourhood of the origin of X_W, satisfying bilinear equations (recall footnote 2), which we consider now globally on X_W for some particular cases.

i) Classical weight system W. The unfolding parameter space S_W is positively graded and the discriminant D_W consists only of $D_{W,+}$. The primitive form for this case is given by $\mathrm{Res}[\frac{dx\wedge dy\wedge dz}{F_W(x,y,z,\underline{g})}]$ up to a constant.

ii) Elliptic weight system W. There is one unfolding parameter g_1 of weight 0. The g_1 describes the deformation of the elliptic curve $E_{\underline{g}}$ which appear as the boundary $\bar{X}_{\underline{g}} = X_{\underline{g}} \cup E_{\underline{g}}$. One has $D_W = D_{W,+} \cup D_{W,0}$ where $D_{W,0}$ describes the loci where $E_{\underline{g}}$ degenerates. The primitive form is given by the proporstion $\mathrm{Res}[\frac{dx\wedge dy\wedge dz}{F_W(x,y,z,\underline{g})}]/\oint_a \mathrm{Res}[\frac{dx\wedge dy\wedge dz}{F_W(x,y,z,\underline{g})}]$ where a is a vanishing cycle, called a *marking*, vanishing at $D_{W,0}$ ([S2]).

iii) E-hyperbolic weight system. There is one unfolding parameter, say g_1, of negative weight $= 2\varepsilon(W) = -2$. One has $D_W = D_{W,+} \cup D_{W,-}$. There are some evidences which support the following conjecture.

"#**conjecture and problem 1**$'$ *The primitive form for an e-hyperbolic weight system W has the expression:*

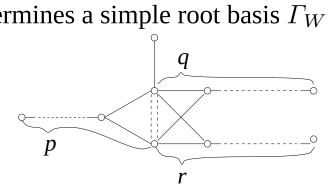
$$\mathrm{Res}\left[\frac{dx \wedge dy \wedge dz}{F_W(x,y,z,\underline{g})}\right] / \oint_\gamma \mathrm{Res}\left[\frac{dx \wedge dy \wedge dz}{F_W(x,y,z,\underline{g})}\right] \tag{5}$$

where γ is a cycle vanishing at the discriminant $D_{W,0} \cup D_{W,-}$. Do similar expressions hold for the primitive forms for W with $\varepsilon(W) \leq 0$?

7. Vanishing Cycles and Generalized Root Systems

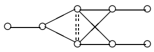
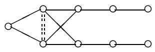
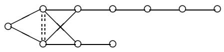
Denote by R_W (resp. $R_{W,-}$) the set of cycles ($\in H_2(X_{\underline{g}}, \mathbb{Z})$ for a generic point $\underline{g}$) vanishing as the parameter $\underline{g}$ tends to a generic point of $D_{W,+}$ (resp. $D_{W,-}$) (see Footnote 6). They form mutually orthogonal root systems with respect to the intersection form I. The root lattice $Q_W(-1) := \mathbb{Z}R_W$ is rank μ. The restriction of I on Q_W has the signature (μ_+, μ_0, μ_-) with $\mu = \mu_+ + \mu_0 + \mu_-$, $\mu_0 = 2\#\{1 \leq i \leq \mu \mid m_i = 0\}$ and $\mu_- = 2\#\{1 \leq i \leq \mu \mid m_i < 0\}$. In Problem 4 we asked to find a reasonable simple root basis of R_W. Let us illustrate some examples.

i) Classical weight system. The R_W is a finite root system and a choice of a Weyl chamber determines a simple root basis Γ_W like this [B].

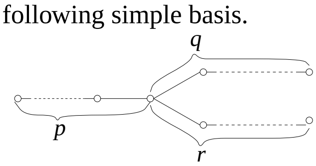


Here, p, q, r is a set of suitable integers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ depending on W.

ii) Elliptic weight system. The R_W is an elliptic root system. *Depending on the marking* a, a simple root basis Γ_W is determined ([S3,I]).

$\tilde{E}_6$:  $\tilde{E}_7$:  $\tilde{E}_8$: 

iii) E-hyperbolic weight system. The R_W is an e-hyperbolic root system. Gabrielov gave the following simple basis.



Here, p, q, r, called Gabrielov numbers, is a set of integers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ depending on W.

For further study of bases, one is referred to the works of W. Ebeling [E1], Gabrielov. On the other hand, it is interesting to observe that *the set of integers* p, q, r *in the above i) and iii) (and, conjecturely, also ii) after a good definition of the virtual dual weight system* W^**) are purely arithmetically determined as the set of signature* $A(W^*)$ *of the dual weight system* W^* (see [S6 §'s 10,12]). This suggests the following strengthening of Problem 4.

"#**problem 4′** *Depending on a choice of a primitive form (i.e. on the cycle* γ *in* (5)*), find a direct arithmetic way of describing one or several simple root basis* Γ_W *of the root system* R_W *such that the Coxeter element as the product of reflections of the basis (in a suitable order) is of finite order* h *with the characteristic polynomial* $\varphi_W(\lambda)$.

