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Towards a categorical construction of Lie algebras

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To the memory of Nguyen Huu Duc
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Preface

This is an introduction to the program which we call "towards a categorical construction of Lie Algebras". That is, from the data of a system of 4 integers $W := (a, b, c; h)$, called a *regular system of weights*, satisfying an arithmetic condition, we want to construct a certain generalization $\mathfrak{g}_W$ of a simple Lie algebra. Precisely, to a weight system, we first associate a surface with a singular point. Then, using the geometry of the singularity, a triangulated category is attached. Finally, we want to read Lie theoretic data from the category and to construct the algebra $\mathfrak{g}_W$.[1] The program is still in its early stages, and, in the present paper, we are mainly concerned with some categorical aspects of the program, and then ask questions on the possible constructions of Lie algebras.

The organization of the paper is as follows. In §1-9, we start by recalling the classical relations of simple or simply elliptic singularities with simple or elliptic Lie algebras, respectively, as the prototype of relations between singularities and Lie algebras. This part is rather

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[1]This is a part of the long program "a categorical construction of primitive forms" (see [Mat][Od1][Sa7] and Footnote 11 for a definition of a primitive form, and consult the overview articles [Sa15]and [Sa19]). We expect that a good class of primitive forms are constructed from the Lie algebra $\mathfrak{g}_W$ associated with regular systems of weights W (see §4 and 12). In the present paper, we are concerned with the part of the program before the construction of the Lie algebra, and most parts are readable without a knowledge of a primitive form.

sketchy and we suggest the reader either look at the references or skip details. In §10-15, we start anew by introducing the concept of a regular system of weights and by associating a singularity to it. We discuss about two geometric (algebraic and topological) aspects of the singularity and about the possibly associated Lie algebra. We discuss also about the $*$-duality on the set of regular weight systems. This part may look somehow loose and involved without a clear focus. However, these considerations seem to get converged to a clearer forcus by introducing a categorical approach in §16-18. In §16, we descripbe the triangulated category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ associated with the singularity. Then we determine the generating structure of the category for two basic cases in §17 and 18, which are the goal of the present paper.

Let us explain the contents in more details. One key observation in §1-9 is that the Lie algebra side data: the Coxeter transformation $\mathbf{c}$ on the root lattice is identified with the singularity side data: the Milnor monodromy action $\mathbf{c}$ on the lattice of vanishing cycles (see §5). As in the classical Lie theory [Bou], we consider exponents $m_i \in \mathbb{Z}_{\geq 0}$ of eigenvalues of $\mathbf{c}$ (see §8), and then, inspired by the theory of primitive forms (see Footnotes 23, 24), we look at the *generating function of the exponents*:

$$\chi(T) \;=\; T^{m_1} + T^{m_2} + \cdots + T^{m_\mu}. \tag{A}$$

Then, we observe that, for any of the simple or elliptic Lie algebras (corresponding to simple or simply elliptic singularities), $\chi(T)$ decomposes as:

$$\chi(T) \;=\; T^{-h}\frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)} \tag{B}$$

for some integers a, b, c and $h :=$ order of $\mathbf{c}$ with

$$0 < a, b, c < h \quad \text{and} \quad \gcd(a, b, c) = 1. \tag{C}$$

In §10, we reverse our view point; we call a system of 4 integers $W = (a, b, c; h)$ satisfying (C) a *regular system of weights* (or, a *regular weight system*), if the rational function in the RHS of (B) becomes a Laurent polynomial. Then, we use the regular weight system as the starting point for all of the later constructions. Actually, the Laurent polynomial becomes a finite sum of monomials as in (A), where the *exponents* m_i of the monomials are allowed to be negative in general.

The regular weight systems are concisely classified by the *smallest exponent* ($= a + b + c - h$), denoted by $\varepsilon_W \in \mathbb{Z}$. In fact, we see $\varepsilon_W \leq 1$ in general, and that regular weight systems with $\varepsilon_W = 1$ or 0 correspond to simple or simply elliptic singularities, respectively. As for the next class, $\varepsilon_W = -1$, we obtain 14+8+9 regular weight systems, which are the objects of our main interest in the present paper.

In §11-15, associated with a regular weight system W, we introduce and study a surface $X_{W,0}$ which has an isolated singular point at the origin 0. Namely, let f_W be a generic weighted homogeneous polynomial in coordinates x, y, z of weights a, b, c with the total degree h. Then, the regularity of W is equivalent to the equation $f_W = 0$ defining a hypersurface $X_{W,0}$ which has an isolated singular point at the origin 0. This is also equivalent to say that $C_W := (X_{W,0} \setminus \{0\})/\mathbb{G}_m$ being a smooth orbifold curve, where the orbifold data (i.e. signature, see §11, a)) is arithmetically determined from W. In other words, the curve C_W is equipped with a fractional $(=\varepsilon_W^{-1})$ power of the canonical bundle, and the blowing down of its zero-section is the surface $X_{W,0}$ with an isolated singular point which we want to study (see §11).

As described in §3-7, in order to get the Lie algebra $\mathfrak{g}_W$ from the simple or simply elliptic singularity, historically, there were two approaches: the algebraic one, using a resolution of the singularity, and the topological one, using the set of vanishing cycles (see §5) in a smoothing (Milnor fiber) of the singularity. Let us see below how these two approaches work for each of the cases ε_W=1 and 0.

Case ε_W=1 (the simple singularity): in the first approach, the *resolution diagram of the simple singularity* is identified with the Dynkin diagram of a simple Lie algebra (Du Val, see §3), and defines its Cartan matrix. Then, as is standard in Lie theory, by the use of Chevalley generators and Serre relations associated to the Cartan matrix, we obtain a simple Lie algebra $\mathfrak{g}_W$. On the other hand, in the second approach, *the set of vanishing cycles in the middle homology group of a smoothing (= Milnor fiber) of the singularity is identified with the set of roots of a finite root system in its root lattice of a simple Lie algebra* (see §7). Then, inside the lattice vertex algebra [Bo1] of the root lattice, we consider the Lie-algebra $\mathfrak{g}'_W$ generated by the vertex operators e^α of the roots α ([S-Y]§1). The Lie algebras $\mathfrak{g}_W$ and $\mathfrak{g}'_W$ constructed by these two approaches are canonically isomorphic, due to the fact that the vertices of the Dynkin diagram obtained by the first approach gives arise a simple basis of the root system obtained by the second approach, because of the existence of the simultaneous resolution of the simple singularity due to Brieskorn (§4 [Br1]). Further, Brieskorn's description of the universal family of the simple singularity enables us to describe a primitive form by the Kostant-Kirillov forms on co-adjoint orbits of a simple Lie group.

Case $\varepsilon_W = 0$ (the simply elliptic singularity): the first approach to use the exceptional set of the resolution of the singularity gives merely a single elliptic curve, and Lie theoretic data is not apparent (see Footnote 3). On the other hand, the data of the second approach, i.e. the set of vanishing cycles of a simply elliptic singularity, is characterized as the set of roots of an elliptic root system ([Sa 14] I, see §7 and Footnote 17). As in the case of $\varepsilon_W = 1$, we get the Lie algebra $\mathfrak{g}'_W$ generated by the vertex operators of elliptic roots inside the lattice vertex algebra of the elliptic root lattice. On the other hand, we construct arithmetically a certain root basis for the elliptic root system, called the elliptic diagram (Table 7). Then, as in the first approach for the case of $\varepsilon_W = 1$, we can construct a Lie algebra $\mathfrak{g}_W$ by generalizing the Serre relations associated to the Cartan matrix of the elliptic diagram. Actually, these two Lie algebras $\mathfrak{g}_W$ and $\mathfrak{g}'_W$ are shown to be naturally isomorphic; we call this algebra the *elliptic Lie algebra* (see §6 and [S-Y]).[2]

At this stage, we remark that there is a third approach for the construction of Lie algebras $\mathfrak{g}''_W$ by use of the representation theory of finite dimensional algebras, which is sometimes called the Ringel-Hall construction. Namely, Ringel [Ri 2,3,4] has determined the structure constant among the Chevalley basis of a simple Lie algebra by using the data of representations of a *hereditary algebra* (c.f. [Ga]). The idea was further extended to the representation theory of *tubular algebras* by Lin-Peng [L-P 1,2], and they obtained the elliptic Lie algebras of types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ (which are exactly the cases when the elliptic Lie algebras are expected to admit primitive forms, [Sa14]II). In fact, those hereditary algebras and tubular algebras are obtained as the path algebras (see §16 **6.**(32)) of quivers associated to the classical Dynkin diagrams or to the elliptic diagrams, respectively. Since the Lie algebra depends only on the derived category of the abelian category of modules over the path algebra, some generalizations of the method in terms of triangulated category are in progress. The reader is referred to [P-X], [Toë], [D-X] and [X-X-Z] for details.

We examine, in the present paper, the "Lie theoretic data" of the above mentioned three approaches for the case $\varepsilon_W = -1$.

The singularities associated with the 14 weight systems with $\varepsilon_W = -1$ are called exceptional uni-modular singularities by Arnold [Ar3]. 1. Topological approach: certain distinguished bases of the lattices of vanishing cycles for them have been obtained by Gabrielov ([Gab2], see

[2]As in simple Lie algebra case, the symplectic structures on the co-adjoint orbits of the elliptic Lie group are expected to form a primitive form. See [Sa14] VI (Integrable Highest Weight Modules), VII (Elliptic Groups and their Invariants), in preparation, and [Ya3]).

Table 12), where the triplet (p,q,r) of lengths of the three branches of the diagram is called the Gabrielov number. 2. Algebraic approach: the exceptional set of the minimal resolution of the 14 singularities is given by a star-shape configuration of 4 rational curves (see Table 11), where the triplet (p,q,r) of the minus of the self-intersection numbers of the three branching curves is called the Dolgachev number. Then Arnold observed that there is an involutive one to one correspondence from the set of 14 exceptional uni-modular singularities to itself, which exchange the Gabrielov number and the Dolgachev number. The involution is called the *Strange duality* ([Ar3],§13). In the other words, the "Lie theoretic data" of the two approaches are exchanged by the strange duality.

The strange duality, which is nowadays understood as an appearance of *mirror symmetry* [3], admitted several interpretations and explanations. Among these, in §14, we introduce *$*$-duality on regular systems of weights*, which is an involution $*$ on a set of regular systems of weights characterized as follows: let us introduce the characteristic polynomial of the weight system W by $\varphi_W(\lambda):=\prod_{i=1}^{\mu}(\lambda-\exp(2\pi\sqrt{-1}\frac{m_i}{h}))\in\mathbb{Z}[\lambda]$. As a cyclotomic polynomial, we decompose it as $\varphi_W(\lambda)=\prod_{i|h}(\lambda^i-1)^{e_W(i)}$. Then, another regular weight system W^* is the $*$-dual of W if and only if $h=h^*$ and $e_W(i)+e_{W^*}(h/i)=0$ for all $i\in\mathbb{Z}_{>0}$ with minor additional conditions.[4] Then, we prove that any weight system with $\varepsilon_W=1$ is selfdual; $W=W^*$, and that the $*$-duality induces the strange duality on the set of 14 weight systems with $\varepsilon_W=-1$. Therefore, we expect in general that the $*$-duality exchanges the algebraic approach for a weight system W with the topological approach for the dual system W^*. Then, instead of the naive study of resolution diagrams of the singularity $X_{W,0}$ in the algebraic side of W, what stands for the lattice and the basis of vanishing cycles of $X_{W^*,0}$ in the topological side of W^*?

Inspired by the recent studies of D-branes on mirror symmetry in mathematical physics ([K-L 1,2], [H-W], [Wal] and [Or1], see §15), we study *the homotopy category* $\mathrm{HMF}^{gr}_{A_W}(f_W)$ *of matrix factorizations of*

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[3]The reader is referred to [Kon],[Yau] for mirror symmetry in general and to [K-Y][Ta1] for the Landau-Ginzburg orbifold case. Already in case of $\varepsilon_W=0$, the algebraic data, i.e. the elliptic curve in the exceptional set in the resolution of the singularity, is not "mirror dual" to the elliptic root system of vanishing cycles obtained topologically. In order to get mirror symmetry here, one should think of the elliptic curve with a group action [Ta1]. A more comprehensive description is obtained by considering the pairs of a regular weight system and a group action. However, in the present paper, we do not get into such details.

[4]The $*$-dual of W may not exist for all W, but is unique if it exists and is denoted by W^* with $W^{**}=W$ [Sa17]. It seems interesting to extend the concept of regular systems of weights (by considering group actions (Footnote 3) and non-hypersurface singularities), which is closed under the $*$-duality.

the polynomial f_W as the algebraic approach.[5] We devote §16 to the descriptions of three different definitions of this category and its basic properties. We expect that the advantage of this approach is that this category carries a "universality" such that it can recover all the three approaches to the Lie algebra, which we have discussed above.[6]

In §17 and 18, we observe and explain this fact in the case of the category for simple singularities with $\varepsilon_W = 1$ and for the exceptional singularities with $\varepsilon_W = -1$.

We show that the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ for $\varepsilon_W=1$ is generated by a strongly exceptional collection $\mathcal{E}$ (see §16 **4.**), whose associated quiver is a Dynkin quiver $\vec{\Delta}$ of type W, and that the path-algebra $\mathbb{C}\vec{\Delta}$ (see §16 **6.**) is isomorphic to the algebra $\mathrm{End}(\mathcal{E})$ consisting of all morphisms among the objects of the exceptional collection. Therefore, we have the equivalence $\mathrm{HMF}^{gr}_{A_W}(f_W) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta})$ due to a theorem of Bondal-Kaplanov (see §16 **4**). Hence, using the classical result by Gabriel [Ga], the K-group $K_0(\mathrm{HMF}^{gr}_{A_W}(f_W))$ and the image set in the K-group of indecomposable objects of the category are isomorphic to the root lattice and the set of roots of a finite root system, respectively. That is, $\mathrm{HMF}^{gr}_{A_W}(f_W)$ recovers all three data for the Lie algebra discussed above, inducing the natural isomorphisms $\mathfrak{g}_W \simeq \mathfrak{g}'_W \simeq \mathfrak{g}''_W$ among them.

In the case $\varepsilon_W = -1$, the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ is generated again by a strongly exceptional collection $\mathcal{E}$ whose associated quiver Δ_A is given in Table 14, where A is the signature set (13) of W (see Footnote 32). We show again an isomorphism $\mathrm{End}(\mathcal{E}) \simeq \mathbb{C}(\Delta_A, R)$ and an equivalence $\mathrm{HMF}^{gr}_{A_W}(f_W) \simeq D^b(\text{mod-}\mathbb{C}(\Delta_A, R))$ of the categories, where $\mathbb{C}(\Delta_A, R)$ is the quotient of the path-algebra $\mathbb{C}\Delta_A$ by the relations R (see (32)and §18 Theorem). Hence, in the 14 uni-modular exceptional cases, comparing Table 12 with 14 and in view of the strange duality, we conclude that the K-group $K_0(\mathrm{HMF}^{gr}_{A_W}(f_W))$ is isomorphic to the lattice of vanishing cycles for the $*$-dual weight system W^*; this is what we expected.

We conjecture that the image set in the K-group of exceptional indecomposable objects of the category coincides with the set of vanishing cycles for the singularity $X_{W^*,0}$, and, hence, the three approaches to the Lie algebra are available from the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$. Whether the three Lie algebras $\mathfrak{g}_W$, $\mathfrak{g}'_W$ and $\mathfrak{g}''_W$ for them are isomorphic to each other or not is an interesting and important open problem.

[5] This was proposed by Takahashi [Ta2] (c.f. Orlov [Orl1]) answering a problem posed by the author [Sa15] (5.3) Problem. The sections §16, 17 and 18 are based on the joint works [K-S-T 1-2].

[6] It is also remarkable that the stability condition space [Br1][H-M-S] on this category seems to have a close relationship with the period domain for period maps of primitive forms [Sa22].

§1. Simple polynomials

There are a finite number of regular polyhedra, namely, the icosahedron, dodecahedron, octahedron, hexahedron and the tetrahedron, known at the time of Platon. The regular dihedron, which has only two faces of the n-gon ($n \geq 3$), is nowadays included in the list of regular polyhedra. The subgroup G of $SO(3)$ consisting of rotations of three dimensional Euclidean space, which moves a regular polyhedron (centered at the origin) to itself, is called the regular polyhedral group. The binary extension $\tilde{G}$ of the regular polyhedral group G is obtained by taking the inverse image of G through the surjective homomorphism $SU(2) \to SO(3)$. It is well-known that the binary regular polyhedral groups (including binary dihedral groups) and the cyclic subgroups $Z_n := \left\langle \begin{pmatrix} \exp \frac{2\pi\sqrt{-1}}{n} & 0 \\ 0 & \exp\left(-\frac{2\pi\sqrt{-1}}{n}\right) \end{pmatrix} \right\rangle$ for $n \in \mathbb{Z}_{>0}$ together form a complete list of finite subgroups of $SU(2)$ up to conjugacy. As an abstract group, all of the groups have a presentation:

$$\langle p, q, r \rangle := \langle\ x, y, z \ \mid\ x^p = y^q = z^r = xyz \rangle$$

for suitable integers $p, q, r \in \mathbb{Z}_{>0}$, given in the next Table 1 (here, x, y and z induces the rotation of the polyhedron centered at the barycentre of an edge, a face and a vertex).

$\langle 1, b, c \rangle$	$\simeq$ Z_n	$\simeq$	cyclic group of order $n = b + c$
$\langle 2, 2, n \rangle$	$\simeq$ $\mathcal{D}_{2n}$	$\simeq$	binary dihedral group of n-gon $n \geq 2$
$\langle 2, 3, 3 \rangle$	$\simeq$ $\mathcal{A}_4$	$\simeq$	binary regular tetrahedral group
$\langle 2, 3, 4 \rangle$	$\simeq$ S_4	$\simeq$	binary regular octahedral group
$\langle 2, 3, 5 \rangle$	$\simeq$ $\mathcal{A}_5$	$\simeq$	binary regular icosahedral group

Table 1.

In fact, these are the only cases when the group $\langle p, q, r \rangle$ is finite (see [C-M]). The group is sometimes called the Kleinean group because of the following result due to A. Schwarz [Sc] and F. Klein [Kl1].

Theorem. *Let $\tilde{G} \subset SU(2)$ be a Kleinean group. Let it act linearly on $\mathbb{C}^2$, and, hence, on the ring $\mathbb{C}[u, v]$ of polynomial functions on $\mathbb{C}^2$ (where u, v ares coordinates of $\mathbb{C}^2$). Then the subring $\mathbb{C}[u, v]^{\tilde{G}} := \{P \in \mathbb{C}[u, v] \mid gP = P\ ^\forall g \in \tilde{G}\}$ of invariants is generated by 3-homogeneous elements, say x, y and z, which satisfy a single relation, say $f_{\tilde{G}} = f(x, y, z)$. That is:*

$$\mathbb{C}[u, v]^{\tilde{G}} \simeq \mathbb{C}[x, y, z]/(f_{\tilde{G}}).$$

The polynomial $f_{\tilde{G}}$ is called a *simple polynomial*, which is listed in the following table.

Type	$f_{\tilde{G}}$	Kleinean group
A_l	$x^{l+1} + yz$	Z_n
D_l	$x^2y + y^{1-1} + z^2$	$\langle 2, 2, n\rangle$
E_6	$x^4 + y^3 + z^2$	$\langle 2, 3, 3\rangle$
E_7	$x^4 + xy^3 + z^2$	$\langle 2, 3, 4\rangle$
E_8	$x^5 + y^3 + z^2$	$\langle 2, 3, 5\rangle$

The Types in the left-side shall be explained in §3.

Table 2.

Note. From the polynomial $f_{\tilde{G}}$, one can recover $\tilde{G}$. See Appendix 3.

F. Klein, in the introduction to his lecture notes on the icosahedron [Kl1], described the time when he and Lie studied together in Berlin and Paris during the years 1869-70: “At that time we jointly conceived the scheme of investigating geometric or analytic forms susceptible of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labors, though these may have appeared to lie far asunder. Whilst I primary directed my attention to groups of discrete operations, and was thus led to the investigation of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations”.

§2. Simple Lie algebras and root systems

Let us explain another stream of mathematics started from Lie and Killing-Cartan.

The Lie algebras describe “the infinitesimal structure of continuous groups”. The series of works [Ki] by Killing starting from the year 1888, determining the structure of *simple Lie algebras* (which was completed by E. Cartan [Ca]) has introduced a new mathematical structure (see [Ha]) which goes far beyond the class of simple Lie algebras, and is strongly influential on the present program.

Killing looked at the adjoint action of the maximal abelian (Cartan) subalgebra of a simple Lie algebra and decomposed the Lie algebra into a direct sum of equi-eigenspaces of the action. Since an equi-eigenvalue (as an element of the dual space of the Cartan subalgebra) is a root of the characteristic eigen-equation, he called it a *root* (*Wurzel*), and showed that the system of roots for a simple Lie algebra satisfies some properties, which are nowadays known as the axioms for a finite root system (see ([Bou]§6n^{o}1)). The classification of simple Lie algebras is reduced to the classification of finite root systems. In fact, it is achieved by determining the matrix $(2I(\alpha,\beta)/I(\alpha,\alpha))_{\alpha,\beta\in\Gamma}$ (called the Cartan matrix), where I is Killing form on the root lattice and Γ is a simple basis of the root system[7].

§3. Du Val diagrams and Coxeter diagrams

Let us see how the two streams of mathematics, one starting with Klein and the other with Lie-Killing, meet again in the year 1934, when Du Val and Coxeter were together at Trinity college in Cambridge. At that time, the concept of the Weyl group, generated by reflections s_α for all roots α of the Lie algebra, was established in connection with the representation theory of simple Lie algebras (Weyl [We] (1925-6) and Cartan [Ca]). The classification of root systems is reduced to the classification of the Weyl group [Wae]. Then Coxeter, by use of the fundamental domain (=Weyl chamber) of the Weyl group, classified all finite reflection groups acting on Euclidean space. Namely, he gave an explicit presentation of the Weyl group in terms of generators and relations, known as the Coxeter relations [Co1].[8] For the classification, he introduced a diagram (tree) Γ, where the vertices correspond to the generators and an edge is drawn between two vertices which are non-commutative (see [Bou] for more details on reflection groups). In Table 3, the Coxeter's diagram for the Weyl groups of types A_l, D_l, or E_l are given by removing i) the vertex ρ_0 of the diagram and ii) the "tilde $\tilde{}$ "

[7]Recall [Bou](chap.6 §1 5.) that a simple basis of a (finite) root system is characterized as a system of linear forms on the Cartan algebra, whose zeros define the system of walls (oriented to the inside) of a Weyl chamber. It is admirable that, even at such an early stage (1888) of the study of simple Lie algebras, Killing (see [Ki]S12,13) began to study root basis Γ, the product $\prod_{\alpha\in\Gamma} s_\alpha$ of the reflections s_α associated to the basis (presently known as the Coxeter-Killing transformation) and its eigenvalues (which presently defines the exponents). However, for their geometric significance in terms of the Weyl group and chambers, one must wait until Weyl's work [We]. As we shall see, finding generalizations of the simple root basis, Coxeter- Killing transformations and the exponents are central problem in the present paper.

[8]The generators are given by the reflections attached to the walls of the chamber (which is bijective to the set Γ of simple basis of Killing) and the relations are given by the dihedral group relations for every pair of generators along 2-codimensional facets of the chamber. The higher codimensional facets of the chamber do not play a role in determining the group.

from the types in RHS of table (see Appendix for more details on the table).

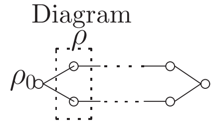
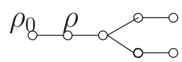
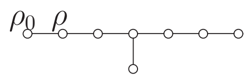
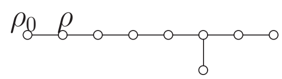
Kleinean group	Diagram	Type
Z_n		$\tilde{A}_{n-1}$
$\langle 2, 2, n\rangle$		$\tilde{D}_{n+2}$
$\langle 2, 3, 3\rangle$		$\tilde{E}_6$
$\langle 2, 3, 4\rangle$		$\tilde{E}_7$
$\langle 2, 3, 5\rangle$		$\tilde{E}_8$

Table 3.

The complex hypersurface X_0 in $\mathbb{C}^3$ defined by the zero-loci of a simple polynomial in the list of Klein (Table 2) has an isolated singular point at the origin 0 (cf. §11 Fact4.), called a *simple singularity* [Dur]. In the year 1934, Du Val [Du] studied the (minimal) resolution $\pi : \tilde{X}_0 \to X_0$ of the simple singularity. He associated a diagram Γ to the resolution: decompose the exceptional set $E := \pi^{-1}(0)$ into irreducible components $\cup_{i=1}^{l} E_i$, then, vertices x_i of the diagram are in one to one correspondence with irreducible components E_i and an edge is drawn between x_i and x_j if and only if $E_i \cap E_j \neq \emptyset$. He observed that for each Kleinean group on the LHS of Table 3, the diagram he obtained is exactly the one given in the middle of the Table 3, deleting the vertex ρ_0. In the introduction of [Du], he wrote “It may be noted that the “trees” of curves which we have had to consider bear a strict formal resemblance to the spherical simplices whose submultiple of π, considered by Coxeter”. In the same volume of the London Journal, Coxeter [Co1] listed diagrams for reflection groups, answering to a request of Du Val (for the definitions of diagrams for a basis of a lattice, see Footnote 41, and for a quiver, see §16, 6).

§4. Universal unfolding of simple singularities by Brieskorn

We observed in §3 that *there is a one to one correspondence between the diagrams of Du Val associated to simple polynomials and those of Coxeter in the classification of simple Lie algebras* (recall Table 3). However, at this stage, their relation remained a "strict resemblance", as Du Val wrote. A more direct and decisive relationship was found 40years later in the work of Brieskorn and Grothendieck. In ICM Nice 1970, Brieskorn [Br4] reported the following result.

Theorem. (Brieskorn [Br4]) *Let $X \to S$ be the universal unfolding* [9] *of a simple singularity, and let $\mathfrak{g}$ be the corresponding simple Lie algebra. Then, one has a commutative diagram:*

$$\begin{array}{ccccc} X & \subset & \mathfrak{g} & & \\ \downarrow & & \downarrow & & \\ S & \simeq & \mathfrak{g}/\!\!/Ad(\mathfrak{g}) & \simeq & \mathfrak{h}/\!\!/W \end{array}$$

where i) the vertical arrow in right side of the diagram is the adjoint quotient morphism due to Chevalley's theorem, and ii) $X \subset \mathfrak{g}$ is an embedding of X onto a transversal slice to the nilpotent subvariety of $\mathfrak{g}$ at a subregular element.

Brieskorn further described the simultaneous resolution (c.f. [Br1,2]) of the universal family.[10] He wrote "Maybe the two theories do not lie so far asunder".

Remark 1. The Brieskorn's description of the universal unfolding $X \to S$ of a simple singularity by use of a simple Lie algebra has the advantage in determining certain global differential geometric structures on the family $X \to S$, since, in the Lie algebra $\mathfrak{g}$, the integrability conditions are already built in. For instance, the primitive form of the family $X \to S$ [11], which is defined by an infinite system of non-linear equations, for the simple singularity is described by the Kostant-Kirirov symplectic form

[9]The concept of an unfolding of a singularity of a function f is due to R. Thom [Th]. We shall give in §5 and in Footnote 12. a brief description of them. From an algebraic geometric view point, it is essentially the same concept as a semi-universal deformation of the hypersurface defined by $f = 0$ near at the singular point (see [Sch] and [Tu]).

[10]This was reproven by a use of representation of quivers [Kr] (see the works by H. Nakajima for further studies on the relationship between Lie algebras and representations of quivers).

[11]For a primitive form, see [Mat][Od1][Sa7][Sa19]. It is a relative de-Rham cohomology class $\zeta \in \mathbf{H}_{DR}(X/S)$ which 1) generates all the other de-Rham

[Sa7] [Yah] [Ya1] [Yo]. The flat structure (Frobenius mfd structure) on the deformation parameter space S is described by the Coxeter-Killing transformation of the Weyl group [Sa16] [He] [Sab].

These facts motivated the author to convince the following: *for a further class of singularities, using suitable Lie algebras, construct primitive forms and flat structures globally.* However, the list of regular polyhedral groups and that of the simple Lie algebras have already been used up. Are these the only cases where singularity theory and Lie theory come happily together?

§5. Universal unfolding of a hypersurface singularity

Before we go further, we prepare some terminologies on vanishing cycles of a hypersurface isolated singular point studied by authors [Br3] [Le1] [Gab1] [Eb1].

Let $f(\underline{x})$ with $\underline{x} := (x_0, \cdots, x_n)$ $(n \geq 0)$ be a holomorphic function defined in a neighborhood $\mathcal{U}$ of the origin 0 of $\mathbb{C}^{n+1}$ with the coordinate $\underline{x}$. Assume that the hypersurface $X_0 := \{(\underline{x}) \in \mathcal{U} \mid f(\underline{x}) = 0\}$ has an isolated singular point at the origin $0 \in X_0$. This is equivalent to that $J_f := \mathbb{C}\{\underline{x}\}/\left(\frac{\partial f(\underline{x})}{\partial x_0}, \cdots, \frac{\partial f(\underline{x})}{\partial x_n}\right)$ is of finite rank over $\mathbb{C}$, where $\mathbb{C}\{\underline{x}\}$ is the local ring of all convergent series in $\underline{x}$.

Theorem. (Milnor [Mi]) *Consider a map $f : X_{\delta,\varepsilon} \to D_\varepsilon$ where $X_{\delta,\varepsilon} := \{\underline{x} \in \mathcal{U} \mid |\underline{x}| < \delta\} \cap f^{-1}(D_\varepsilon)$ and $D_\varepsilon := \{t \in \mathbb{C} \mid |t| \leq \varepsilon\}$ for positive real numbers δ, ε such that $0 < \varepsilon \ll \delta \ll 1$. Then, $f|_{X \setminus f^{-1}(0)} : X \setminus f^{-1}(0) \to D_\varepsilon \setminus \{0\}$ is a locally trivial topological fibration such that the general fiber is homotopic to a bouquet of μ_f-copies of n-sphere S^n, where $\mu_f := \dim_{\mathbb{C}} J_f$ is called the Milnor number.*

The fibration is called the Milnor fibration, whose general fiber over a base point $1 \in D_\varepsilon$, denoted by X_1, is called the *Milnor fiber*. *If f is globally defined weighted homogeneous polynomial of positive weights, then we may choose $\delta = \varepsilon = \infty$.*

As a consequence of this result, the (reduced) homology group of the Milnor fiber is non-trivial only in dimension n, and we have $\tilde{\mathrm{H}}_n(X_1, \mathbb{Z}) \simeq \mathbb{Z}^{\mu_W}$. Let us introduce particular elements of $\tilde{\mathrm{H}}_n(X_1, \mathbb{Z})$, called *vanishing*

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cohomology classes as a $\mathcal{D}_S$-module, and 2) satisfies an infinite system of bilinear differential equation (by means of residue pairings). Its local existence on S is known by [Sai]. Global existence on S is known only for simple or simply elliptic singularities. It is believable that $\mathfrak{g}$ is the Cartan prolongation of X with respect to the primitive form. Such global construction of primitive forms by means of globally defined integrable systems (such as Lie algebras) is the basic motivation in the present paper. However, we shall not discuss the primitive form itself in any further detail.

cycles: let us consider a *universal unfolding* of f (Thom [Th]), which is a function $F(\underline{x},\underline{t})$ in $\underline{x}\in\mathbb{C}^{n+1}$ and $\underline{t}=(t_1,\cdots,t_{\mu_f})\in\mathbb{C}^{\mu_f}$ defined in a neighborhood of the origin $(\underline{0},\underline{0})\in\mathbb{C}^{n+1}\times\mathbb{C}^{\mu_f}$ satisfying i) $F(\underline{x},\underline{0})=f(\underline{x})$, and ii) $\frac{\partial F(\underline{x},0)}{\partial t_i}$ $(i=1,\cdots,\mu_f)$ span the $\mathbb{C}$-vector space J_f.

For a small value of $\underline{t}$, again by choosing δ and ε suitably for $f_{\underline{t}}(\underline{x})=F(\underline{x},\underline{t})$, we consider the map $f_{\underline{t}}:X_{\delta,\varepsilon}\to D_\varepsilon$ such that, excluding finite number of its fivers over the critical values, it gives a locally trivial fibration, whose general fiber is homeomorphic to the Milnor fiber. If $\underline{t}$ is general, then $f_{\underline{t}}|_X$ has exactly μ_f-number of non-degenerate critical points and the (critical) values are distinct (that is, $f_{\underline{t}}$ is a Morsification of f). We may choose the "base point" 1 whose fiber $f_{\underline{t}}^{-1}(1)$ is the Milnor fiber X_1 on the boundary of the disc D_ε. Let $g:[0,1]\to D_\varepsilon$ be any continuous path starting at the base point $1\in D_\varepsilon$ and ending at a critical value c, without passing any critical points on $[0,1)$. Then the pull-back $X_{[0,1]}$ of the fibration $X\to D_\epsilon$ over the interval $[0,1]$ retracts to X_c. Thus, the natural inclusion $X_1\subset X_{[0,1]}$ induces a homomorphism $\iota:\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\to\tilde{\mathrm{H}}_n(X_c,\mathbb{Z})$ whose kernel $\ker(\iota)$ is rank 1 module $\mathbb{Z}$ (since the Hessian of $f_{\underline{t}}$ at the critical point is non-degenerate).

Definition Let the setting be as above. A base e (up to sign) of the kernel $\ker(\iota)$ in $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$ is called a *vanishing cycle* along the path g. We denoteby R_f the set of all vanishing cycles running all possible paths g and the critical values c.

Let γ be a path in D_ε which starts at the base point 1 and move along g close to the critical value c and then turns once around c counterclockwisely, and then return to 1 along g. This path induces the monodromy $\rho(\gamma)\in\mathrm{Aut}(\tilde{\mathrm{H}}_n(X_1,\mathbb{Z}))$, whose action on $u\in\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$ is described by the following Picard-Lefschetz formula:

$$\rho(\gamma)(u) \;=\; u-(-1)^{\frac{n(n-1)}{2}}(u,e)e$$

where $(\cdot,\cdot):\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\times\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\to\mathbb{Z}$ is the intersection form on the middle homology group (see Footnote 35). If n is even, it is symmetric and $(e,e)=(-1)^{n/2}2$ so that $\rho(\gamma)$ is a *reflection action* with respect to the vector e, denoted by w_e.

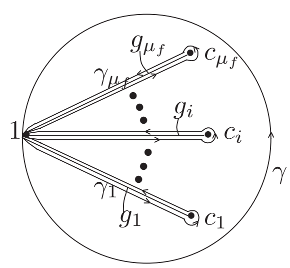


Table 4.

Now, we describe the *distinguished basis* of the middle homology group $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$, depending on two choices: i) to give a numbering of the critical values, say $c_1,\cdots,c_{\mu_f}$, of $f_{\underline{t}}$, ii) to choose μ_f paths $g_1,\cdots,g_\mu$ in D_ε such that a) each g_i is a path connecting 1 with c_i as above, which is not self-intersecting, b) distinct paths g_i and g_j are intersecting only at 1, and c) the passes $g_1, \cdots, g_\mu$ are starting at the point 1 in the linear order $1, \ldots, \mu_f$ counter-clock wisely (see Table 4).

Fact-Definition. Under the above the setting, the set $e_1,\cdots, e_{\mu_f}$ of vanishing cycles (up to choices of sign) associated to the paths $g_1,\cdots, g_{\mu_f}$ form an ordered basis of $\tilde{\mathrm{H}}_n(X_1, \mathbb{Z})$, called a *distinguished basis* (see [Br3], [Le1], [Gab1], [Eb1])

Monodromy. Let γ be the path starting at 1 turning once around the boundary of D_ε counter-clock wisely and comes back to 1. The monodromy of this path $\mathbf{c} := \rho(\gamma) \in \mathrm{Auto}(\tilde{\mathrm{H}}_n(X_1, \mathbb{Z}))$ is called the *Milnor monodromy.* Since γ is homotopic to the product $\gamma_1 \cdots \gamma_{\mu_f}$ of paths γ_i (see Table 4), we express the monodromy $\mathbf{c}$:

$$\mathbf{c} \; = \; w_{e_1} \cdots w_{e_{\mu_f}}$$

as a product of reflections associated to a distinguished basis $e_1, \cdots, e_{\mu_f}$.

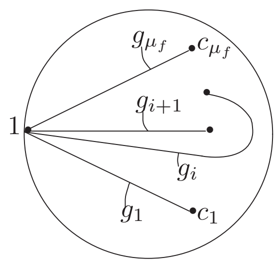


Table 5.

Braid group B_{μ_f} action on distinguished basis: First, we remark that the homotopy classes of the paths $\gamma_1, \cdots, \gamma_{\mu_f}$ give a free generator system of the group $\pi_1(D_\varepsilon \setminus \{c_1,\cdots, c_{\mu_f}\}, 1)$. Thus the choice of the paths $g_1,\cdots, g_{\mu_f}$, up to homotopy, corresponds to a choice of a free generator system of the free group. On the other hand, the braid group B_{μ_f} acts on the set of free generator systems, as usual as follows: for $1 \le i < \mu_f$, define an action σ_i : $\gamma_1, \cdots, \gamma_{\mu_f} \mapsto \gamma_1, \cdots, \gamma_{i-1}, \gamma_i\gamma_{i+1}\gamma_i^{-1}, \gamma_i, \gamma_{i+2}, \cdots, \gamma_{\mu_f}$. This causes an action of σ_i on paths $g_1, \cdots, g_{\mu_f}$ to those given in Table 5. and on the distinguished basis $e_1, \cdots, e_{\mu_f}$ to the distinguished basis $e_1, \cdots, e_{i-1}, w_{\gamma_i}(e_{i+1}), e_i, e_{i+2}, \cdots, e_{\mu_f}$. One can immediately verify that σ_i $(1 \le i < \mu_f - 1)$ satisfy Artin braid relations (see [Ar]) so that we obtain *a braid group action on the set of distinguished basis.*

Remark 2. Even if we start with a globally defined weighted homogeneous polynomial f of positive weights, in order to construct the fibration $f_{\underline{t}} : X \to D_\varepsilon$ above, we need to shrink the domain of $f_{\underline{t}}$ suitably by a use of δ and ε as above. In fact, if one of the coordinate t_i has negative weight (c.f. §11,b),4)), the embedding of a Milnor fiber $X_{\underline{t}}$ into the global affine surface $\hat{X}_{\underline{t}} := \{\underline{x} \in \mathbb{C}^{n+1} \mid F(\underline{x}, \underline{t}) = 0\}$ induces a non-trivial extension $\tilde{\mathrm{H}}_n(X_{\underline{t}}, \mathbb{Z}) \subset \tilde{\mathrm{H}}_n(\hat{X}_{\underline{t}}, \mathbb{Z})$. The extension is achieved by adding the lattice of the vanishing cycles "coming from ∞" and is expected to play key role in analytic theory of primitive forms (see [Sa19]§6 Conjecture and Problem I').

Remark 3. In mathematical physics, hypersurface singularity is studied under the name of Landau-Ginzburg model.

§6. Simply elliptic singularities

We return to the main stream of our considerations in the present paper: to seek for a connection of primitive forms with Lie theory.

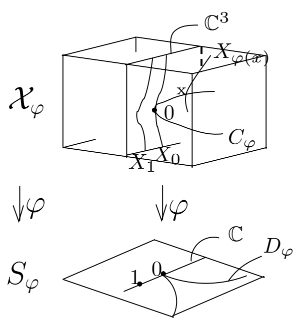
In the year 1974, the author [Sa2] came up with a new class of normal surface singularities, which are "located on the boundary" of the deformation space of simple singularities. They are called the *simply elliptic singularities*, which include the following three types of hypersurfaces:

Type	equation f_W	$E \cdot E$	(μ_+, μ_0, μ_-)
$\tilde{E}_6$ or $E_6^{(1,1)}$	$x^3 + y^3 + z^3 + \lambda xyz$	-3	0, 2, 6
$\tilde{E}_7$ or $E_7^{(1,1)}$	$x^4 + y^4 + z^2 + \lambda xyz$	-2	0, 2, 7
$\tilde{E}_8$ or $E_8^{(1,1)}$	$x^6 + y^3 + z^2 + \lambda xyz$	-1	0, 2, 8

Table 6.

The simple elliptic singularities X_0 are characterized from two different view points: a) by the resolution of the singularity X_0: *a normal singular point 0 of a surface X_0 is simply elliptic if and only if,* by definition, *the exceptional set $E = \pi^{-1}(0)$ of the minimal resolution $\pi : \tilde{X}_0 \to X_0$ of the singularity contains only a single elliptic curve*, and b) by deformation of the singularity: *a singular point 0 of a hypersurface surface X_0 is either simple or simply elliptic if and only if any singularity in a local deformation of X_0 admits a weighted homogeneous structure.*[12]

[12]Let us explain what do we mean by 1. "singularity in a local deformation of X_0", and 2. "weighted homogeneous structure" on a singularity X_0.



Local deformation of X_0

1. Recall §5 the universal unfolding $F(\underline{x}, \underline{t})$ defined in a neighborhood $\tilde{\mathcal{U}}$ of the origin of $\mathbb{C}^{n+1} \times \mathbb{C}^{\mu_f}$. Then, it defines a local analytic flat family of analytic varieties $\varphi : \mathcal{X}_\varphi \to S_\varphi$ where $\mathcal{X}_\varphi := \{(\underline{x}, \underline{t}) \in \tilde{\mathcal{U}} \mid F(\underline{x}, \underline{t}) = 0\}$, S_φ is a neighborhood of the origin of $\mathbb{C}^{\mu_f}$, and φ is the projection to the second factor. The fiber $\varphi^{-1}(0)$ over 0 is nothing but the original singular surface X_0 so that the family $\{X_{\underline{t}} := \varphi^{-1}(\underline{t})\}_{\underline{t} \in S_\varphi}$ is called the semi-universal deformation of the singularity X_0 ([K-S], [Sch]). One can show that the critical set C_φ of the map φ is (locally near at the origin 0) a smooth subvariety of dimension $\mu_f - 1$, which is finite over S_φ so that the image $D_\varphi := \varphi(C_\varphi)$ is (locally near at 0) is a hypersurface in S_φ, called the

Here in the case of simply elliptic singularity, a) the resolution diagram in the sense of Du Val consists only of a single elliptic curve E and Lie theoretic data are hardly seen, in contrast with the case of the simple singularity. However, b) they show a new relation (in a symbolical level) with Lie theory through deformation theory as follows: *in the local deformation (see* 1. *of* Footnote 12*) of an elliptic singularity of type* $\tilde{\Gamma} \in \{\tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$[13]*, only an elliptic singularity of the same type* $\tilde{\Gamma}$ *or a simple singularity can appear. The simple singularity of type* Γ *can appear if and only if* Γ *is a subdiagram of* $\tilde{\Gamma}$. This fact was explained soon after its finding by use of the lattice $(\mathrm{H}_2(X_1,\mathbb{Z}), I)$ (here, $I = -(\cdot,\cdot)$, see Footnote 35).[14] Thus, for a simply elliptic singularity X_0, a relationship with Lie theory begun to appear from the lattice of the smoothing X_1, instead of the resolution $\tilde{X}_0$. Do we need to change our view point? [15] We shall come back to this question of "change of view-points" later when we discuss $*$-duality in §14 and 15.

discriminant of φ. Then, for any point $x \in C_\varphi$, the variety $X_{\varphi(x)} = \varphi^{-1}(\varphi(x))$ is singular at the point x. This is a singularity in a local deformation of X_0. As we saw already, for a generic point $x \in C_\varphi$, $(X_{\varphi(x)}, x)$ is an ordinary double point (i.e. Morse singularity).