8. Coxeter Element

We restrict our attention to Problem 5 i). The flat (or, Frobenius) structure on (a covering of) S_W is determined by a choice of the primitive form (see Footnote 4). On the other hand, we saw that in the classical case, due to a solution of the second inversion problem, one has $S_W \simeq \mathfrak{h}/W_W$ for the Weyl group W_W (=the group generated by reflections of R_W) and the flat structure is determined only inside the Weyl group invariant theory (I.4, 5). Recall further that it was crucial for the description that i) a Coxeter element has primitive hth root of unities as its eigenvalue, and ii) the associated eigenvectors are regular w.r.t. the Weyl group. It is marvellous that one has already a partial generalization of this fact [S6]: *for any regular weight system W, there exist exponents which are either equal to 1 or -1. In particular, $\varphi_W(\lambda) = 0$ has always hth primitive roots of unity as its roots.* Accordingly, we conjecture the following:

"#**problem 5$'$** *The eigenspace belonging to the eigenvalue* $\exp(2\pi\sqrt{-1}/h)$ *(or,*$\exp(-2\pi\sqrt{-1}/h)$*) is regular with respect to the Weyl group W_W.*

Of course, this is true for the classical case. For the elliptic and e-hyperbolic cases, it is also true by modifying the formulation in a stronger form as will be explained in the following period domain.

9. Period Maps, Period Domain

Consider the period maps for even dimensional vanishing cycles.

i) Elliptic weight system W. The root system R_W is semidefinite and has two dimensional radical. So, the period map (defined by the solutions of the Gauss-Legendre equation for W) has one component which cannot be given by integration of the primitive form. Accordingly, the period domain is a complex one dimensional extension $\tilde{\mathbb{E}}_W$ of the complex half-space $\mathbb{E}_W := \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(Q_{W,\mathbb{R}}, \mathbb{C}) \mid \ker(\varphi) > 0\}_0/\mathbb{C}^\times$ (here > 0 means positive definite and '$_0$' means a connected component). Then Problem 5' is modifiedly true that *the 1-eigenspace for the Coxeter element defined by the diagram 7.ii) defines a 1-dimensional affine line in $\mathbb{E}_W$ which does not belong to any affine reflection hyperplane.* This fact leads to the construction of the flat structure on $\tilde{E}_W//\tilde{W}_W$ (see [S3,II], and Satake [Sat2] for the descriptions of the flat coordinates. For a description of the action of Γ_0 on the flat structure, see [Sat1]. See also Looijenga [Lo1]). The full answer to the second inversion problems is still in progress (see Helmke-Slodowy [H-S], Saito-Yoshii [S-Y], [S-T] and [Y2]).

ii) E-hyperbolic weight system W. The root lattice Q_W has the signature $(l+2,0,2)$ with $l+4=\mu_W$. So, the intersection form I is non degenerate, the period map is defined only by integration of the primitive form, and the period domain is given by $\tilde{B}_W := \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(Q_{W,\mathbb{R}},\mathbb{C}) \mid \ker(\varphi) > 0\}_0$ where the meanings of >0 and '$_0$' are as before. The problem 5' is affirmatively answered that *the two eigenspaces*$\setminus\{0\}$ *for the two eigenvalues* $\exp(\pm 2\pi\sqrt{-1}/h)$ *belong either to the period domain or to its orientation reversed component, and they are regular with respect to the Weyl group action on the period domain.*

Even though $\tilde{B}_W$ is not a classical domain, its double cover is a classical domain of type IV as follows. Recall the big lattice $L_W := Q_W \oplus \mathbb{Z}R_{W,-}(-1)$ (recall 7. for $R_{W,-}$) of signature $(l+3,0,2)$, and consider the attached symmetric domain $\tilde{\mathcal{D}}_W := \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(L_W,\mathbb{C}) \mid \varphi\cdot\varphi = 0,\ \ \varphi\cdot\bar{\varphi} > 0\}/\mathbb{C}^\times$ where we denote by '$\cdot$' the inner product induced from I on the dual space of $L_{W,\mathbb{R}} = L_W \otimes \mathbb{R}$. The restriction $\varphi \to \varphi|Q_W$ induces the quotient map $\tilde{\mathcal{D}}_W \to \tilde{B}_W$ by the action of $W(R_{W,-}) \simeq \mathbb{Z}/2\mathbb{Z}$. Since the integral of the primitive form (5) over the big lattice L_W maps the complement of the discriminant loci to the domain $\tilde{\mathcal{D}}_W$, we are naturaly lead to consider the inversion map: $\tilde{\mathcal{D}}_W \to S_W \setminus D_W$, whose flat linear factors are the primitive automorphic forms. Recall that there is a unique unfolding parameter, say g_1, of the negative weight -2 corresponding to the quadratic form $I(x,x)$, which is the unique algebraic primitive automorphic form. The rest of the primitive automorphic forms have positive weights and are transcendental forms coming from the cover $\tilde{\mathcal{D}}_W$. The descriptions of the automorphic forms on $\tilde{B}_W$ as well as on $\tilde{\mathcal{D}}_W$ are in progress (see Aoki [Ao], Zagier). For a geometric description of the period map, see [Lo2].

10. Primitive Automorphic Forms

We are still far from a complete description of the flat structure and the primitive automorphic forms for the e-hyperbolic weight systems. It seems reasonable to expect them to be described as combinations of characters of certain representations of the Lie algebras discussed in I.5. The work of Borcherds [Bo] seems to suggest a description of the discriminant form Δ for the dual weight system W^*, where the duality between the root systems of Witt index 1 and 2 seems to play a role. The clarifications of the relations of the primitive automorphic forms and the discriminant form with the 'character formula' and the 'denominator formula' for the Lie algebra $\mathfrak{g}_W$ attached to the root system R_W of the signature $(l+2, 0, 2)$ remains as the subject of study in the future.

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