2. Let X_0 be a hypersurface in a neighborhood of the origin 0 of $\mathbb{C}^{n+1}$ defined by an analytic equation $f(\underline{x}) = 0$ with an isolated singular point at 0. We say that X_0 admits a weighted homogeneous structure at 0 if there is a local analytic coordinate change at 0 such that the defining equation $f(\underline{x})$ is transformed to a weighted homogeneous polynomial $P(\underline{x})$ (i.e. $P(\underline{x}) = \sum_{a_0 i_0 + \cdots a_n i_n = h} c_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n}$ for some positive integers $a_0, \cdots, a_n$ and h). Then, the following i), ii) and iii) are equivalent [Sa1]: i) X_0 admits an weighted homogeneous structure, ii) The sequence: $0 \to \mathbb{C} \to \mathcal{O}_{X_0,0} \xrightarrow{d} \Omega^1_{X_0,0} \to \cdots \xrightarrow{d} \Omega^{n+1}_{X_0,0} \to 0$ is exact, where $(\Omega^{\cdot}_{X_0,0}, d)$ is the Poincaré complex over X_0 at 0, and iii) f belongs to the ideal $\left(\frac{\partial f(\underline{x})}{\partial x_0}, \cdots, \frac{\partial f(\underline{x})}{\partial x_n}\right)$ in the local ring $\mathbb{C}\{\underline{x}\}$.

[13] The names $\tilde{E}_i$ are taken from that of the affine Coxeter diagrams (Table 3) for the reason explained in this section. They are nowadays called also $E_i^{(1,1)}$ for the reason explained in the next §7.

[14] This is shown by using the fact that the lattice $(\mathrm{H}_2(X_1,\mathbb{Z}), I)$ is isomorphic to $Q_{\tilde{\Gamma}} \oplus \mathbb{Z}$ (see [Ga], [Eb1,2]) where $Q_{\tilde{\Gamma}}$ is the affine root lattice of a type $\tilde{E}_6, \tilde{E}_6$ and $\tilde{E}_8$. See next §7.

[15] This question is supported by the fact that the period domain for the period map $\int \zeta$ of the primitive form is determined from the lattice $\mathrm{H}_2(X_1,\mathbb{Z})$ [Sa7], [Sa14]II.

§7. Vanishing cycles for simple and simply elliptic singularities

In order to sharpen the new view point, i.e. to study the lattice $(\mathrm{H}_2(X_1,\mathbb{Z}),I)$ of the middle homology group of the smoothing X_1 of singular surface X_0, we consider a particular subset $R\subset \mathrm{H}_2(X_1,\mathbb{Z})$, the set of *vanishing cycles* introduced in §5 (c.f. [Sa15](5.2),(5.3)). From this view point, let us state some consequences of Brieskorn's description [Br4] on simple singularities:

1) *The minimal resolution $\tilde{X}_0$ and the smoothing X_1 of a simple singularity X_0 of type Γ are homeomorphic. Hence one obtains an isomorphism of lattices:*

$$*) \qquad \mathrm{H}_2(X_1,\mathbb{Z}) \quad \simeq \quad \mathrm{H}_2(\tilde{X}_0,\mathbb{Z}) \ .$$

Here, the homotopy type of the homeomoprhims, and hence the isomorphism of lattices $*$) depend on the Weyl group of type Γ. In fact, the ambiguity of the isomorphism can be resolved (up to an outer automorphism of the Weyl group) by choosing the base point 1 in the totally real region of the deformation parameter space S_φ (see Footnote 16).

2) *The set of vanishing cycles R in $\mathrm{H}_2(X_1,\mathbb{Z})$ (see* §5*) forms a finite root system of type Γ, and $\mathrm{H}_2(X_1,\mathbb{Z})$ is identified with the root lattice Q_Γ of the root system.*

3) *The homology classes $[E_i]\in \mathrm{H}_2(\tilde{X}_0,\mathbb{Z})$ $(i=1,\cdots,l)$ of the exceptional curves E_i in the resolution $\tilde{X}_0$ are mapped by the homomorphism $*$) to a simple root basis Γ of the root system R, which are also distinguished basis in the sense in §5.*[16]

If X_0 is a simply elliptic singularity, none of 1), 2) or 3) holds. However, 2) suggests *to regard the set of vanishing cycles in* $\mathrm{H}_2(X_1,\mathbb{Z})$ *for a Milnor fiber X_1 of an elliptic singularity as a generalization of root systems*. In fact, we can generalize the root systems[17] by removing

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[16]The paths $g_1,\cdots,g_{\mu_f}$ in S_φ (Footnote 12), with whom associated distinguished basis $e_1,\cdots,e_{\mu_f}$ is the simple root basis, is given in [Sa20] §4.3 Figure 6. and Theorems 4.1 and 4.2, using semi-algebraic geometry of the real discriminant $D_{\varphi,\mathbb{R}}$ of the universal deformation of the simple singularity. Furthermore, the associated paths γ_i $i=1,\cdots,\mu$ (Table 4) generate the fundamental group $\pi_1(S_\varphi\backslash D_\varphi,1)$ and satisfy Artin braid relations of type Γ so that the fundamental group becomes an Artin group ([Br5] [B-S]). Then, *the intersection matrix $(I(e_i,e_j))_{ij=1,\cdots,\mu}$ is shown to become the Cartan matrix of type Γ by solving the braid relations where $\gamma_1,\cdots,\gamma_\mu$ are substituted by Picard-Lefschetz formula for $\rho(\gamma_1),\cdots,\rho(\gamma_\mu)$ in* §5.

[17]A subset R of a real vector space equipped with a symmetric form I is called a (generalized) root system if $\mathbb{Z}R$ is a full lattice, $2I(\alpha,\beta)/I(\beta,\beta)\in\mathbb{Z}$ and $\alpha-2I(\alpha,\beta)/I(\beta,\beta)\beta\in R$ for $\forall\alpha,\beta\in R$, and irreducible in a suitable sense

the finiteness axiom from the classical one for a finite root system [Bou] Chap. VI §1 so that the set of vanishing cycles for any even dimensional hypersurface isolated singularity becomes a generalized root system. In particular, the set of vanishing cycles for a simply elliptic singularity is characterized as an *elliptic root system*, that is, a root system belonging in a semipositive lattice with radical of rank 2 (see [Sa14] I).

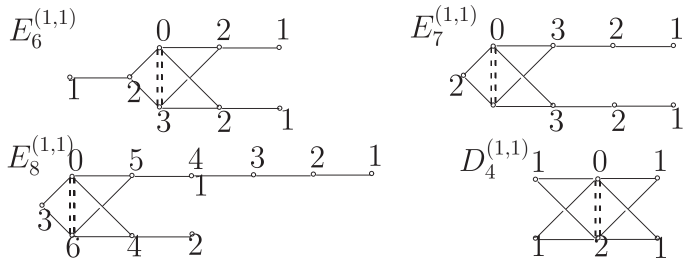
However, by the lack of 1) and 3) for the case of simple singularity, we cannot find a generalization of "the simple root basis" of the elliptic root system naively from the resolution of X_0. Also, no geometric method to choose one particular distinguished basis (see §5) is knowm.[18] However, we choose some root basis arithmetically[19] such that the elliptic Coxeter-Killing transformation defined as a product of reflections associated with the basis is of finite order. As in the case of classical finite root systems, we associate a diagram, called an elliptic diagram, to the basis (see Footnote 41). Some of the simply-laced elliptict diagrams are given in following Table 7.

([Sa14]I). A root system is finite or affine if I is positive definite or semidefinite and rank(radical)=1, respectively. A root system is called *elliptic* if I is positive semidefinite and rank(radical)=2. The set of vanishing cycles for a simply elliptic singularity of type $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$ is the elliptic root system of type $E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$.

[18]Gabrielov [Gab2] (Fig. 10 and 11.) obtained the diagrams in Table 7. for certain distinguished basis as one of the possible choices after the braid group action under the guiding principle to find the diagrams containing small number of triangles. On the other hand, in the simple singularity case, the semialgebraic geometry of the discriminant ([Sa20]) can yield the distinguished basis which corresponds to the simple root basis of the finite root system (see also A'Campo's [AC]). There seems a gap between topology and semi-algebraic geometry.

[19]There does not exist elliptic Weyl chambers and, hence, there seemed no a priori definition of a simple basis for an elliptic root system (see [Klu]). However, the elliptic diagram in Table 7. is defined by duplicating the vertex of the affine diagram at the largest exponent (see [Sa14]I(8.6)). We define the elliptic Coxeter-Killing transformation $\mathbf{c}_e$ as the product of reflections (acting on $\mathrm{H}_2(X_1, \mathbb{Z})$) attached to the vertices of the elliptic diagram (in a suitable order). Then one has: i) $\mathbf{c}_e$ is of finite order h, and the eigenvalues of $\mathbf{c}_e$ determine the exponents of the elliptic root system (see §8 and Table 9), ii) the eigenvector of $\mathbf{c}_e$ belonging to the eigenvalue 1 is regular in the elliptic Cartan algebra $\mathfrak{h}_e$ with respect to the elliptic Weyl group W_e and iii) the universal central extension $\tilde{W}_e$ of W_e is generated by a lift $\tilde{\mathbf{c}}_e^h$. Using i), ii) and iii), a flat structure on the quotient space $\tilde{\mathfrak{h}}_e // \tilde{W}_e$ is constructed ([Sa15]II, [Sat,1,2]).

Table 7. Simply laced Elliptic diagrams of Codim=1 ([Sa14] I, Table 1).



The numbers attached at vertices are the exponents of the root system (see §7).

The diagrams plays basic role, as in the finite root system case, in describing the elliptic root systems [Sa14]I, elliptic Weyl groups [ibid]III, elliptic Lie algebras [S-Y]. The construction of the primitive forms from the elliptic Lie algebras is a work in progress.[20]

§8. Exponents and weight systems

In this section, we first introduce the *exponents* for a finite or elliptic root system, which play important role in the classical and elliptic Lie theory[21]. Then, we try to extend the definition of exponents for a generalized root system, and meet with a problem of "choice of the phases"

[20]In [S-Y] the following three algebras are shown to be isomorphic: a) an algebra generated by vertex operators [Bo1] for all elliptic real roots, b) an algebra generated by the Chevalley triplets attached to the elliptic diagram (Table 7) satisfying certain generalized Serre relations, and c) an amalgamation of an affine algebra and a Heisenberg algebra. An algebra isomorphic to any one of them is called an elliptic algebra. It is also a universal central extension of a 2-toroidal algebra. We remark that the elliptic root systems and the Lie algebras are found also from the representation theory of tubular algebras (see Y. Lin and L. Peng [L-P,1&2]). Works on highest weight representations and Chevalley type invariant theory for an elliptic algebra and group are in progress (see Footnote 2). Due to the existence of the regular element (see Footnote 19), several properties similar to classical algebraic groups and its invariant theory hold for the elliptic Lie algebras and its adjoint groups. These facts supports the program that the elliptic primitive forms are constructed on the elliptic Lie algebras (see references in Footnote 2).

[21]The exponents are equal to the degrees of basic $\mathfrak{g}$- or W-invariants and play basic roles in Lie theory (see [Ko],[Sp],[St1]), and also in the study of the flat structures ([Sa16],[Sa14]II,[Sa7]).

of the exponents. In order to solve the problem, we are lead to introduce a new concept: the *regular system of weights*.

First, we recall a definition of exponents for a finite or elliptic root system. In both cases, we define a Coxeter-Killing transformation as a product **c**, in a suitable order, of reflection actions on the lattice $H_2(X_1,\mathbb{Z})$ attached to a simple root basis (recall §5). The **c** is of finite order h (called the *Coxeter number*, see §19 Remark)[22]. Then the exponents $m_1,\cdots,m_\mu$ are integers such that $\exp(2\pi\sqrt{-1}\frac{m_i}{h})$ $(i=1,\cdots,\mu)$ are the eigenvalues of **c** (see [Bou]Ch.v,n^{o}6.2 and [Sa14] I (9.7) Lemma A.iii)). However, this determines the exponents only up to modulo h. In case of finite root systems and elliptic root systems, one poses further the constraint on the range $0\leq m_i\leq h$ and on the symmetricity $m_i+m_{\mu-i+1}=h$ for $i=1,\cdots,\mu$. Under these constraints, we determine uniquely the exponents as in the next tables.

Type	$(a,b,c;h)$	exponents
A_l $(l\geq 1)$	$(1,b,c;l+1)$	$1,2,\ldots,l$ $(b+c=l+1)$
D_l $(l\geq 3)$	$(2,l-2,1-1;2(l-1))$	$1,3,5,\ldots,2l-3,l-1$
E_6	$(3,4,6;12)$	$1,4,5,7,8,11$
E_7	$(4,6,9;18)$	$1,5,7,9,11,13,17$
E_8	$(6,10,15;30)$	$1,7,11,13,17,19,23,29$

Table 8.

Type	$(a,b,c:h)$	exponents
$E_6^{(1,1)}$	$(1,1,1:3)$	$0,1,1,1,2,2,2,3$
$E_7^{(1,1)}$	$(1,1,2:4)$	$0,1,1,2,2,2,3,3,4$
$E_8^{(1,1)}$	$(1,2,3:6)$	$0,1,2,2,3,3,4,4,5,6$

Table 9.

We try further to introduce the exponents through Coxeter-Killing transf. (Milnor Monodromy) for root systems of singularities (since they are necessary data for primitive forms; see discussions below). In fact,

[22]The Coxeter-Killing transformation has distinguished properties: i) **c** is of finite order h, ii) the primitive hth roots of unity (or, 1 for the case of an elliptic root system) are eigenvalues of **c**, and iii) the eigenvectors of **c** belonging to them are *regular* (i.e. they are not fixed by the Weyl group and the adjoint group of the Lie algebra, [Col], [Bou] chap.V§6 n^{o}2, [Sa14]II §10 Lemma B). This existence of regular eigenvectors is basic for the construction of the adjoint quotient morphism $\mathfrak{g}\to\mathfrak{g}/\!\!/Ad(\mathfrak{g})\simeq\mathfrak{h}/\!\!/W$ ([Ko],[Sp],[St1]) and of the flat structure on $\mathfrak{h}/\!\!/W$ ([Sa16], [Sa14]II).

we shall obtain in §18 quite interesting class of generalized root systems of Witt index 2 together with some distinguished root basis. However, we meet here at present a subtle problem: *phase of exponents*, which lead the author to introduce the concept of the weight system below. To explain the problem concretely, we cite some results from later sections as follows.

1. Consider a polynomial in LHS of Table 10 in §13. The zero loci of the polynomial in $\mathbb{C}^3$ defines a hypersurface X_0 with an isolated singular point at the origin.

2. The generalized root system (= the set of vanishing cycles) in $\mathrm{H}_2(X_1, \mathbb{Z})$ in the middle homology group of a Milnor fiber X_1 of X_0 has a root basis whose associated diagram is given in Table 12 (where p, q, r, called the Gabrielov#, are given in Table 13).

3. Define the Coxeter-Killing transformation $\mathbf{c}$ as the product of reflection actions on $\mathrm{H}_2(X_1, \mathbb{Z})$ associated with the vertices of the diagram in a suitable order. Then, $\mathbf{c}$ is of finite order h and the characteristic polynomial of $\mathbf{c}$ is given in the form (15) for a suitable choice of a system of integers m_i called exponents given in Table 10.

4. Observes that m_i's in Table 10 is exceeding the interval $[0, h]$. Thus, *the Coxeter-Killing transformation is unable to determine their phases* $(:= [m_i/h])$ for these new class of root systems. On the other hand, these m_i/h without the ambiguity "modulo 1" are well defined directly from a choice of a primitive form. [23]

Concern: The root system with basis may not have sufficient data to determine the phases of exponents and to construct the primitive forms.

We shall discuss again on this issue (see §14 Remark 7). This fact, due to the important role of exponents [Sa7][Sai], leads the author to handle them directly (but not through eigenvalues of Coxeter-Killing transformation) as follows.

Consider the generating function (called a characteristic function) [24] for each type of exponents in Tables 8 and 9.

$$\chi(T) \ := \ T^{m_1} + T^{m_2} + \cdots + T^{m_\mu}. \tag{1}$$

[23]The proportions m_i/h are eigenvalues of an operator N in the flat structure associated to a primitive form, and are called exponents of the flat structure ([Sa4] and [Sa7] (3.3) Definition). Therefore, we should have stated more exactly that, conjecturally, there exist a primitive form (constructed from the Lie algebra which we shall study) such that the associated flat structure determines the set of exponents m_i/h.

[24]It is introduced as the Fourier transform of the distribution of the exponents (see [Sa4] (3.1.1) and [Sa7] (3.3.14)) in order to study the zero-loci of χ.

Then, these generating functions for the finite and elliptic root systems of types A_l, D_l, E_6, E_7, E_8 and $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ have a decomposition[25] of the form:

$$\chi(T) \;=\; T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)} \tag{2}$$

where a, b, c, called weights, are integers and h, called the Coxeter number, is the order of the Coxeter element $\mathbf{c}$ such that

$$0 < a, b, c < h \quad \text{and} \quad \gcd(a, b, c) = 1. \tag{3}$$

Note that the set of weights a, b, c are uniquely determined from the characteristic function $\chi(T)$, except for the type A_{h-1}.[26] See Tables 8 and 9 for explicit lists of $(a, b, c; h)$. The generating function (1) of exponents for a finite or an elliptic root system are characterized by the factorization (2) without a pole as follows. Consider abstractly a system:

$$W \;:=\; (a, b, c; h) \tag{4}$$

of 4 integers satisfying (3) (and additionally, $a=1$ if $b+c=h$ called type A_{h-1}), and call it a *weight system*, where a, b, c are called the *weights* and h is called the *Coxeter number*.

Fact 1. ([Sa11]Theorem 2) *If the function* χ_W (2) *for* W *has no poles, then it is equal to a generating function* (1) *of exponents either for a finite root system of type* A_l, D_l, E_6, E_7, E_8 *or for an elliptic root system of type* $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$.

Let us call the rational function $\chi_W := \chi$ in (2) the *characteristic function* associated to W, and call a weight system W is *simple* (resp. *elliptic*) if its characteristic function χ_W is equal to a generating function (1) for a finite root system (resp. elliptic root system) (explicitly, see Table 8 and 9). [27]

---

[25] In the present paper, we are interested in only the cases when all roots of $\chi(T) = 0$ are on the unit circle. But, this is not the case in general for a general primitive form (see [Sa4]).

[26] The characteristic function for the type A_{h-1} is expressed as $\chi_{A_{h-1}}(T) = T + \cdots + T^{h-1} = \frac{T^h-T}{T-1} = T^{-h}\frac{(T^h-T)(T^h-T^b)(T^h-T^c)}{(T-1)(T^b-1)(T^c-1)}$ for $a = 1$ and for any integer b, c with $b + c = h$.

[27] To be exact, one should add the diagram for $D_4^{(1,1)}$ (recall Table 7.) in the list. A diagram is called simply-laced if it does not contain a multiple edges. Any other diagrams for simple (or, elliptic) root system is obtained by the foldings of these simply-laced diagrams.

Before analyzing the characteristic function χ_W further, we state another fact, which gives a geometric meaning to the weights a, b, c and to the Coxeter number h in case of a simple and elliptic weight system (see Table 2 and 4 for a proof):

Fact 2. *A simple polynomial $f_{\tilde{G}}(x, y, z)$ in* Table 2 *(resp. an equation for an elliptic singularity in* Table 6*) is a weighted homogeneous polynomial of degree h with the weights a, b, c on the variables x, y, z for a simple (resp. elliptic) weight system $(a, b, c; h)$. The simple weight system determines the simple polynomial, up to a homogeneous coordinate change, uniquely. The elliptic weight system determines the equation up to one parameter (=the modulus parameter of elliptic curves).*

§9. Triangle Δ of weight system, geometry and algebra

Summarizing the results of previous sections, we obtain the following triangle among three mathematical objects: weight system, geometry and algebra:

$$\begin{array}{ccc} & \{ \text{ Simple weight systems } \} & \\ & \Downarrow \qquad\qquad \Nwarrow & \\ \{ \text{ Kleinean groups } \} & \Rightarrow & \left\{ \begin{array}{c} \text{Simple Lie algebras} \\ \text{with simple root basis} \end{array} \right\}. \end{array} \tag{5}$$

Here, the three arrows are constructed as follows.

1) The correspondence $\Downarrow$ (denoted by $\Phi_{\Downarrow}$) is given by the pair of the fundamental group $\pi_1(X_0 \setminus \{0\})$ for the hypersurface X_0 defined by the polynomial in Table 2 and its action to the covering space $\tilde{X}_0$ (use §1 Theorem, Fact 2 and a theorem due to Mumford [Mu1], see Appendix).

2) The correspondence $\Rightarrow$ (denoted by $\Phi_{\Rightarrow}$) is given in three different ways (depending on the view points), all of which give the same result:

a) Use the Du Val diagram for the simple singularity (§1 and 2) and obtain the diagram of the simple root basis of the simple Lie algebra,

b) Use the set of vanishing cycles for the singularity (§5) and obtain the set of real roots of the simple Lie algebra,

c) Use the McKay correspondence ([Mc], see Appendix) and obtain the Dynkin graph for the simple Lie algebra.

Here, the first two approaches a) and b) are equivalent due to Brieskorn's theorem (recall §7 1),2) and 3)). The third approach c) gives the dual basis of the basis given by a) with respect to the Killing form (see Appendix), but is more direct algebraic construction.

3) The correspondence $\Nwarrow$ (denoted by $\Phi_{\Nwarrow}$) is given by the decomposition (2) of the generating function (1) of the exponents (Table 7) of the root system of the simple Lie algebra.

By a direct inspection of the cases, we see that *a composition of the three arrows* $\Phi_{\Swarrow}$, $\Phi_{\Rightarrow}$ *and* $\Phi_{\Nwarrow}$ *starting at any corner of the triangle* (5) *is an identity.*[28] Here we stress that the key step among the three arrows is the horizontal correspondence $\Phi_{\Rightarrow}$. The others are rather straight forward. As a consequence of this observation, we conclude that

The datum of the set of exponents for a finite root system, which, a priori, is a very small part of the information of the root system, is sufficient to recover the whole root system and the simple Lie algebra. In the same way, the datum of a system W *of weights* (4) *is sufficient to reconstruct the simple Lie algebra.*

A similar triangle as (5) holds for the triple of elliptic weight systems, Heisenberg groups of rank 2 ([Sa14] II, Appendix) and elliptic Lie algebras ([Sa14] IV). This supports the construction of the elliptic primitive forms and the flat structures from the elliptic Lie algebras. This motivates the author to generalize the triangle by starting with a wider class of weight systems and search for corresponding Lie algebras.

We propose to use the top corner of the triangle (5) as the key to uncover a new class of objects: consider any system W (4) of 4 integers, *relaxing the condition on* $\chi_W(T)$ (2) *to be a polynomial to to be a Laurent polynomial. Then, associated to the new weight system, we look for new geometric objects in the left corner and new algebras in the right corner, respectively.* That is: we try to recover the triangle:

$$
\begin{array}{ccc}
 & \{\text{ Weight system } W\ \} & \\
\Swarrow & & \Nwarrow \\
\{\text{Geometry of } X_W\} & \Longrightarrow & \{\text{Algebra } \mathfrak{g}_W\}
\end{array}
\tag{6}
$$

with *the goal to construct primitive forms and their associated period mappings and automorphic forms* (see [Sa19] for the details on the goal). Actually, without this setting of the goal, the objects and the correspondences in the triangle (6) are ambiguous (see §12). Note that each corner of the triangle is not a category and the correspondences $\Rightarrow, \Swarrow$ and $\Nwarrow$

[28] A similar triangle is obtained by replacing the three corners by {elliptic weight systems}, {Heisenberg groups of rank 2 with the extension classes -3,-2,-1} and {Elliptic Lie algebras of type $E_6^{(1,1)}$,$E_7^{(1,1)}$, $E_8^{(1,1)}$ with their simple basis}, where we choose the correspondence b) as for the arrow $\Rightarrow$.

are not functors. However, *we expect a sort of "functoriarity" (yet to be defined) due to the deformation relations among X_W's.*

§10. Top corner of the triangle: regular systems of weights

We start anew by introducing the concept of a regular system of weights.[29]

Definition. A weight system $W = (a, b, c; h)$ (4) satisfying (3) is called *regular* if the function $\chi_W(T) := T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$ is a Laurent polynomial in $\mathbb{Z}[T, T^{-1}]$.

We give two basic properties of a regular system of weights in the following Fact 3. and in Fact 4. in the next section. The two properties are equivalent to the definition of the regular systems of weights, and they already attribute to the properties in the right and left corners of the triangle (6), respectively.

We first discuss about the new definition of exponents.

Fact 3.([Sa11]Theorem 1) *A weight system W* (4) *is regular, if and only if there exist integers $m_1, \cdots, m_\mu$ with $\mu = \mu_W = \frac{(h-a)(h-b)(h-c)}{abc}$ called the rank of W, such that χ_W* (2) *is developed into the sum of monomials of the form* (1).

We call $m_1, \cdots, m_\mu$ the *exponents* of W[30], which we order: $m_1 \leq \cdots \leq m_\mu$ linearly. By use of the functional equality $T^h \chi_W(T^{-1}) = \chi_W(T)$, one has the duality of exponents:

$$m_i + m_{\mu-i+1} = h \qquad (i = 1, \cdots, \mu). \tag{7}$$

A fact which is not used in the present paper but shall be of basic importance (see Footnote 22, ii)), is that there exists always an exponent prime to h [Sa,13,18].

The advantage to start from a weight system is that the exponents are a priori defined without an ambiguity of their phases (i.e. $[m_i/h] \in$

[29]This is slightly modified ([Yas]) from the original definition [Sa11]: $\chi_W(T)$ has a pole at most only at $T = 0$. Using a relation: $T^h \chi_W(T^{-1}) = \chi_W(T)$, the two definitions are equivalent.

[30]In order to agree with the classical convention in Lie theory (e.g. [Bou]), we have called the integers m_i exponents. However, from a view point of the flat structure on S_φ (recall Footnote 23), one should better call the rational numbers m_i/h exponents. This view point becomes important again, when we consider the category of graded matrix factorizations §16.

$\mathbb{Z}$). The smallest exponent $\min\{m_1,\cdots,m_\mu\}$ is given and denoted by

$$\varepsilon_W \ := \ a+b+c-h. \tag{8}$$

Actually, §8 Fact 1. implies that if $\varepsilon_W>0$ (resp. $=0$), then automatically one has $\varepsilon_W=1$ (resp. $=0$) and W is a simple (resp. elliptic) weight system, whose exponents coincide with the exponents of the corresponding finite or elliptic root system.

For each negative integer $\varepsilon<0$, there always exist a finite number of regular systems of weights having ε as the smallest exponent (see [Sa12, Sa17] Appendix 1,2. for many interesting examples of W with $\varepsilon_W<0$). In particular, there exist 14+8 regular systems of weights for the case $\varepsilon_W=-1$ having no 0 exponents (see Table 10), on which we shall discuss more in details in the present paper.

We are now to analyze the other corners of the triangle (6). Recall that the finite or elliptic root system cannot be directly constructed from the weight system, but we needed to turn the triangle (5) counter-clockwisely. Similarly, we start with analyzing the left corner of (6) in the next section.

§11. Left corner of the triangle: a geometry of X_W

Finding the objects in the left corner of the triangle (6) and $\Phi_{\swarrow}$ follows from the following characterization Fact 4. of the regularity of a weight system W.

For any given weight system $W=(a,b,c;h)$, consider a weighted homogeneous polynomial

$$f_W(x,y,z) \ := \ \textstyle\sum_{ai+bj+ck=h} c_{ijk}x^iy^jz^k. \tag{9}$$

Fact 4. ([Sa11]Theorem 3) *The weight system W (4) is regular, if and only if there exists a polynomial f_W of the form (9) such that the quotient ring:*

$$J_W \ := \ \mathbb{C}[x,y,z]/\left(\frac{\partial f_W}{\partial x},\frac{\partial f_W}{\partial y},\frac{\partial f_W}{\partial z}\right), \tag{10}$$

called the Jacobi ring of f_W, is of finite rank μ_W over $\mathbb{C}$.

"If" part of the statement is trivial. Actually, any polynomial (9) with generic coefficients carries this property.

In fact, Fact 4. is trivially equivalent to that the hypersurface

$$X_{W,0} \ := \ \{(x,y,z)\in\mathbb{C}^3 \mid f_W(x,y,z)=0\} \tag{11}$$

has an isolated singular point at the origin, i.e. $X_{W,0}$ is smooth except at the origin $0\in X_{W,0}$, due to the Nullstellensatz of Hilbert.

Let us call f_W in Fact 4. a *polynomial of type* W. We employ the hypersurface $X_{W,0}$ (11) with an isolated singular point at 0 and admitting a $\mathbb{C}^\times$-action[31]:

$$\lambda \in \mathbb{C}^\times \ : \ (x,y,z) \ \mapsto \ (\lambda^a x, \lambda^b y, \lambda^c z)$$

as for the object in the left corner of the triangle (6). Following the history in §2-7, we analyze $X_{W,0}$ from two a) algebraic and b) topological view points.

a) **Orbi-bundle $K_{C_W}^{\frac{1}{\varepsilon_W}}$ over the curve C_W.**

There are many studies on surface singularities with a good $\mathbb{C}^\times$-action (e.g. [Dol1,2,3,4], [Pin4,5], [Sa11,12,16], [Wa,1,2]). We recall a few results of them, which are necessary in our purpose. First, we remark that the smoothness of $X_0 \setminus \{0\}$ implies that the quotient variety

$$C_W \ := \ (X_{W,0} \setminus \{0\})/\mathbb{C}^\times \ = \ \mathrm{Proj}(\mathbb{C}[x,y,z]/(f_W(x,y,z)) \tag{12}$$

is a smooth curve. However, the $\mathbb{C}^\times$-bundle $X_{W,0} \setminus \{0\} \overset{\mathbb{C}^\times}{\to} C_W$ has some finite number of singular fibers (i.e. fixed by some non-trivial finite subgroups, called isotropy groups, of $\mathbb{C}^\times$). In this sense, C_W carries also a structure of an *orbifold curve* (to be precise, an *algebraic stack*). The pair $(g : p_1, \cdots, p_r)$ of the genus g of the curve C_W and the *set*, called the signature set, of the orders of the isotropy groups:

$$A(W) \ = \ \{p_1, \cdots, p_r\} \tag{13}$$

is called the *signature* of the orbifold ([F-K]pp.182-190). In fact, we have

Fact 5. ([Sa11]Theo.6) *The genus g of the curve C_W is equal to the multiplicity $a_0 := \#\{1 \le i \le \mu \mid m_i = 0\}$ of exponents equal to 0.*

The signature set $A(W)$, up to some $p_i = 1$, is explicitly determined from the weights W arithmetically.[32]

The orbifold Euler number: $2 - 2g + \sum(1/p_i - 1)$ is positive, 0 or negative according to whether ε_W is positive, 0 or negative. Accordingly, the orbifold universal covering of C_W is either $\mathbb{P}_1$, the complex plane $\mathbb{C}$

[31]The action is said good since the the exponents of the action a, b, c are positive (or, equivalently, the coordinate ring $R_W := \mathbb{C}[x,y,z]/(f_W)$ is non-negatively graded.

[32]The genus and the signature set of the orbi-curve C_W is explicitly give as follows.

$$a_0 := \#\{(i,j,k) \in \mathbb{Z}^3_{\ge 0} \mid ai + bj + ck = h\} = \#\{1 \le i \le \mu \mid m_i = 0\},$$
$$A(W) := \{a_i \mid \ a_i \not| h, 1 \le i \le 3\} \amalg \{ \ \gcd(a_i, a_j) * (m(a_i, a_j; h) - 1), \ 1 \le i < j \le 3\}$$

where $\{a_1, a_2, a_3\} = \{a, b, c\}$ and $m(a, b; h) = \#\{(u,v) \in \mathbb{Z}^2_{\ge 0} \mid au + bv = h\}$.

or the complex upper half plane $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Then, for a weight system W with $\varepsilon_W \neq 0$, we have the description of the $\mathbb{C}^\times$-bundle: $X_{W,0} \setminus \{0\} \to C_W$ ([Dol3]Prop.1, [Sa11](5.5)Lemma)

Fact 6. *Let W be a regular system of weights. According as $\varepsilon_W > 0$ or < 0, one has the following natural commutative diagrams, respectively.*

$$(14) \quad \begin{array}{ccc} K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}} \setminus \text{0-section} & \underset{/\,\tilde{\Gamma}_W}{\longrightarrow} & X_{W,0} \setminus \{0\} \\ \big\downarrow /\,\mathbb{C}^\times & & \big\downarrow /\,\mathbb{C}^\times \\ \mathbb{P}^1 & \underset{/\,\tilde{\Gamma}_W}{\longrightarrow} & C_W \end{array} \quad \text{and} \quad \begin{array}{ccc} K_{\mathfrak{H}}^{\frac{1}{\varepsilon_W}} \setminus \text{0-section} & \underset{/\,\tilde{\Gamma}_W}{\longrightarrow} & X_{W,0} \setminus \{0\} \\ \big\downarrow /\,\mathbb{C}^\times & & \big\downarrow /\,\mathbb{C}^\times \\ \mathfrak{H} & \underset{/\,\tilde{\Gamma}_W}{\longrightarrow} & C_W. \end{array}$$

Here, 1) $K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}}$ *and* $K_{\mathfrak{H}}^{\frac{1}{\varepsilon_W}}$ *is a* ε_W*th root of the canonical bundle of* $\mathbb{P}_1$ *or* $\mathfrak{H}$*, respectively, and* 2) $\tilde{\Gamma}_W$ *is a co-compact discrete subgroup of* $SU(2)$ *or* $PSL(2,\mathbb{R})$*, whose actions on* $\mathbb{P}^1$ *or* $\mathfrak{H}$ *are liftable to the bundles (Footnote* 33*), respectively.*

The action of $\tilde{\Gamma}_W$ *on* $\mathbb{P}_1$ *or* $\mathfrak{H}$ *may have fixed points such that the quotient map* $/\tilde{\Gamma}_W$ *gives the orbifold universal covering of* C_W*. That is: the signature of the group* $\tilde{\Gamma}_W$ *(*[Ma]*) coincides with that* $(a_0 : A(W))$ *of the orbifold curve* C_W*.*[33]

These imply that C_W in a Deligne-Mumford stack. They give the "algebraic data" of the geometry of $X_{W,0}$ for $\varepsilon_W \neq 0$.[34]

Example. Case $\varepsilon_W > 0$ (i.e. W is a simple weight system in Table 8). Then, we naturally have $K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}} \setminus \{0\} \simeq \mathbb{C}^2 \setminus \{0\}$, and the $\tilde{\Gamma}_W$ action in LHS is identified with the Kleinean group $\tilde{G}$-action in RHS (recall §1). I.e. the liftablity condition in Fact 6. is automatically satisfied). The induced action of $\tilde{\Gamma}_W$ on $\mathbb{P}_1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$ is identified with the

[33] We have a similar geometry for $\varepsilon_W = 0$. Namely, the three simply elliptic singularities of types $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$ are quotients of the trivial $\mathbb{C}^\times$ bundle over $\mathbb{C}$ by an action of a Heisenberg group of rank 2 of characteristic class -3,-2 and -1, respectively ([Sa2],[Sa14]II Appendix).

[34] To be exact, there remains still the problem to characterize (or, to list up) the pair $(\tilde{\Gamma}_W, \varepsilon)$ of a number $\varepsilon \in \mathbb{Z}_{<0}$ and a co-compact Fuchsian group $\tilde{\Gamma}_W \subset PSL(2,\mathbb{R})$ such that the action of $\tilde{\Gamma}_W$ on $\mathfrak{H}$ is liftable to that on $K_{\mathfrak{H}}^{1/\varepsilon}$. This condition on $(\tilde{\Gamma}_W, \varepsilon)$ (in order to obtain a Gorenstein normal surface singularity $K_{\mathfrak{H}}^{1/\varepsilon} // \tilde{\Gamma}_W^*$) is equivalent to finding a splitting factor $\tilde{\Gamma}_W^*$ in $\tilde{\Gamma}_d$ of the central extension $1 \to \mathbb{Z}/\varepsilon\mathbb{Z} \to \tilde{\Gamma}_d \to \tilde{\Gamma}_W \to 1$ (see [Sa12] (5.2)(5.3) and (5.4)). To list the cases when $K_{\mathfrak{H}}^{1/\varepsilon} // \tilde{\Gamma}_W^*$ is a hypersurface requires further works (e.g. [Dol2] [Wa1]) which is generally unsolved yet. To remain in the category of Gorenstein singularities seems theoretically easier and natural.

regular polyhedral group G-action on S^2. So, there are three singular orbits {centers of faces of the polyhedron}, {centers of edges of the polyhedron} and {vertices of the polyhedron} of fixed points on $\mathbb{P}_1$. Therefore, the signature set $A(W)$ (13) consists of the three numbers p, q, r in Table 1.

Similarly, in case $\varepsilon_W < 0$, the Fuchsian group $\tilde{\Gamma}_W$ has elliptic fixed points in $\mathfrak{H}$, whose orbits correspond in 1:1 to the elements of $A(W)$.

b) **Generalized root system and Coxeter-Killing transformation.**

We discuss about some of the topological data obtained from the semi-universal deformation (also, called universal unfolding) of $X_{W,0}$.

1) Generalized root system

Let us denote by Q_W the lattice $(\mathrm{H}_2(X_{W,1},\mathbb{Z}), I = -(\cdot,\cdot))$ of vanishing cycles[35] and by R_W the set of vanishing cycles for f_W (which depend only on W but not on a choice of f_W). As is explained already, it is easy to see that R_W *satisfies the axiom of generalized root system having* Q_W *as its root lattice in the sense* [Sa14] I. The following 2) and 3) describe some strong properties carried by R_W. However, we do not know a characterization of a root system which arises as the set of vanishing cycles associated to a singular point.

2) Coxeter-Killing transformation

The Milnor monodromy induces an automorphism $\mathbf{c}$ of finite order h of the lattice Q_W, which we shall call also the Coxeter-Killing transformation of the root system R_W. Using the weighted homogeneity of the defining equation f_W, it is easy to see that the characteristic polynomial $\det(\lambda \cdot \mathrm{id}_Q - \mathbf{c})$ is given by

$$\varphi_W(\lambda) \;=\; \textstyle\prod_{i=1}^{\mu}(\lambda - \exp(2\pi\sqrt{-1}\frac{m_i}{h})) \quad \in \quad \mathbb{Z}[\lambda]. \tag{15}$$

[35]The middle homology group $\mathrm{H}_2(X_{W,1},\mathbb{Z})$ admits the symmetric bilinear form, called the intersection form, $(u,v) := \langle u, P(v)\rangle$ obtained from the Poincare duality $P : \mathrm{H}_2(X_{W,1},\mathbb{Z}) \to \mathrm{H}^2(X_{W,1},\mathbb{Z})$. In the above definiton of the lattice Q_W, we put the minus sign factor in order to adjust with the classical convention in the Killing form that $I(e,e) = 2$ for any vanishing cycle e. The signature (μ_-, μ_0, μ_+) of I is given by $\mu_- = \#\{1 \le i \le \mu \mid m_i < 0 \text{ or } h < m_i\}$, $\mu_0 = \#\{1 \le i \le \mu \mid m_i = 0 \text{ or } h\} = 2a_0$, $\mu_+ = \#\{1 \le i \le \mu \mid 0 < m_i < h\}$, ([Sai]). Then the *Witt index* (=the maximal rank of totally isotropic subspace) of $\mathrm{H}_2(X_{W,1},\mathbb{Z}) = \mu_0 + \mu_- = \#\{\text{exponents exceeding the interval } (0,h)\}$ is always *even.* This fact supports the existence of the Coxeter-Killing transformation of finite order and to ask for Chevalley type invariant theory to the algebra $\mathfrak{g}_W$ in §12 iv).

The set $\{\exp(2\pi\sqrt{-1}m_i/h) \mid i=1,\cdots,\mu\}$ is closed under the action of the Galois group over $\mathbb{Q}$.[36] Recall that:

Fact 7. ([Sa, 13(Theorem 1), 18(Theorem 5.1)]) *Let us denote by $e_W(h)$ the multiplicity of the hth primitive roots of unity in the roots of the equation $\varphi_W(\lambda)=0$. Then, for any regular system of weights W, one has $e_W(h)>0$.*

Remark 4. In the classical simple Lie algebra case, the eigenvector of the Coxeter-Killing transformation belonging in to the hth primitive root of unity (in the Cartan subalgebra of $\mathfrak{g}_W$) is regular with respect to the adjoint action of the simple Lie group and that of the Weyl group. This gives a key role to the vector in the invariant theory by Kostant [Ko], Springer [Sp], Steinberg [St1] as well as in the construction of the primitive form and the flat structure [Sa18].

3) Root basis

Any distinguished basis $(e_1,\cdots,e_{\mu_W})$ (recall §5) gives a root basis of the root system R_W in the sense: i) $R_W=\cup_{i=1}^{\mu_W}\langle w_{e_1},\cdots,w_{e_{\mu_W}}\rangle\cdot e_i$, and ii) the Coxeter-Killing transformation is given by the product $w_{e_1}\cdots,w_{e_{\mu_W}}$. This implies: iii) $Q_W=\oplus_{i=1}^{\mu_W}\mathbb{Z}e_i$ and iv) $\langle w_{e_1},\cdots,w_{e_{\mu_W}}\rangle$ coincides with the group generated by reflections for all $e\in R_W$ (=the Weyl group of the root system R_W).

As we saw already in §5, the braid group of μ_W-strings acts on the set of distinguished basis. It is desirable to find some “simple” basis for the root system R_W by the use of the action. There are several works in the direction by Gabrielov [Gab 1,2], Ebeling [Eb 1,2], Kluitman [Klu] and others. However, purely topological data of the braid group action alone seems insufficient to choose some distinguished ones. On the other hand, one may still have a hope to choose some particular basis, either by a use of semi-algebraic geometry of the discriminant of the family $\mathcal{X}_\varphi\to S_\varphi$ (see Footnote 12 and [Sa20]), or by the algebraic approach a) by a use of the orbifold structure on C_W given in the first half of the present §. The study of this subject belongs still to a future work.

4) Cycles from ∞.

We already discussed about the cycles from infinity in §5 Remark2. Under the setting of a regular system of weights W, let us discuss again about it.

Let us define explicitely a universal unfolding of f_W by

$$F(\underline{x},\underline{t}) \;:=\; f_W(\underline{x})+\varphi_1(\underline{x})\cdot t_1+\varphi_2(\underline{x})\cdot t_2+\cdots+1\cdot t_{\mu_W}$$

[36]This is shown as follows. Substitute any power of $\exp(2\pi\sqrt{-1}/h)$ in (1). (2) implies that it is a rational number.

where $\varphi_1, \varphi_2, \cdots, \varphi_\mu \equiv 1$ are weighted homogeneous polynomials in $\mathbb{C}[x,y,z]$ (with respect to the weights (a,b,c)) such that their images in the Jacobi ring J_W (10) gives its $\mathbb{C}$-basis. Clearly, the function in a neighborhood of origin gives the universal unfolding in the sense explained in §5. However, we remark that $F(\underline{x},\underline{t})$ is affine globally defined, where, by putting $\deg(t_i) := h - \deg(\varphi_i) = m_i + \varepsilon_W$ $(1 \le i \le \mu_W)$, it is a weighted homogeneous polynomial. The lowest degree coordinate is t_1 and its degree is equal to $2\varepsilon_W$. That is, the unfolding parameter $\underline{t}$ gets negative weights if (and only if) $\varepsilon_W < 0$. Consider the affine global family of affine surfaces: $\hat{\varphi}_W : \hat{\mathcal{X}}_W \to S_W$, where $\hat{\mathcal{X}}_W := \{(\underline{x},\underline{t}) \in \mathbb{C}^3 \times \mathbb{C}^{\mu_W} \mid F(\underline{x},\underline{t}) = 0\}$, $S_W := \mathbb{C}^{\mu_W}$ and $\hat{\varphi}$ is the projection to the second factor. The discriminants of $\hat{\varphi}_W$ is a divisor of S_W and decomposes into a union of $D_{W,+}$, $D_{W,0}$ and $D_{W,-}$ according as the behavior of the vanishing cycles vanishing at the components (see [Sa19]II §6). Then, as was shown in §5, the lattice Q_W of middle homology group of the Milnor fiber is generated by the vanishing cycles which are degenerating to the discriminant $D_{W,+}$. Then, the extension $\hat{Q}_W := (\mathrm{H}_2(\hat{X}_{W,\underline{t}}, \mathbb{Z}), -I)$ for a generic parameter value $\underline{t}$ such that the coordinate component $t_1 \neq 0$ is a orthogonal direct sum of the lattice Q_W with the lattice Q_W^∞ generated by the vanishing cycles which are degenerating to the discriminant $D_{W,-}$. It is expected that the periods of the cycles in Q_W^∞ give the denominators for primitive forms ([Sa19]II §6 Conjecture and Problem).

Remark 5. The concept of the generalized root system of vanishing cycles and the braid group action on its basis may better be lifted to a categorical level due to the recent developments of the study of Floer homology groups of Lagrangean subvarieties in symplectic varieties [Sei].[37]

Remark 6. As we shall see in §16, for weight systems W having its $*$-dual, the lattices Q_W and Q_W^∞ are expected to have a categorical construction as the K-groups of the category of the graded and un-graded matrix factorizations, respectively, where the Coxeter-Killing transf. is defined as the A-R translation.

§12. Right corner of the triangle: an algebra $\mathfrak{g}_W$

We now come to the main question of the present paper:

[37]A comprehensive treatment of this subject shall appear in: K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono: Lagrangian intersection Floer theory - anomaly and obstruction, in preparation.

Question. *For any regular system of weights W, define the correspondences $\Phi_{\Rightarrow}$ and $\Phi_{\Nwarrow}$ which make the triangle* (6) *commutative. Precisely, construct the algebra $\mathfrak{g}_W$ from the data of the geometry of X_W, satisfying the following conditions* i) - vi).

Then, we automatically have $\Phi_{\Nwarrow}$ and commutativity of the triangle.

We impose some working hypothetical conditions i)-vi) on the algebra $\mathfrak{g}_W$; otherwise the question is ambiguous. Under these constraints, we expect a sort of functoriality and uniqueness for the correspondence $\Phi_{\Rightarrow}$ (recall §9).[38]

i) The algebra $\mathfrak{g}_W$ should be a simple Lie algebra for the case $\varepsilon_W > 0$ and a elliptic Lie algebra for the case $\varepsilon_W = 0$.

ii) The algebra $\mathfrak{g}_W$ should carry an integrability structure, generalizing the Jacobi identity for Lie algebras (i.e. $\mathfrak{g}_W$ should be the prolongation of X_W with respect to the equations for a primitive form; see the last paragraph in §4).

iii) $\mathfrak{g}_W$ should contain an abelian subalgebra $\mathfrak{h}_W$ isomorphic to $\mathrm{Hom}(Q_W, \mathbb{C})$ (which we may call the Cartan-Killing subalgebra of $\mathfrak{g}_W$). The adjoint action of $\mathfrak{h}_W$ on $\mathfrak{g}_W$ induces the root space decomposition of $\mathfrak{g}_W$ so that R_W should be the set of real roots (i.e. a root $\alpha \in Q_W$ such that $I(\alpha, \alpha) > 0$), whose multiplicities are equal to 1. The real root spaces $\mathfrak{g}_{W,\alpha}$ for $\alpha \in R_W$ generate the algebra $\mathfrak{g}_W$.

iv) Depending on a choice (*Note* 3. below), one should have a family of Chevalley type invariant theories for the adjoint group G_W action on $\mathfrak{g}_W$ and the adjoint quotient morphism with the identification of the quotient varieties $\tilde{\mathfrak{g}}_W // \mathrm{Ad}(\tilde{\mathfrak{g}}_W) \simeq \tilde{\mathfrak{h}}_W // \tilde{W}_W$. Here, $\tilde{\mathfrak{g}}_W$, $\tilde{\mathfrak{h}}_W$ and $\tilde{W}_W$ are suitable *hyperbolic extensions* of $\mathfrak{g}_W$, $\mathfrak{h}_W$ and W_W, if the Killing form I has a degeneration .[39]

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[38]Beside the classical construction of semi-simple Lie algebras, there are several new approaches, e.g. using vertex operators [Bo1], or using Ringel-Hall algebras [P-X], as was discussed in Preface. However, in connection with our final goal (the construction of primitive forms), we would like to be cautious in choosing the type of construction.

[39]One supporting reason for this condition is the following fact ([Sa13](2.2) Theorem1, [Sa17] Theorem5.1 and (5.6)): *for any regular system of weights, there always exists an exponent which is prime to the Coxeter number h.* This generalizes the existence of an eigenvalue of a primitive hth root of unity of the Coxeter-Killing transformation $\mathbf{c}$ in the classical case [Col] [Bou]Ch.v§6 Theorem 1. This is a key fact for the construction of the adjoint quotient morphism and for the global construction of the flat structure (see Footnotes 18,19).

v) The universal unfolding $\mathcal{X}_W \to S_W$ of the singularity $X_{W,0}$ should be embedded into the adjoint quotient map $\mathfrak{g}_W \to \tilde{\mathfrak{h}}_W//\tilde{W}_W$ (c.f. §4 when $\varepsilon_W > 0$). The relative (with respect to the adjoint quotient map) symplectic form on $\mathfrak{g}_W$ (Kostant-Kirirov form when $\varepsilon_W > 0$) induced from the involutive structure given in ii) should (up to a unit factor) induce a primitive form on the family $\mathcal{X}_W \to S_W$, whose exponents (recall Footnote 23) coincide with the exponents of the weight system W (up to the factor h).

vi) The flat structure on the quotient variety $\tilde{\mathfrak{h}}_W//\tilde{W}_W$ (c.f. [Sa16], [Sa14] II) and the flat structure on S_W defined from the theory of primitive forms [Sa7] should be identified by the isomorphism in iv). This, in particular, requires that the set of exponents for the primitive form on $X_{W,0}$ should coincides with the set of exponents associated to the flat structure of the algebra $\mathfrak{g}_W$.

The last condition vi) implies that the generating function (1) ([Sa7] (3.3.14)) of the exponents for the flat structures of the algebra $\mathfrak{g}_W$ decomposes as in (2), and defines the weight system $W=(a,b,c;h)$, which we had at the beginning. That is, *the correspondence* $\Phi_{\Nwarrow}$ *of the triangle* (6) *is defined by the use of the decomposition of the generating function* (1) *of the exponents of the algebra. Then, the composition* $\Phi_{\Nwarrow} \circ \Phi_{\Rightarrow} \circ \Phi_{\Swarrow}$ *is the identity on the top of the triangle* (6)*.* Thus, we shall obtain a family of primitive forms having the exponents given at the beginning by a regular system of weights, when the problem is solved.

Obviously, the simple Lie algebra $\mathfrak{g}_W$ of type W for a simple weight system W satisfies all conditions i)-vi). The elliptic algebra $\mathfrak{g}_W$ for an elliptic weight system W satisfies i), ii) and iii), and the flat structure on $\tilde{\mathfrak{h}}_W//\tilde{W}_W$ has been constructed. However, the construction of the adjoint quotient space $\tilde{\mathfrak{g}}_W//\mathrm{Ad}(\tilde{\mathfrak{g}}_W)$ is still a work in progress (see Footnote 20).

For general weight system W, we introduce in §16 a category $\mathrm{HMF}^{gr}_{A_W}(f_W)$, which is expected to give three constructions of Lie algebras. We ask to clarify the relationship among the constructions, and whether they satisfy i)-vi) (up to the $*$-duality which we shall introduce in §14) (see Problem at the end of §18).

On the other hand, elliptic root systems have a radical of rank 2. Then, depending on the choice of its rank 1 subspace, called a marking, one defines the extensions $\tilde{\mathfrak{g}}_W$, $\tilde{\mathfrak{h}}_W$ and $\tilde{W}_W$ (see [S-Y],[Sa14]I,II,[S-T]). These extensions, called hyperbolic, are necessary for the construction of the flat structure [Sa14]II as well as in the representation theory and invariant theory of the elliptic algebra.

Note. 1. If the Killing form I on the root lattice $Q_W = \mathrm{H}_2(X_{W,1}, \mathbb{Z})$ degenerates ($\Leftrightarrow$ the genus a_0 of the curve C_W is positive, see Fact 5, Footnotes 32 and 35), then the algebra $\mathfrak{g}_W$ may have a "radical" (corresponding to the moduli parameter of the curve C_W). In that case, as for $\mathfrak{g}_W$, we assign the universal algebra (i.e. the one having the largest radicals) for the unicity of the notation $\mathfrak{g}_W$.

2. The other problem in answering **Question** is, which view-point a) or b) in §9 do we generalize? It seems likely that, in the above iii), the two view points a) and b) give two different root systems and two different algebras. Let us tentatively denote by $\Phi^{a)}_{\Rightarrow}$ the correspondence using the algebraic geometric data of the singularity $X_{W,0}$ and by $\Phi^{b)}_{\Rightarrow}$ the correspondence using the topological data of the deformations of $X_{W,0}$. In fact, these two different view points are, nowadays, called mirror symmetric to each other (see [Kon1], [Yau] for mirror symmetry in general). There is a duality operation on the set of weight systems, called the $*$-duality, which conjecturally exchanges the two approaches (see §14 **Addition to Question**). Then, the conditions iv) and v) on the period map seem to choose $\Phi^{b)}_{\Rightarrow}$ for the correspondence $X_W \Rightarrow \mathfrak{g}_W$.

3. The denominator of an elliptic primitive form depends on a choice of a primitive element in the radical of the root lattice ([Sa7] (3.1) Example), which determines the polarization (marking [Sa14] I) of the elliptic root system. Similarly, the primitive form for the 14 exceptional unimodular singularities is conjectured to be a proportion of a form with its integral over the cycle coming from infinity (see [Sa19]II 6. Conjecture, §11b) 4) and Footnote 49).

§13. Strange duality of Arnold

In order to get an insight to the Question in §12 and also to sharpen it by Addition to Question in the next §14, we look closely at the case $\varepsilon_W = -1$ in this section where the singularities are called exceptional unimodular singularties. We recall the *strange duality* among the 14 cases due to Arnold [Ar4].

There are 14+8+9 regular systems of weights of $\varepsilon_W = -1$, where the first 14+8 cases are genus $a_0 = 0$ and the remaining 9 cases are positive genus $a_0 > 0$. The multiplicity $e_W(h)$ of the first 14 weight systems is equal to 1 and that of the next 8 weight systems are either equal to 2 or 3. Accordingly, the signature set $A(W)$ (Footnote 32) consists of 3 elements for the first 14 cases, and of 4 or 5 elements for the 8 cases (where f_W depends on parameter(s)). (see [Sa11, Tables 3,4 and 5] for details on the geometry of them in the sense of §11). In the present paper, we study only the 14+8 cases where genus a_0 is zero.

Table 10. 14+8 regular systems of weights of genus $a_0 = 0$ and $\varepsilon_W = -1$

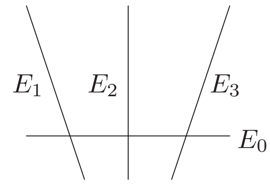
Polynomial f_W	$(a, b, c; h)$	exponents
$x^7 + y^3 + z^2 :$	$(6, 14, 21; 42)$	$-1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 43$
$yx^5 + y^3 + z^2 :$	$(4, 10, 15; 30)$	$-1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 31$
$x^4 z + y^3 + z^2 :$	$(3, 8, 12; 24)$	$-1, 2, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 25$
$x^5 + xy^3 + z^2 :$	$(6, 8, 15; 30)$	$-1, 5, 7, 11, 13, 15, 17, 19, 23, 25, 31$
$yx^4 + xy^3 + z^2 :$	$(4, 6, 11; 22)$	$-1, 3, 5, 7, 9, 11, 11, 13, 15, 17, 19, 23$
$x^3 z + xy^3 + z^2 :$	$(3, 5, 9; 18)$	$-1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 19$
$x^5 + y^2 z + z^2 :$	$(4, 5, 10; 20)$	$-1, 3, 4, 7, 8, 9, 11, 12, 13, 16, 17, 21$
$yx^4 + y^2 z + z^2 :$	$(3, 4, 8; 16)$	$-1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17$
$x^4 + y^3 + xz^2 :$	$(6, 8, 9; 24)$	$-1, 5, 7, 8, 11, 13, 16, 17, 19, 25$
$x^3 y + y^3 + xz^2 :$	$(4, 6, 7; 18)$	$-1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 19$
$x^3 z + y^3 + xz^2 :$	$(3, 5, 6; 15)$	$-1, 2, 4, 5, 5, 7, 8, 10, 10, 11, 13, 16$
$x^4 + y^2 z + z^2 x :$	$(4, 5, 6; 16)$	$-1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 17$
$x^3 y + y^2 z + z^2 x :$	$(3, 4, 5; 13)$	$-1, 2.3.4.5.6.7.8.9.10.11.14$
$x^4 + yz(y - z) :$	$(3, 4, 4; 12)$	$-1, 2, 3, 3, 5, 6, 6, 7, 9, 9, 10, 13$
$y(y - x^3)(y - \lambda x^3) :$	$(2, 6, 9; 18)$	$-1, 1, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, 15, 17, 19$
$xy(y - x^2)(y - \lambda x^2) + z^2 :$	$(2, 4, 7; 14)$	$-1, 1, 3, 3, 5, 5, 7, 7, 7, 9, 9, 11, 11, 13, 15$
$y(y - x^2)(y - \lambda x^2) + xz^2 :$	$(2, 4, 5; 12)$	$-1, 1, 3, 3, 4, 5, 5, 7, 7, 8, 9, 9, 11, 13$
$(y^2 - x^3)(y^2 - \lambda x^3) + z^2 :$	$(2, 3, 6; 12)$	$-1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 8, 9, 10, 11, 13$
$x(z - x^2)(z - \lambda x^2) + y^2 z :$	$(2, 3, 4; 10)$	$-1, 1, 2, 3, 3, 4, 55, 6, 7, 7, 8, 9, 11$
$x^3 y + z(z - y)(z - \lambda y) :$	$(2, 3, 3; 9)$	$-1, 1, 2, 2, 3, 4, 4, 5, 5, 6, 7, 7, 8, 10$
$xy(x - y)(y - \lambda_1 x)(y - \lambda_2 x) + z^2 :$	$(2, 2, 5; 10)$	$-1, 1, 1, 3, 3, 3, 5, 5, 5, 5, 7, 7, 7, 9, 9, 11$
$y(y - x)(y - \lambda_1 x)(y - \lambda_2 x) + xz^2 :$	$(2, 2, 3; 8)$	$-1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 9$

Here λ, λ_1 and λ_2 are parameters $\neq 0, 1$ and $\lambda_1 \neq \lambda_2$.

In order to construct $\Phi_{\Rightarrow}$, remember a) and b) of §9, 1) for the case $\varepsilon_W = 1$. An immediate analogy of $\Phi_{\Rightarrow}^{a)}$ is to study the resolution of the singularity $X_{W,0}$. The minimal resolution $\pi : \tilde{X}_{W,0} \to X_{W,0}$ of the singularity is determined by [Dol1] as follows: the exceptional set $\pi^{-1}(0) \subset \widetilde{X}_0$ of the minimal resolution is a union of 4-rational curves E_0, E_1, E_2 and E_3, which intersect transversely as illustrated in Table 11 and are self-intersecting as

$$-1 = E_0^2, \quad -p = E_1^2, \quad -q = E_2^2, \quad -r = E_3^2$$

where p, q, r are positive integers such that $(0 : p, q, r)$ is the signature of the orbifold curve C_W (§11 Fact 5., [Dol 1,2,3,4], [Pin4,5],[Sa 11,12]). The signature set $A(W) = \{p, q, r\}$ (13), in this particular case, was called the *Dolgachev numbers* [Ar3] [Dol1], which are listed in Table 13.



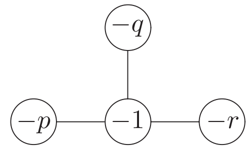
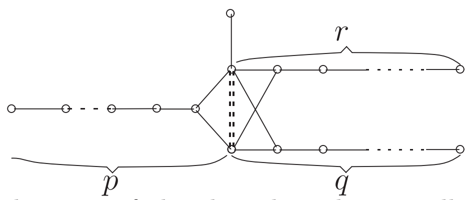


Table 11. Exceptional set Resolution diagram

On the other hand, as for the correspondence $\Phi^{b)}_{\Rightarrow}$, one should know the set of vanishing cycles in the lattice $\mathrm{H}_2(X_{W,1}, \mathbb{Z})$, whose signature is $(2, 0, *)$ (Footnote 35.). The *distinguished basis* of the lattices were studied by the authors A.M. Gabrielov [Gab1,2] and W. Ebeling [Eb1,2]. In particular, they found certain "simple"[40] distinguished basis for each of the 14 exceptional singularities, which is expressed by the following diagram[41]:

Table 12. Distinguished basis for the exceptional unimodular singularities.



where the length p, q, r of the three branches is called the *Gabrielov numbers* [Ar3].[42]

[40]Here, we mean by "simple" the following: 1) the vertices of the diagram is a $\mathbb{Z}$-basis of the lattice $H_2(X, \mathbb{Z})$, 2) a product in suitable order of the reflections on the lattice attached to the vertices of the diagram (i.e. a Coxeter-Killing transformation) is of finite order h and its eigenvalues are $\exp(2\pi\sqrt{-1}m_i/h)$ for the exponents m_i, 3) consider the group W acting on the lattice generated by the reflections attached to the vertices of the diagram. Then the set of the vanishing cycles is equal to the union of the W-orbits of the vertices of the diagram.

[41]Let $e_1, \cdots, e_\mu$ be a basis of a lattice Q_W such that $I(e_i, e_i) = 2$ for $i = 1, \cdots, \mu$. Then, we associate a *diagram* to the basis as follows: to each basis element e_i for $i = 1, \cdots, \mu$, we associate ith vertex of the diagram. Between ith and jth vertices of the diagram, we draw $-I(e_i, e_j)$ edges if $I(e_i, e_j) < 0$, $I(e_i, e_j)$ dotted edges if $I(e_i, e_j) > 0$ and no edges if $I(e_i, e_j) = 0$.

[42]There is a strong reason to suspect that the diagram should be (a part of) the correspondence $\Phi_{\Rightarrow}$ for the 14 weight systems, since it partially answers to the questions iv) and v) in §12 as follows. Let $\xi, \overline{\xi}$ be the eigenvectors of the Coxeter-Killing transformation belonging to the eigenvalues $\exp(\pm 2\pi\sqrt{-1}/h)$. Then each belongs to the two connected component of the cone $\{x \in Q_W \otimes_{\mathbb{Z}} \mathbb{C} \mid I(x, \overline{x}) < 0, I(x, x) = 0\}$ over a symmetric domain of type IV and is regular

Then, Arnold [Ar3] observed the following duality and called it the **strange duality**: *there exists an involutive bijection σ on the set of 14 exceptional singularities, by which Dolgachev numbers and Gabrielov numbers interchange.*

In the next table, we indicate the involution σ by the two-sided arrows $\leftrightarrow$ between the weight systems corresponding to the singularities.

	Weights	$A(W)$=Dolgachev#	Gabrielov#	$\varphi_W(\lambda)$
⟲	(3, 4, 4; 12)	4, 4, 4	4, 4, 4	$\frac{(\lambda^{12}-1)(\lambda^4-1)}{(\lambda^3-1)(\lambda-1)}$
⟲	(3, 4, 5; 13)	3, 4, 5	3, 4, 5	$\frac{(\lambda^{13}-1)}{(\lambda-1)}$
	(4, 5, 6; 16)	2, 5, 6	3, 4, 4	$\frac{(\lambda^{16}-1)(\lambda^4-1)}{(\lambda^8-1)(\lambda-1)}$
⟲	(3, 5, 6; 15)	3, 3, 6	3, 3, 6	$\frac{(\lambda^{15}-1)(\lambda^3-1)}{(\lambda^5-1)(\lambda-1)}$
	(4, 6, 7; 18)	2, 4, 7	3, 3, 5	$\frac{(\lambda^{18}-1)(\lambda^3-1)}{(\lambda^9-1)(\lambda-1)}$
	(6, 8, 9; 24)	2, 3, 9	3, 3, 4	$\frac{(\lambda^{24}-1)(\lambda^4-1)(\lambda^3-1)}{(\lambda^{12}-1)(\lambda^8-1)(\lambda-1)}$
	(3, 4, 8; 16)	3, 4, 4	2, 5, 6	$\frac{(\lambda^{16}-1)(\lambda^2-1)}{(\lambda^4-1)(\lambda-1)}$
⟲	(4, 5, 10; 20)	2, 5, 5	2, 5, 5	$\frac{(\lambda^{20}-1)(\lambda^5-1)(\lambda^2-1)}{(\lambda^{10}-1)(\lambda^4-1)(\lambda-1)}$
	(3, 5, 9; 18)	3, 3, 5	2, 4, 7	$\frac{(\lambda^{18}-1)(\lambda^2-1)}{(\lambda^6-1)(\lambda-1)}$
⟲	(4, 6, 11; 22)	2, 4, 6	2, 4, 6	$\frac{(\lambda^{22}-1)(\lambda^2-1)}{(\lambda^{11}-1)(\lambda-1)}$
	(6, 8, 15; 30)	2, 3, 8	2, 4, 5	$\frac{(\lambda^{30}-1)(\lambda^5-1)(\lambda^2-1)}{(\lambda^{15}-1)(\lambda^{10}-1)(\lambda-1)}$
	(3, 8, 12; 24)	3, 3, 4	2, 3, 9	$\frac{(\lambda^{24}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^8-1)(\lambda^6-1)(\lambda-1)}$
	(4, 10, 15; 30)	2, 4, 5	2, 3, 8	$\frac{(\lambda^{30}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^{15}-1)(\lambda^6-1)(\lambda-1)}$
⟲	(6, 14, 21; 42)	2, 3, 7	2, 3, 7	$\frac{(\lambda^{42}-1)(\lambda^7-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^{21}-1)(\lambda^{14}-1)(\lambda^6-1)(\lambda-1)}$

Table 13. The strange duality and the $*$-duality.

The strange duality captured the attention of many authors and was interpreted by Dolgachev, Nikulin and Pikham in terms of duality between algebraic cycles and transcendental cycles on certain K3 surfaces [Pin1,2]. Further generalizations of the duality were studied by several authors [N-G][Pin4,5][Lo4][E-W].

In §14, we induce the strange duality from the $*$-duality of weight systems [Sa17], which is interpreted as a mirror symmetry [Ta1].

§14. $*$-duality of regular systems of weights

We introduce one key operation $*$ of the present paper: the $*$-duality on regular systems of weights [Sa17]. It induces the strange duality in the arithmetical level, and induces, a much wider class of dualities among weight systems beyond the strange duality.

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w.r.t. the Weyl group (i.e. does not belong to any reflection hyperplane ([Sa15] (5.6) Lemma 2). We remark also that the diagram defines a splitting hyperbolic plane of the lattice Q_W.

Recall the *characteristic polynomial* $\varphi_W(\lambda)$ (15) of a regular system of weights W. Since $\varphi_W \in \mathbb{Z}[\lambda]$ (recall §11 b) and Footnote 36) and is a cyclotomic polynomial, there exists a unique expression:

$$\varphi_W(\lambda) \; = \; \prod_{i|h} (\lambda^i - 1)^{e_W(i)} \tag{16}$$

for some integer $e_W(i) \in \mathbb{Z}$ for all $i \in \mathbb{Z}_{>0}$ with $i|h$, where h is the Coxeter number of W.

Definition. A regular system of weights W' is called $*$-*dual* to W if 1) its Coxeter number h' coincides with that h of W, and 2) one has the duality relation:

$$e_W(i) + e_{W'}(h/i) \; = \; 0 \qquad \text{for } {}^\forall i \in \mathbb{Z}_{>0}. \tag{17}$$

Here, we put $e_W(i) = e_{W'}(i) = 0$ for an integer i with $i \not| h$.

Example. 1. The characteristic polynomial for the type E_8 decomposes as

$$\varphi_{E_8}(\lambda) \; = \; \frac{(\lambda^{30}-1)(\lambda^5-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda-1)(\lambda^6-1)(\lambda^{10}-1)(\lambda^{15}-1)}.$$

Then $e_{E_8}(30)+e_{E_8}(1) = e_{E_8}(5)+e_{E_8}(6) = e_{E_8}(3)+e_{E_8}(10) = e_{E_8}(2)+e_{E_8}(15) = 0$. This implies E_8 is selfdual. This is a special case of the next 2.

2. All regular weight systems W with $\varepsilon_W > 1$ (i.e. simple weight systems) are selfdual ([Sa17] Theo.7.10.1). This fact resemble the result of Brieskorn in §7, 1). However, the $*$-duality, in general, implies neither of the the homeomorphisms $\tilde{X}_{W,0} \simeq X_{W^*,1}$ nor $\tilde{X}_{W^*,0} \simeq X_{W,1}$ (see the examples below). Therefore, it seems interesting to ask what the natural generalization of [O-O] is for the $*$-dual pair?

3. Any regular system of weights W of rank μ_W equal to 24 is selfdual. It is curious to observe that there are 11 such weight systems with $\varepsilon_W < 0$, and the set of their characteristic polynomials is exactly equal to the set of all selfdual characteristic polynomials of the conjugacy classes of the Conway group $\cdot 0$ ([Sa17] Appendix 1) except for the four 6A, 10A, 15D and 18A.

We have the following uniqueness of the $*$-dual weight system.

Theorem. ([Sa17] The.7.8) 1. *For a regular system of weights W if there exists a $*$-dual weight system, then it is uniquely determined from W, which we denote by W^*. By definition, we have $(W^*)^* = W$.*

2. *The smallest exponent of W^* is equal to that of W, $\varepsilon_W = \varepsilon_{W^*}$.*

3. *The multiplicities $e_W(h)$ and $e_{W^*}(h)$ are equal to 1.*

In general, there does not exist a dual weight system W^* for a given regular system of weights W (eg. if the multiplicity $e_W(h)$ is larger or equal than 2, then W cannot have the dual weight system), but if the $*$-dual for W exists, then it is purely arithmetically determined from W. In fact, we have a complete list of dual pair of regular systems of weights ([Sa17] The.7.9). As a consequence, we can prove the following.

Fact 7. ([Sa17] Theo.7.10.2 & §12) *Any of the 14 regular systems of weights W in* Table 10 *(i.e. $\varepsilon_W=-1$, $a_0=0$ and $e_W(h)=1$) is $*$-dual to a weight system in the same* Table. *Further, if W and W^* are dual, then $\mu_W+\mu_{W^*}=24$*[43] *and*

$$\text{Dolgachev } \# \text{ of } W = \text{Gabrielov } \# \text{ of } W^*$$
$$\text{Dolgachev } \# \text{ of } W^* = \text{Gabrielov } \# \text{ of } W$$

That is: the strange duality of Arnold is induced from the $*$-duality.

Remark 7. Whether W^* is dual to W or not is determined only by the characteristic polynomials φ_{W^*} and φ_W, and hence, in view of (15), it is determined only by the exponential: $\exp(2\pi\sqrt{-1}m_i/h)$ $(i=1,\cdots,\mu)$ and $\exp(2\pi\sqrt{-1}m_i^*/h)$ $(i=1,\cdots,\mu^*)$. That is, *the information of the phases $[m_i/h]$, $[m_i^*/h]$ of the exponents are unnecessary to determine the $*$-duality.*

This brings us to a puzzle: we had mentioned (§8 Concern) that the eigenvalues $\exp(2\pi\sqrt{-1}m_i/h)$ of a Coxeter-Killing transformation may not be sufficient to recover the phases of the exponents. This was the main reason why we introduced the concept of regular systems of weights in §10 (but not a root system with a simple basis) as our starting point, since a regular system of weights carries the full data of the set of exponents. From this starting point, we arrive at a result that the phase is unnecessary for the definition of duality among them.

The author does not have a good answer to this puzzle. The only fact, we can mention here is that *a regular system of weights W, which admits the dual W^*, has a peculiarity such that the datum of the set of exponentials $\{\exp(2\pi\sqrt{-1}m_i/h) \mid i=1,\cdots,\mu\}$ is sufficient to recover* $\{m_i \mid i=1,\cdots,\mu\}$ (see the proof of [Sa17] The.7.9).

Namely, the uniqueness of the dual weight system can be shown briefly as follows: if a weight system W admits a dual weight system, then the characteristic polynomial $\varphi_W(\lambda)$ is reduced (i.e. $e_W(i)\in\{0,\pm1\}$ for $i\in\mathbb{Z}_{>0}$). This is equivalent to $e_W(h)=1$ and we call such W *simple.* On the other hand, a simple weight system W is arithmetically

[43] In the original proof [Sa17] Theorem 7.10, 2., the rank relation was not stated explicitly.

reconstructed from its characteristic polynomial φ_W ([ibid] The.6.3). This proves the uniqueness of W (and W^*).

Perhaps, the above puzzle is closely related to another puzzle: the $*$-duality is described purely in terms of arithmetic whereas the strange duality in general exchanges the algebraic and the transcendental structures in geometry.

Remark 8. It seems an interesting and important problem to find a reasonable extension of regular systems of weights, which is closed under the $*$-duality. For instance, the end remark of Footnote 34 suggests that Gorenstein surface singularities with good $G \times \mathbf{A}^\times$-action should be included (see [Ta1]). However, in the present paper, we do not go into any details of the question. Instead, we proceed here as if we were already in the extended category which is closed under the $*$-duality, and ask the following follow-up to the question in §12.

For two decades, inspired from mathematical physics, one observes "symmetry relations" called *mirror symmetry* between some symplectic topological varieties, called the A-model side and some algebraic (or complex analytic) varieties, called the B-model side. Mirror symmetry is formulated at different levels: from identities of numerical invariants of the varieties to the equivalence of categories associated to the varieties. In the present paper, we do not go into any details of the subject but just refer the reader to some of the literature (see for instance [Kon1],[Yau],[H-V]). It is expected that the models on both sides finally should induce an isomorphic flat structures (recall the condition vi) in §12). Mirror symmetry on topological Landau-Ginzburg orbifold model (which corresponds to the singularity theory in mathematics) is described by Kawai-Yang [K-Y] in terms of the duality of orbifoldized Poincaré polynomials. Therefore, it is natural to ask whether (and this was actually proven by A. Takahashi [Ta1]) *the $*$-duality of weight systems is equivalent to mirror symmetry in the Landau-Ginzburg orbifold model in mathematical physics.*[44]

Having these background, we ask the following mathematical question.

Addition to §12 Question. Does there exist an involutive correspondence $\mathfrak{g} \mapsto \mathfrak{g}^*$ on the set of algebras in the right corner of the triangle (6) so that one has the isomorphism: $\mathfrak{g}_{W^*} \simeq (\mathfrak{g}_W)^*$? That is: does

[44] Accordingly, the definition of the $*$-duality for the weight systems of type V in the classification of [Sa17] §5 is modified.

there exist a $*$-duality in the right corner of the triangle (6) making the following diagram commutative?

(18)

$$\begin{array}{ccc} \{\text{Regular systems of weights } W\} & \Leftrightarrow & \{\text{Algebras } \mathfrak{g}_W\} \\ \Updownarrow * & & \Updownarrow * \\ \{\text{Regular systems of weights } W^*\} & \Leftrightarrow & \{\text{Algebras } \mathfrak{g}_{W^*} \simeq (\mathfrak{g}_W)^*\}\ . \end{array}$$

This means that we seek duality operations in each corner of the triangle (6) (i.e. the $*$-duality on the top, the mirror symmetry in the left and the new conjectural $*$-duality in the right) so that the arrows are compatible with them. Likewise, the domain of the definition of the $*$-duality in the top is restricted, so a similar constraint on the domain of the definition in the RHS might exist. Note also that the $*$-duality in the RHS does not keep the rank μ_W (of the Killing-subalgebra $\mathfrak{h}_W$, recall §12 iii)), but is rather complementary in the sense that $\mu_W + \mu_{W^*} = 24$ in case of $\varepsilon_W = -1$.

What seems remarkable here is the fact that the $*$-duality in the RHS exchanges the algebras which are constructed from algebraic data with that from transcendental data, whereas the $*$-duality in the LHS is purely arithmetically defined. In §15, we shall discuss the duality at a categorical level.

***η*-product.** In the rest of this section, we give a digression on the reformulation of the $*$-duality in terms of eta products ([Sa17] §13).

1. Let $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ (where $q^a := \exp(2\pi\sqrt{-1}a\tau)$ be Dedekind eta function. To the product (16), we associate a product

$$\eta_W(\tau) \ := \ \prod_{i|h} \eta(i\tau)^{e_W(i)}. \tag{19}$$

Assertion ([Sa17](13.3)) *Two weight systems W and W^* are dual to each other if and only if one of the following (equivalent) relations holds:*

$$\begin{aligned} \eta_W(-1/h\tau)\cdot\eta_{W^*}(\tau) &= (\tau/\sqrt{-1})^{a_0}/\sqrt{d_{W^*}}, \\ \eta_{W^*}(-1/h\tau)\cdot\eta_W(\tau) &= (\sqrt{-1}/\tau)^{a_0}/\sqrt{d_W}, \end{aligned}$$

where d_W is the discriminant defined by $\prod_{i|h} i^{e_W(i)}$.

2. We observe the following behavior of the coefficients of the expansion of $\eta_W(\tau)$ in the powers of q (called the Fourier coefficients of $\eta_W(\tau)$ at ∞).

i) Fourier coefficients of the eta-products of type A_l ($l \geq 1$), D_l ($l \geq 4$) and E_l ($l = 6, 7, 8$) are positive and are exponentially growing.

ii) Fourier coefficients of the eta-products of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$ are non-negative and are polynomially growing. For example:

$$\begin{aligned} & \eta_{E_8^{(1,1)}}(12\tau) \ := \ \tfrac{\eta(30\cdot 12\tau)\eta(5\cdot 12\tau)\eta(3\cdot 12\tau)\eta(2\cdot 12\tau)}{\eta(12\tau)\eta(6\cdot 12\tau)\eta(10\cdot 12\tau)\eta(15\cdot 12\tau)} \\ = \ & q^5+q^{17}+q^{29}+q^{41}+q^{53}+2q^{65}+q^{89}+q^{101}+q^{113}+2q^{125}+q^{137}+q^{149}+\cdots . \end{aligned}$$

Here, i) is trivially checked but ii) requires some work [Sa14]V, where the Melin transform $L_W(s)$ of η_W is expressed by the L-functions of the cyclotomic field $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1},\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-1},\sqrt{-3})$ with abelian Galois group according to W is of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$. Then, their L-functions have quadratic expressions by Dirichlet L-functions. For example: using some Dirichlet characters ε, χ on $\mathbb{Z}/12\mathbb{Z}$, the L-function of type $E_8^{(1,1)}$ is expressed as

$$L_{E_8^{(1,1)}}(s)=\frac{1}{4}\prod_{p\in P_1}\frac{1}{(1-p^{-s})^2}\prod_{p\in P_\varepsilon\cup P_{\varepsilon\chi}}\frac{1}{1-p^{-2s}}\Big\{\prod_{p\in P_\chi}\frac{1}{(1-p^{-1})^2}-\prod_{p\in P_\chi}\frac{1}{(1+p^{-1})^2}\Big\} .$$

A direct inspection on this Euler product shows the non-negativity of all Dirichlet coefficients of them.

For each elliptic root system, we associate the eta-product (19) using the decomposition (16) of the characteristic polynomial of its Coxeter-Killing transformation. Then,

Fact 8. *The Fourier coefficients are non-negative if and only if the root system is of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$, which are exactly the types of simply-laced elliptic root systems admitting the flat structure* (compare [Sa14]V, Theorem with [Sa14]II, §11 Theorem.)

Finally, we remark that a stronger form[45] of the following statement was conjectured in [Sa17](Conjecture 13.5) and is proven by S. Yasuda [Yas].

Fact 9. *Let us define the dual rank ν_W of W by $\nu_W := -\sum_{i|h} e_W(h/i)i$ ($\nu_W = \mu_{W^*}$ if the $*$-dual of W exists). Then, all Fourier coefficients of $\eta_W(\tau)$ at ∞ are non-negative integers if and only if $\nu_W \geq 0$.*

In particular, if a weight system W admits the $*$-dual, then all Fourier coefficients of η_W are non-negative.

Question: Interpret the Fourier series of η_W as the generating function of counting of some objects either from the geometry of X_W or from the algebra $\mathfrak{g}_W$.

[45] An eta product $\eta(h\tau)^\nu\eta_W(\tau)$ has non-negative Fourier coefficients if and only if $\nu_W \geq \nu$.

§15. Towards algebraic construction of the correspondence $\Phi_{\Rightarrow}$

We return to the question of constructing the correspondence $\Phi_{\Rightarrow}$ posed in §12. According the program §12 iii), the algebra $\mathfrak{g}_W$ should have the root space decomposition with respect to the adjoint action of the Cartan-Killing subalgebra $\mathfrak{h} \simeq \mathrm{H}^2(X_1, \mathbb{C})$ with the generalized root system R_W of the vanishing cycles in $\mathrm{H}_2(X_1, \mathbb{Z})$ (§5,7). So, our first task should be to give a good description of the set of vanishing cycles R_W and to find a good basis for it.

For the 14 cases of $\varepsilon_W = -1$, we explained already that the diagram in Table 12. due to Gabrielov is a good candidate for the simple root basis for the generalized root system (recall Footnotes 40 and 42). However, the diagrams were obtained after several braid actions on the basis of the lattice of vanishing cycles ([Gab2], see also [Eb1]). It seems as if the diagrams are obtainded ad hoc, and hard to find an explanation on their naturarity and necessity from purely topological machinery alone.

On the other hand, once we introduce the use of $*$-duality in §14, the situation changes drastically. Namely, owing to §14 Fact 7., the Gabrielov numbers of the diagram of the 14 weight systems W with $\varepsilon_W = -1$ are given by the signature set $A(W^*)$ ((13) and Footnote 32) of the $*$-dual weight system W^*. That is, the Gabrielov number for W is determined "intrinsically" by two arithmetic steps: step 1. calculate the $*$-dual weight system W^* from W ([Sa17] The.7.9) and step 2. calculate the signature set $A(W^*)$ of W^* ([Sa11] The.6), which can be done without any ambiguity. That is: the diagram in Table 12. for W is, at least in its numerical level, determined from the algebraic approach through the $*$-dual W^*. Actually, the same phenomenon occurred already for the simple singularities, where $\varepsilon_W > 0$ and the weight system W is self-dual (§14 Example 2) and then the signature set $A(W^*)$ $(= A(W))$ gives the branch lengths of the diagram of the simple basis of the finite root system R_W (recall §7, 3), §11 a) Example and Table 3).

These facts led the author to ask the following question: [46]

Problem: *Construct the root system R_W and its basis through the algebraic approach $\Phi^{a)}_{\Rightarrow}(X_{W^*})$ instead of the topological approach $\Phi^{b)}_{\Rightarrow}(X_W)$.*

[46] Problem ([Sa15], in English translation p.124). Construct directly from the system of weights $(a, b, c; h)$, without pathing through the homology group of the Milnor fiber, arithmetically or combinatorially, the generalized root system (Q, I, R, c) given above. That is to say, give a basis $\alpha_1, \cdots, \alpha_\mu$ and their inner products $I(\alpha_i, \alpha_j)$ $(1 \le i \le j \le \mu)$ directly from the data $(a, b, c; h)$.

The numerical data $A(W^*)$ alone are not sufficient, and we need to find a structural approach to reconstruct the root system R_W and its basis. Based on recent developments in mathematical physics[47], A. Takahashi [Ta2] gave an answer to the first part of the problem (i.e. the construction of the root system R_W). He conjectured that the K-group of the category of graded matrix factorizations for f_{W^*} should be isomorphic to the lattice of vanishing cycles $(\mathrm{H}_2(X_1, \mathbb{Z}), I)$. He has shown that the category for the case of the polynomial $f_{A_l} = x^{l+1}$ of type A_l is derived equivalent to the category of modules over the path algebra for the Dynkin quiver of type A_l, so that the set of indecomposable objects in the K-group gives the set of roots of type A_l, and he further conjectured that this should hold for all the other simple polynomials.

In the following three sections 16, 17 and 18, we report the results of some joint works of H. Kajiura, A. Takahashi and the author along these line and on its further development. We introduce in §16 the homotopy category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ of graded matrix factorizations for f_W, in three different formulations.

In §17, we study the category for a simple weight system W for $\varepsilon_W = 1$, and show that it is generated by a strongly exceptional collection (see §16, 4. for a definition) whose associated quiver is a classical Dynkin quiver of the type $W = W^*$ [K-S-T 1]. Then, due to a classical result by Gabriel [Ga], the set of indecomposable objects in the category form the classical finite root system in the associated Grothendieck group (= K-group), as was expected.

In §18, we study [K-S-T 2] the category for a weight system W of $14 + 8$ weight systems of $\varepsilon_W = -1$ with $a_0 = 0$. We show that it is generated by a strongly exceptional collection whose associated quiver is of the form Table 14, where the set of lengths of branches of the quiver is given by the signature set $A(W)$ (13) of the weight system W. We show further that *the path algebra for the quiver with relations is isomorphic to the finite dimensional algebra consisting of morphisms among the objects of the exceptional collection.* Then, owing to a result of Bondal-Kapranov [B-K], the category is equivalent to the bounded

[47] A hint was given by the Gonzalez-Verdier interpretation of c) ([G-V], see Appendix), where the dual basis of the simple root basis was constructed by certain vector bundles on $\tilde{X}_{W,0}$. Then, the derived category of the abelian category of coherent sheaves was acknowledged in the recent development in mirror symmetry of D-branes due to Kapustin-Li [K-L 1,2], Hori-Walcher [H-W] and Walcher [Wal].

The category of graded D-branes of type B in Landau-Ginzburg models was formulated by D. Orlov [Orl2] as the triangulated category of the singularity X.

derived category of that of modules over the path algebra of wild type with two relations. In particular, in the 14 exceptional modular cases, in view of the $*$-duality (§14, Fact 7.) and by the comparison of Table 12 with 14, *the Grothendieck group* $\mathrm{K}_0(\mathrm{HMF}^{gr}_{A_W}(f_W))$ *is isomorphic to the lattice of vanishing cycles* $\mathrm{H}_2(X_{W^*,1},\mathbb{Z})$ *for the $*$-dual weight system* W^* *of* W. *"Whether the generalized root system* R_{W^*} *in* $\mathrm{H}_2(X_{W^*,1},\mathbb{Z})$ *(i.e. the set of vanishing cycles, see* §11,b), 1) *) is exactly the image of the set of exceptional indecomposable objects in* $\mathrm{HMF}^{gr}_{A_W}(f_W)$ *or not*" is an open and interesting question.[48]

The above results on the category of graded matrix factorization for $\varepsilon_W{=}{\pm}1$ seem to suggest that the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ for W with $a_0{=}0$ may possibly have certain canonical strongly exceptional collections, which are liftings, at the categorical level, of an answer to the latter half of Problem in Footnote 46.

§16. The category of graded matrix factorizations

In this section, we introduce the triangulated category $\mathcal{T}_W$ associated with a regular system of weights W in three equivalent forms: by the homotopy category of the graded matrix factorizations,[49] by the stable category of maximal Cohen-Macaulay modules, and by the category of singularities ([Bu],[Orl1],[Ta2]). We discuss some basic properties of the category such as Serre duality, the generation of the category,

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[48]One should lift the question into the categorical level as follows: since R_{W^*} is a union of the Weyl group orbits of a distinguished basis due to the irreducibility of the discriminant D_φ (Footnote 12), and a distinguished basis is the image of the objects of an exceptional collection, we ask "whether any exceptional indecompsable object in $\mathrm{HMF}^{gr}_{A_W}(f_W)$ is obtained by a successive application of mutations on the objects of the exceptional collection or not?".

[49]The concept of a matrix factorization is introduced by D. Eisenbud [Ei] in order to describe the *two periodic resolutions of maximal Cohen-Macaulay modules*. It was applied in the study of hypersurface isolated singularities ([Kn1,2],[Gr],[Sch]). It obtained a new impetus through mathematical physics ([K-L], [H-W]) and found new application to the categorification of link invariants ([K-R]). From a graded matrix factorization, forgetting about its grading one obtains a ungraded Marx factorization. This induces a comparison of the categories of graded and ungraded Matrix factorizations. This forgetful functor induces the embedding of the corresponding K-groups, which should conjecturally mirror dual to the embedding of the lattice of vanishing cycles to that of *cycles coming from infinity*. However, in the present paper, we shall not discuss this subject further (see §11 b) 4) and §12 Note 3.).

exceptional collections and Auslander-Reiten translation. For basic terminology and concepts in category theory, one is referred to [Ke].

Let $W=(a,b,c;h)$ be a regular system of weights. We regard the polynomial ring

$$A_W \ := \ \mathbb{C}[x,y,z]$$

to be graded by the weight $\deg(x)=2a/h$, $\deg(y)=2b/h, \deg(z)=2c/h$.[50] Fix a polynomial $f_W \in A_W$ of type W (9), which is of degree 2. Put [51]

$$R_W \ := \ A_W/(f_W) \ = \ \mathbb{C}[x,y,z]/(f_W).$$

Obvious remarks are that A_W is a regular ring and R_W is a Gorenstein ring. By definition, both A_W and R_W are graded rings graded by $\frac{2}{h}\mathbb{Z}_{\geq 0}$. In the present paper, by a graded module M over A_W or R_W, we always mean a module which is graded by $\frac{2}{h}\mathbb{Z}$, i.e. $M=\oplus_{d\in\frac{2}{h}\mathbb{Z}}M_d$. A graded homomorphism $f : M \to N$ of degree a between graded modules is defined as usual a homomorphism with $f(M_d)\subset N_{d+a}$ for any $d \in \frac{2}{h}\mathbb{Z}$. We denote by gr-$A_W$ or gr-R_W the category of *finitely generated* graded A_W or R_W-modules, respectively, whose morphisms are homogeneous of degree 0. We denote by τ the degree shift operator on the set of graded modules to itself defined by $(\tau M)_d = M_{d+\frac{2}{h}}$. For a morphism f, we associate the same morphism $\tau(f): \tau M \to \tau N$.

For $M,N \in$ gr-A_W, the module $\mathrm{Hom}_{A_W}(M,N)$ of all A_W-homomorphisms naturally belongs to gr-A_W by letting $\mathrm{Hom}_{A_W}(M,N)_{\frac{2n}{h}} := \mathrm{Hom}_{\text{gr-}A_W}(M,\tau^n N)$. The same statement replacing A_W by R_W holds also.

1. The homotopy category of graded matrix factorizations for f_W.

Definition. i) A graded matrix factorization for f_W is a system

$$M \ := \ (P_0 \overset{p_0}{\underset{p_1}{\rightleftarrows}} P_1)$$

where $P_1,\ P_2$ are graded free A_W-modules of finite rank and $p_0,\ p_1$ are graded A_W-homomorphisms such that $p_0p_1 = f_W\cdot id_{P_0},\ p_1p_0 = f_W \cdot id_{P_1}$ and $\deg(p_0)=0,\ \deg(p_1)=2$. The set of all graded matrix factorizations for f_W is denoted by

$$\mathrm{MF}^{gr}_{A_W}(f_W) \ := \ \{\text{graded matrix factorizations for } f_W\}.$$

[50]In order to compare with the conventions of matrix factorizations, we have to duplicate the grading compared with that for the flat structure. Hence, one should note that $\deg(f_W)=2$.

[51]The reader is notified with the fact that there is an unfortunate coincidence of this notation with that for the set of vanishing cycles in §11 b) 1).

ii) A graded homomorphism from $M=(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1)$ to $M'=(P'_0 \underset{p'_1}{\overset{p'_0}{\rightleftarrows}} P'_1)$ is a pair $\Phi=(\phi_0,\phi_1):(P_0,P_1)\to(P'_0,P'_1)$ of graded A_W-homomorphisms homogeneous of degree 0 making the following diagram commutative.

$$\begin{array}{ccccccccc} & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & \\ \cdots & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \cdots \\ & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & P'_1 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & \end{array}$$

The set of all graded homomorphisms is denoted by $\mathrm{Hom}_{\mathrm{MF}^{gr}_{A_W}(f_W)}(M, M')$.

iii) We denote also by $\mathrm{MF}^{gr}_{A_W}(f_W)$ the additive category of all matrix factorizations with respect to above defined homomorphisms.

Definition. We denote by $\mathrm{HMF}^{gr}_{A_W}(f_W)$ the homotopy category of $\mathrm{MF}^{gr}_{A_W}(f_W)$. That is, the objects of $\mathrm{HMF}^{gr}_{A_W}(f_W)$ are the same as $\mathrm{MF}^{gr}_{A_W}(f_W)$. The module of homomorphisms is defined as the quotient space by the homotopy equivalence

$$\mathrm{Hom}_{\mathrm{HMF}^{gr}_{A_W}(f_W)}(M,M') := \mathrm{Hom}_{\mathrm{MF}^{gr}_{A_W}(f_W)}(M,M')/\sim$$

where a morphism $\Phi=(\phi_0,\phi_1)$ is homotopic to zero, denoted by $\Phi\sim 0$, if there exists A_W-homomorphisms $h_0 : P_0 \to P'_1$ and $h_1 : P_1 \to P'_0$ with $\deg(h_0) = -2$ and $\deg(h_1) = 0$ such that $\phi_0 = p'_1 h_0 + h_1 p_0$ and $\phi_1 = p'_0 h_1 + h_0 p_1$.

$$\begin{array}{ccccccccc} & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & \\ \cdots & \swarrow & \phi_0\downarrow & \swarrow{\scriptstyle h_1} & \phi_1\downarrow & \swarrow{\scriptstyle h_0} & \phi_0\downarrow & \swarrow & \cdots \\ & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & P'_1 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & \end{array}$$

Example. The $A_W \underset{f}{\overset{1}{\rightleftarrows}} A_W$ and $A_W \underset{1}{\overset{f}{\rightleftarrows}} \tau^h A_W$ are matrix factorizations which are homotopic to 0, since we have the following commutative diagram:

$$\begin{array}{ccccccc} \rightarrow & A_W & \xrightarrow{f} & A_W & \xrightarrow{1} & A_W & \xrightarrow{f} \\ \swarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 0} & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 0} \\ \rightarrow & A_W & \xrightarrow{f} & A_W & \xrightarrow{1} & A_W & \rightarrow \end{array}$$

Any 0-object M (i.e. $1_M \sim 0$) in the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ is a direct sum of copies of some τ-powers shifts of $(A_W \underset{f}{\overset{1}{\rightleftarrows}} A_W)$ and $(A_W \underset{1}{\overset{f}{\rightleftarrows}} A_W)$.

Definition. (Shift functors) We introduce two auto-equivalence functors:

$$\tau(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) := (\tau P_0 \underset{\tau p_1}{\overset{\tau p_0}{\rightleftarrows}} \tau P_1), \qquad \tau(\phi_0,\phi_1)=(\tau\phi_0,\tau\phi_1),$$

$$T(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) := (P_1 \underset{-\tau^h p_0}{\overset{-p_1}{\rightleftarrows}} \tau^h P_0), \quad T(\phi_0,\phi_1)=(-\phi_1,-\tau^h\phi_0).$$

By definition, they satisfy an obvious but basic relation:

$$T^2 = \tau^h \tag{20}$$

Here are some elementary properties of the category $\mathrm{HMF}_A^{gr}(f_W)$.

i) $\mathrm{HMF}_A^{gr}(f_W)$ *is a Krull-Schmidt category: i.e. if* $e \in End_{\mathrm{HMF}_R^{gr}(f)}(M)$ *for an object* M *is idempotent* $e^2=e$*, then there exist an object* M' *and morphisms* $\Phi' : M' \to M$ *and* $\Phi : M \to M'$ *such that* $\Phi' \circ \Phi = e$ *and* $\Phi \circ \Phi' = \mathrm{id}_{M'}$.

ii) $\mathrm{HMF}_A^{gr}(f_W)$ *is of Ext-finite type: i.e.* $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}(M,T^nN)$ *is finite dimensional vector space for all objects* M *and* N *of the category.*

Sketch of proof. The direct sum $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{HMF}_A^{gr}(f_W)}(M,\tau^nM')$ is a finitely generated A_W-module. Since the sum is annihilated by multiplications by $\partial_xf_W, \partial_yf_W, \partial_zf_W$, it is a finite module over $J_W := A_W/(\partial_xf_W, \partial_yf_W, \partial_zf_W)$. Since f_W is of type W and in view of §11 Fact 4., it is of finite rank over $\mathbb{C}$. □

Definition. (Mapping cone) For any morphism $\Phi = (\phi_0,\phi_1) \in \mathrm{Hom}_{\mathrm{MF}_R^{gr}(f)}(M,M')$, we introduce the *mapping cone* $C(\Phi) \in \mathrm{MF}_R^{gr}(f)$ as follows.

$$\begin{pmatrix} P_1 & \begin{pmatrix} -p_1 & 0 \\ \phi_1 & p_0' \end{pmatrix} & \tau^h P_0 \\ \oplus & \rightleftarrows & \oplus \\ P_0' & \begin{pmatrix} -\tau^h p_0 & 0 \\ \tau^h\phi_0 & p_1' \end{pmatrix} & P_1' \end{pmatrix}$$

and obtains a sequence: $*)\quad M \xrightarrow{\Phi} M' \xrightarrow{inclusion} C(\Phi) \xrightarrow{-projection} TM$. Then, we have the following general fact (c.f. [G-M], [K-S], [B-K2], [Ta2]).

Theorem. *The additive category* $\mathrm{HMF}_{A_W}^{gr}(f_W)$ *endowed with the shift function* T *and distinguished triangles isomorphic to* $*)$ *for all morphisms* Φ *forms an enhanced triangulated category of Ext-finite type.*

See [B-K2] for a definition of the *enhanced triangulated category.*

2. The stable category of maximal Cohen-Macaulay modules over R_W.

Definition. A graded module $M \in \text{gr-}R_W$ is a maximal Cohen-Macaulay module over R_W if $\text{depth}(M) = \dim(R_W)$ $(=: d = 2)$ ($\Leftrightarrow$ $\text{Ext}^i_R(R_W/(x,y,z), M) = 0$ for $i < d = 2$). The full subcategory of gr-R_W consisting of all graded maximal Cohen-Macaulay modules over R_W is denoted by $\text{CM}^{\text{gr}}(R_W)$.

For an element M of gr-R_W and $n \geq d$, the n-th syzygy $syz^n(M)$ ($:=n$th kernel of a graded free resolution of M) up to a free module factor becomes a maximal Cohen-Macaulay module and doubly periodic in n. Hence, one sees $\text{CM}^{\text{gr}}(R_W)$ is a Frobenius category (i.e. it has enough injective and projective objects which coincide to each other). Then, *the stable category* $\underline{\text{CM}}^{\text{gr}}(R_W)$*, defined below, becomes a triangulated category* (c.f. [Ke]): the objects of the stable category $\underline{\text{CM}}^{\text{gr}}(R_W)$ is the same as $\text{CM}^{\text{gr}}(R_W)$ and the space of morphisms is given by

$$\underline{\text{Hom}}_{\text{gr-}R_W}(M,N) := \text{Hom}_{\text{gr-}R_W}(M,N)/\mathcal{P}(M,N),$$

where $\mathcal{P}(M,N)$ is the subspace of $\text{Hom}_{\text{gr-}R_W}(M,N)$ consisting of morphisms which factor through projective modules.

Fact 10. *For a graded matrix factorization* $M \in \text{MF}^{\text{gr}}_{A_W}(f_W)$*, we associate a maximal Cohen-Macaulay module* $\text{coker}(P_1 \overset{p_1}{\to} P_0) \in \text{CM}^{\text{gr}}(R_W)$ *over* R_W*. This correspondence induces an equivalence of the triangulated categories:*

$$\text{HMF}^{\text{gr}}_{A_W}(f_W) \quad \simeq \quad \underline{\text{CM}}^{\text{gr}}(R_W). \tag{21}$$

The advantage of the category $\underline{\text{CM}}^{\text{gr}}(R_W)$ is that it easily admits the concepts: *Auslander-Reiten triangles* and *Serre duality*, which we explain below. For details on the subject, the reader is referred to textbooks, e.g. [Hap], [Yos].

We first define the Auslander transpose $\text{tr}(M)$ (up to free module factor) of $M \in$gr-R_W by putting $\text{tr}(M) := \text{Coker}({}^t f)$ where $F_1 \overset{f}{\to} F_0 \to M \to 0$ is a finite presentation of M and ${}^t f$ is the contragradient homomorphism of f. Let us denote by $\text{syz}^d(\text{tr}(M))$ the reduced dth syzygy of $\text{tr}(M)$ obtained by avoiding all graded free summands from a dth syzygy of $\text{tr}(M)$. Then, the Auslander-Reiten translation, or A-R translation, $\tau_{AR}(M) \in$ gr-R_W is defined by

A-R translation : $\qquad \tau_{AR}(M) \ := \ \text{Hom}_{R_W}(\text{syz}^d(\text{tr}(M)), \text{K}_{R_W})$

where $\text{K}_{R_W} = \text{Res}_{X_{W,0}} \begin{bmatrix} A_W\ dxdydz \\ f_W \end{bmatrix} = \tau^{-\varepsilon_W} R_W$ is the canonical module of $R_W = A_W/(f_W)$. If M is a maximal Cohen-Macaulay module without a free direct summand, then we easily see that $\mathrm{syz}^2(\mathrm{tr}(M)) \simeq \mathrm{Hom}_{R_W}(M, R_W)$, and, hence,

$$\tau_{AR}(M) \ \simeq \ \tau^{-\varepsilon_W} M. \tag{22}$$

The auto-equivalence of the category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ induced by τ_{AR} is denoted again by τ_{AR}. In view of the relation (20), we have the following relation:

$$\tau_{AR}^h \ = \ (T^2)^{-\varepsilon_W}.$$

The following duality was shown by Auslander and Reiten [A-R3]:

$$\mathrm{Ext}^d_{\text{gr-}R_W}(\underline{\mathrm{Hom}}_{R_W}(M,N), \mathrm{K}_{R_W}) \ \simeq \ \mathrm{Ext}^1_{\text{gr-}R_W}(N, \tau_{AR}(M))$$

for $M, N \in \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$. This, in particular, implies the following

Serre duality : $\mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{Hom}}_{\text{gr-}R_W}(M,N), \mathbb{C}) \ \simeq \ \underline{\mathrm{Hom}}_{\text{gr-}R_W}(N, SM)$

as a bi-functorial isomorphism of vector spaces for $M, N \in \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$, where S is an auto-equivalence of the category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$, called Serre functor [B-K1], defined by

$$S := T\tau_{AR}. \tag{23}$$

As a consequence of Serre duality, one can show that, for any indecomposable object Z of $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$, there exists the *AR-triangle of* Z in the following sense: let $Z \xrightarrow{w} T\tau_{AR}(Z)$ be the morphism, which, by Serre duality, corresponds to the dual of the identity element in $\mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{Hom}}_{\text{gr-}R_W}(Z,Z), \mathbb{C})$. Then, *there exists an object* $AR(Z)$ *and the triangle, called A-R triangle, in* $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$*:*

A-R triangle : $\tau_{AR}(Z) \ \xrightarrow{u} \ AR(Z) \ \xrightarrow{v} \ Z \ \xrightarrow{w} \ T\tau_{AR}(Z)$

such that, for any morphism $g : W \to Z$ *in* $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ *which is not a split epimorphism, there exists* $h : W \to AR(Z)$ *with* $vh = g$.

3. The category of the singularity $X_{W,0} := \mathrm{Spec}(R_W)$.

Definition. ([Orl1]) The triangulated category of the singularity $X_{W,0}$ is

$$D^{\mathrm{gr}}_{Sg}(R_W) := D^b(\text{gr-}R_W)/D^b(\text{gr proj-}R_W)$$

where $D^b(\text{gr-}R_W)$ is the bounded derived category of the abelian category gr-R_W with the natural triangulated structure and $D^b(\text{gr } \textit{proj-}R_W)$ is its full triangulated subcategory consisting of objects which are isomorphic to the bounded complexes of projectives. Actually, the subcategory is the derived category of the exact category of graded projective modules [Ke], and is called the subcategory of perfect complexes.

Since R_W defines the hypersurface $X_{W,0}$ and is Gorenstein, we have

Theorem. (Buchweitz [Bu], Orlov [Orl2]§1.3) *The natural inclusion map* $\mathrm{CM}^{\mathrm{gr}}(R_W) \to \mathrm{gr}\text{-}R_W$ *induces the equivalence of triangulated categories:*

$$\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W) \quad \simeq \quad D^{\mathrm{gr}}_{Sg}(R_W). \tag{24}$$

Orlov [Orl2] gave further a comparison Theorem of $D^{\mathrm{gr}}_{Sg}(R_W)$ with the quotient abelian category $\mathrm{qgr}\text{-}R_W := \mathrm{gr}\text{-}R_W/\mathrm{tors}\text{-}R_W$ where $\mathrm{tors}\text{-}R_W$ is the full subcategory of $\mathrm{gr}\text{-}R_W$ consisting of all finite dimensional R_W-modules. Actually, in case when C_W is a rational curve, we may regard it as a weighted projective line in the sense of Geigle and Lenzing [G-L 1]. Then $\mathrm{qgr}\text{-}R_W$ is derived equivalent to the category of coherent sheaves on the weighted projective line [G-L 2].[52]

4. The triangulated category $\mathcal{T}_W$ associated with a regular system of weights W.

Owing to (21) and (24), we introduce an enhanced triangulated category

$$\mathcal{T}_W \ := \ \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W) \ \simeq \ \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W) \ \simeq \ D^{\mathrm{gr}}_{Sg}(R_W) \tag{25}$$

associated to a regular system of weights W up to equivalences. The advantage of the third expression is that we have the following generation theorem ([K-S-T 2]), which we shall use in the proof of our main theorem in §18.

Theorem. *Let $\mathcal{T}$ be a right-admissible full triangulated subcategory of $\mathcal{T}_W$ satisfying:*

i) The shift functor τ induces an auto-equivalence of $\mathcal{T}$.

ii) There is an object of $\mathcal{T}$ which is isomorphic to the pure complex of the torsion (sky-scraper) module $R_W/(x,y,z)$ in $\mathcal{T}_W$.

Then the natural inclusion $\mathcal{T} \subset \mathcal{T}_W$ induces the triangulated equivalence.

Here, a subcategory $\mathcal{T}'$ of a triangulated category $\mathcal{T}$ is called *right-admissible* if, for any object X of $\mathcal{T}$, there exist $N \in \mathcal{T}'$, $M \in \mathcal{T}'^{\perp} := \{M \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(N,M) = 0 \ \forall N \in \mathcal{T}'\}$ and a triangle: $N \to X \to M \to TN$ in $\mathcal{T}$.

For a later use, we recall some terminologies and results from [Bon].

[52] [G-L 2] treats the case corresponding to the regular weight systems with $\varepsilon_W = 1$. A general proof which covers the case for any regular weight system of genus 0 shall appear in: H. Kajiura, K. Saito and A. Takahashi: Weighted projective lines associated to regular systems of weights, in preparation.

Definition. i) An object E in a triangulated category $\mathcal{T}$ over $\mathbb{C}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(E, T^pE) \simeq \mathbb{C}$ if $p=0$ and 0 if $p \neq 0$.

ii) An *exceptional collection* is a sequence $(E_1, \cdots, E_\mu)$ of exceptional objects satisfying $\mathrm{Hom}_{\mathcal{T}}(E_i, T^pE_j) = 0$ for $\forall p \in \mathbb{Z}$ and $0 \le j < i \le \mu$.

iii) An exceptional collection $(E_1, \cdots, E_\mu)$ is called *strongly exceptional* if $\mathrm{Hom}_{\mathcal{T}}(E_i, T^pE_j) = 0$ for all $1 \le i, j \le \mu$ and $p \neq 0$.

iv) For an exceptional collection $\mathcal{E} := (E_1, \cdots, E_\mu)$, we denote by $\langle \mathcal{E} \rangle := \langle E_1, \cdots, E_\mu \rangle$ the smallest triangulated full subcategory containing $E_1, \cdots, E_\mu$. We say that $\mathcal{E}$ generates $\mathcal{T}$ if $\langle \mathcal{E} \rangle$ is equivalent to $\mathcal{T}$.

v) For a strongly exceptional collection $\mathcal{E} := (E_1, \cdots, E_\mu)$, let us introduce a finite dimensional agebra

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \ := \ \oplus_{0 \le i,j \le \mu} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$$

and call it the *hom-algebra* of the collection $\mathcal{E}$.

Theorem. ([Bon],[B-K2]) *Let $\mathcal{T}$ be an enhanced triangulated category of Ext-finite type, and let $\mathcal{E}$ be a strongly exceptional collection. Then, $\langle \mathcal{E} \rangle$ is right admissible and is, as an enhanced triangulated category, equivalent to the bounded derived category*

$$D^b(\text{mod-Hom}(\mathcal{E}, \mathcal{E})). \tag{26}$$

5. K-group and Auslander-Reiten translation of $\mathcal{T}_W$.

In this paragraph, we show that the Auslander-Reiten translation induces an automorphism of the K-group of the category, which is expressed as the product of reflections. This expression is presumably the mirror dual of the expresion given in §5 of the Milnor monodromy **c** by the product of reflections.

For a triangulated category $\mathcal{T}$, let $\mathrm{K}_0(\mathcal{T})$ be its Grothendieck group (or K-group), i.e. the quotient group of the free abelian group generated by the equivalence classes $[X]$ of objects X of $\mathcal{T}$ divided by the submodule generated by $[X]+[Z]-[Y]$ for all triangles $X \to Y \to Z \to TX$. We denote by $[X]$ the image element in K_0 of X. If a set $E_1, \cdots, E_\mu$ of objects generates the triangulated category, then their images $[E_1], \cdots, [E_\mu]$ generates the K-group over $\mathbb{Z}$.

The shift functor T on $\mathcal{T}$ induces an action $[T] = -\mathrm{id}_{\mathrm{K}_0(\mathcal{T})}$ on $\mathrm{K}_0(\mathcal{T})$, since $[X]+[TX]=0$ for any object X because of the triangle $X \xrightarrow{1} X \to 0 \to TX$. In particular T^2 induces identity on the K-group

The Auslander-Reiten translation τ_{AR} is an auto-equivalence of the triangulated category, so that it induces an automorphism of the group $\mathrm{K}_0(\mathcal{T})$, denoted by $[\tau_{AR}]$. For the category $\mathcal{T}_W$ associated to a regular weight system W, it is of finite order h, since, using the expression (22) and the fact (20), we calculate as

$$[\tau_{AR}]^h = [T^{-2\varepsilon_W}] = (-id_{\mathrm{K}_0(\mathcal{T}_W)})^{-2\varepsilon_W} = id_{\mathrm{K}_0(\mathcal{T}_W)}.$$

If $\mathcal{T}$ is of Ext-finite type over $\mathbb{C}$, the Euler pairing is defined by

$$\chi(X,Y) \ := \ \textstyle\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{hom}_{\mathcal{T}}(X, T^nY)$$

for any two objects X and Y of $\mathcal{T}$. Because of the (co-)homological property of $\mathrm{hom}_{\mathcal{T}}$, it induces a bilinear form on $\mathrm{K}_0(\mathcal{T})$, which we denote again by χ. We equip the K-group with the symmetric bilinear form (e.g. see [Ri1] 2.4) [53]:

$$I(e,f) \ := \ \chi(e,f) \ + \ \chi(f,e). \tag{27}$$

for $e,f \in \mathrm{K}_0(\mathcal{T})$. We remark that if $e=[E]$ where E is an exceptional object of $\mathcal{T}$, then $\chi(e,e)=1$ and, hence, $I(e,e)=2$. Then, similarly to Picard-Lefschetz formula in §5, we can define the reflection $w_e \in O(\mathrm{K}_0(\mathcal{T}), I)$ by letting

$$w_e(u) \ := \ u - I(u,e)e \qquad \text{for } u \in \mathrm{K}_0(\mathcal{T}).$$

The $[\tau_{AR}]$ preserves the bilinear form χ, i.e. $[\tau_{AR}] \in O(\mathrm{K}_0(\mathcal{T}), I)$. Let us express now $[\tau_{AR}]$ as a product of reflections on $\mathrm{K}_0(\mathcal{T})$.

Let $\mathcal{E} := (E_1, \cdots, E_\mu)$ be a strongly exceptional collection of $\mathcal{T}$. Assume that $E_1, \cdots, E_\mu$ generate $\mathcal{T}$ and, hence $[E_1], \cdots, [E_\mu]$ is a basis of $\mathrm{K}_0(\mathcal{T})$. Associated to $[E_1], \cdots, [E_\mu]$, we consider two basis: $f_1, \cdots, f_\mu$ and $g_1, \cdots, g_\mu$ of $\mathrm{K}_0(\mathcal{T})$ defined by the following relations:

$$[E_i] \ = \ \textstyle\sum_{j=1}^{\mu} \chi(E_i, E_j) f_j \ = \ \sum_{j=1}^{\mu} g_j \chi(E_j, E_i) \tag{28}$$

Here, we remark that the matrix $\chi_{\mathcal{E}} := (\chi(E_i, E_j))_{ij=1,\cdots,\mu}$ is an upper triangular matrix with 1 at each diagonal entry so that χ is invertible. Let us denote by $C_{\mathcal{E}} = (C_{\mathcal{E},ij})_{ij=1,\cdots,\mu}$ the inverse matrix $\chi_{\mathcal{E}}^{-1}$ (which is also an upper triangular integral matrix). In fact, using the mapping cone constructions, one can find objects F_i and G_i in $\mathcal{T}$ such that $f_i = [F_i]$ and $g_i = [G_i]$ for $i = 1, \cdots, \mu$.[54] The intersection matrix of them are given

[53] For the purpose of the period map for odd dimensional Milnor fiber, we need to study the skew symmetric bilnear form: $I_{odd}(e,f) := \chi(e,f) - \chi(f,e)$ (see [Sa18]§6 (6.2.2)), [Sa19] latter half of §4). However, we shall not treat them in the present paper.

[54] Actually, these objects F_i and G_i are constructed by use of mutations and are shown to be exceptional objects ([Bon]).

by $C_{\mathcal{E}}$ as follows:
(29)
$$\chi([F_i],[F_j]) \ = \ \chi([G_i],[G_j]) \ = \ C_{\mathcal{E},ji} \qquad \text{for } i,j=1,\cdots,\mu.$$

Since $I([F_i],[F_i])=I([G_i],[G_i])=2$ for $1 \le i \le \mu$, we define reflections $w_{[F_1]},\cdots,w_{[F_\mu]}$ and $w_{[G_1]},\cdots,w_{[G_\mu]}$. Then, one can easily verify the formula:

Fact 11. *Let $(E_1,\cdots,E_\mu)$ be a strongly exceptional collection, then the transformation $[\tau_{AR}]$ is expressed as the product of reflections associated to the basis:*

$$[\tau_{AR}] \ = \ w_{[F_1]}\cdots w_{[F_\mu]} \ = \ w_{[G_1]}\cdots w_{[G_\mu]} \ . \tag{30}$$

6. Quiver and path algebra associated with $\mathcal{E}$.

In this paragraph, associated with a strongly exceptional collection, we consider a slight generalization of a quiver, and then, associated to the (generalized) quiver, we introduce a path algebra with relations, which we shall use in §17 and 18 (see [Ri1] for quivers and path algebras).

Let $\mathcal{E}=(E_1,\cdots,E_\mu)$ be a strongly exceptional collection of a triangulated category $\mathcal{T}$. Then, we associated a quiver $\Delta_{\mathcal{E}}$ given by a pair

$$\Delta_{\mathcal{E}} \ = \ (\Delta_0,\Delta_1), \tag{31}$$

where $\Delta_0 = \{v_1,\cdots,v_\mu\}$ is a set of μ elements, called the *vertices*, and Δ_1, called the set of *allows*, is a multi-set of triplet $(v_i,v_j,\epsilon) \in \Delta_0\times\Delta_0\times\{\pm\}$ where $(v_i,v_j,+)$ appears in Δ_1 only when $i\neq j$, $C_{\mathcal{E},ij}<0$ and $-C_{\mathcal{E},ij}$-times, and $(v_i,v_j,-)$ appears in Δ_1 only when $i\neq j$, $C_{\mathcal{E},ij}>0$ and $C_{\mathcal{E},ij}$-times. We regard $(v_i,v_j,+)\in\Delta_1$ as an arrow (with positive sign) from the vertex v_i to the vertex v_j, and similarly $(v_i,v_j,-)\in\Delta_1$ as a dotted arrow from v_i to v_j.

Remark 16.1. If one forgets the directions of the arrows from the quiver $\Delta_{\mathcal{E}}$ and leaves only lines or dotted lines together with the vertices, then one obtains automatically the intersection diagram Γ of the symmetric bilinear form I with respect to the basis $[F_1],\cdots,[F_\mu]$ or $[G_1],\cdots,[G_\mu]$ of $\mathrm{K}_0(\mathcal{T})$, i.e. Γ is the intersection diagram for the symmetrization of the matrix (29)) (for instance, [Sa14]I (8.2)).

Associated with the above given quiver $\Delta_{\mathcal{E}}$ (31), the path algebra

$$\mathbb{C}(\Delta_{\mathcal{E}},R) \tag{32}$$

with relations R is defined as follows. Let $\Delta_1=\Delta_1^+ \amalg \Delta_1^-$ be the decomposition of the set of arrows into those of positive and negative signs.

We regard $\Delta^+_{\mathcal{E}} := (\Delta_0, \Delta^+_1)$ as the quiver in the classical sense (e.g. [Ri1] 2.1), then by concatenating arrows, one defines paths and the path algebra $\mathbb{C}\Delta^+$ as usual ([ibid]). Let

$$R \ : \ \Delta_1^- \ \longrightarrow \ \mathbb{C}\Delta^+_{\mathcal{E}}$$

be a map such that the image of an arrow $(v_i, v_j, -)$ belongs in the subspace $(v_j \mid v_j) \cdot \mathbb{C}\Delta^+_{\mathcal{E}} \cdot (v_i \mid v_i)$ spanned by all paths from v_i to v_j (here, we denote by $(v \mid v)$ the path of length 0 at a vertex v). Then, we put

$$\mathbb{C}(\Delta_{\mathcal{E}}, R) \ := \ \mathbb{C}\Delta^+_{\mathcal{E}} \ / \ (\mathbb{C}\Delta^+_{\mathcal{E}} \cdot R(\Delta_1^-) \cdot \mathbb{C}\Delta^+_{\mathcal{E}}), \tag{32}$$

where $\mathbb{C}\Delta^+_{\mathcal{E}} \cdot R(\Delta_1^-) \cdot \mathbb{C}\Delta^+_{\mathcal{E}}$ is the both-sided ideal of the path algebra $\mathbb{C}\Delta^+_{\mathcal{E}}$ generated by the image set $R(\Delta_1^-)$. We call $\mathbb{C}(\Delta_{\mathcal{E}}, R)$ the *path-algebra with relations* R.

Remark 16.2. Assigning to each arrow $(v_i, v_j; +) \in \Delta^+_1$ a morphism $f_{ij} \in \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$, we can define a ring homomorphism:

$$\mathbb{C}(\Delta_{\mathcal{E}}, R) \ \longrightarrow \ \mathrm{Hom}(\mathcal{E}, \mathcal{E}) := \oplus_{0 \leq i,j \leq \mu} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$$

for a suitable choice of relations R. In general, the homomorphism can neither be isomorphic nor induce derived equivalence for any choices of f_{ij} and R.

Example. Let us consider a strongly exceptional collection $\mathcal{E} = (E_1, E_2, E_3)$ such that $\chi_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $C_{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Then the associated quiver is a Dynkin quiver $\Delta_{\mathcal{E}} = \circ \rightarrow \circ \rightarrow \circ$ of type A_2 and $\mathbb{C}\Delta_{\mathcal{E}}$ is a path algebra of type A_2. On the other hand, there are two cases of the structure of the hom-algebra $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) := \oplus_{1 \leq i \leq j \leq 3} \mathrm{Hom}(E_i, E_j)$ depending on whether the product $\mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_3) \to \mathrm{Hom}(E_1, E_3)$ is a) non-zero or b) zero. Then the homomorphism $\mathbb{C}\Delta_{\mathcal{E}} \to \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ assigning the two arrows in $\Delta_{\mathcal{E}}$ to the base of $\mathrm{Hom}(E_1, E_2)$ and $\mathrm{Hom}(E_2, E_3)$, respectively, is isomorphic in the case a), but neither isomorphic nor derived equivalent in the case b).

§**17.** The category of matrix factorizations: the case $\varepsilon_W = 1$.

In this section, we study the category $\mathcal{T}_W = \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$ associated with a weight system W with $\varepsilon_W = 1$. Recall that, in this case, the weight systems are classified into types A_l $(l \geq 1)$, D_l $(l \geq 4)$, E_6, E_7 and E_8 (see Table 8), and that the associated polynomials f_W of type W are called the simple polynomials (see Table 2).

Then the following theorem is proven in [K-S-T 1] (c.f. also [Ue]).

Theorem. *Let W be a regular system of weights of type ADE. For any Dynkin quiver $\vec{\Delta}$ of type W (see Note below), there exists a unique strongly exceptional collection $\mathcal{E}_{\vec{\Delta}}$ of the category $\mathcal{T}_W$* (25) *such that*

i) *the $\mathcal{E}_{\vec{\Delta}}$ generate the triangulated category $\mathcal{T}_W$,*

ii) *the quiver associated with the collection $\mathcal{E}_{\vec{\Delta}}$ is isomorphic to $\vec{\Delta}$,*

iii) *the path algebra $\mathbb{C}\vec{\Delta}$ is isomorphic to the hom-algebra $\mathrm{Hom}_{\mathcal{T}_W}(\mathcal{E}_{\vec{\Delta}}, \mathcal{E}_{\vec{\Delta}})$.*

Note. By a Dynkin quiver, we mean an oriented Dynkin diagram of type ADE.

Sketch of proof. According to the works [Ei], [A-R1] and [Au], the Auslander-Reiten quiver for the triangulated category $\mathrm{HMF}_{\hat{\mathcal{O}}}(f_W)$ of ungraded matrix factorizations over the local rings $\mathcal{O}$ and $\hat{\mathcal{O}}$ are well-known to be isomorphic to the both side oriented Dynkin quiver $\overset{\leftrightarrow}{\Delta}$ of type W. We consider the natural forgetful functor: $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W) \to \mathrm{HMF}_{\hat{\mathcal{O}}}(f_W)$ forgetting the gradings on matrix factorizations. Then, by "lifting" the results on $\mathrm{HMF}_{\hat{\mathcal{O}}}(f_W)$ back to the graded category together with the knowledge of the Serre duality, in [K-S-T 1], we determine the list of all indecomposable objects and all irreducible morphisms in $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$. Using these data, we can verify directly the existence (up to τ-shift) of a strongly exceptional collection $\mathcal{E}_{\vec{\Delta}}$ of $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$, and of the natural isomorphism: $\mathbb{C}\vec{\Delta} \simeq \mathrm{Hom}(\mathcal{E}_{\vec{\Delta}}, \mathcal{E}_{\vec{\Delta}})$ (i.e. the non-vanishing of compositions of morphisms corresponding to concatenations of arrows in $\vec{\Delta}$).□

Applying a theorem of Bondal-Kaplanov to the enhanced category $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$, we see the equivalence among the triangulated categories:

$$D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}) \simeq D^b(\mathrm{mod}\text{-}\mathrm{Hom}(\mathcal{E}_{\vec{\Delta}}, \mathcal{E}_{\vec{\Delta}})) \simeq \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W). \tag{33}$$

Combining with the well known results on the representations of the hereditary algebra $\mathbb{C}\vec{\Delta}$ (c.f. [Ga], [Ri1], [Hap]), we obtain the following expected results.

Corollary. *Let the setting be as in Theorem. Then,* i) *the K-group $\mathrm{K}_0(\mathcal{T}_W)$ of $\mathcal{T}_W$ is isomorphic to the root lattice of type $W = W^*$,* ii) *the image set in $\mathrm{K}_0(\mathcal{T}_W)$ of indecomposable objects form the root system R_{W^*} of type W^*, and iii) the image in $K_0(\mathcal{T}_W)$ of a strongly exceptional collection $\mathcal{E}_{\vec{\Delta}}$ forms a simple root basis of the root system R_{W^*}.*

Remark. As in the A_l case [Ta2], a stability condition (Bridgeland [Bri 1]) can be naturally given by the grading of matrix factorizations.

The abelian category associated to the stability condition (as a full subcategory of $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$) is equivalent to the category $\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}$ of finite modules over the path algebra of a Dynkin quiver $\vec{\Delta}$ of the principal orientation introduced in [Sa21].

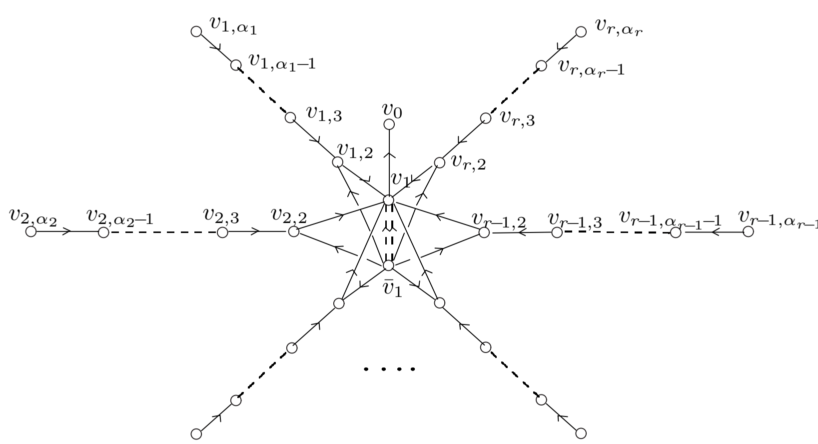
§18. The category of matrix factorizations: the case $\varepsilon_W = -1$

In this section, we describe the category $\mathcal{T}_W = \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$ associated with a regular system of weights W with $\varepsilon_W{=}-1$ and $a_0{=}0$. Recall that the orbifold curve C_W (12) is of genus a_0 so that we are considering the case of rational orbifold curves. There are 14+8 such weight systems, which are listed in Table 10. The associated polynomials f_W of type W are also listed in Table 10, where we remark that, in the first 14 weight systems, there are 3 orbifold points on the curve C_W so that the polynomial f_W contains no moduli parameter, whereas, in the latter 8 weight systems case, there are either 4 or 5 orbifold points on the curve C_W so that the polynomial f_W contains either one or two moduli parameters λ or λ_1, λ_2, respectively.

In order to recall Theorem in [K-S-T 2] 5.4, we introduce some particular quiver $\Delta_{A(W)}$ depending only on the signature set $A(W)$ (13) (see Footnote 32) for the orbifold structure on C_W. Slightly more generally, let us define

Definition. Let $$A = \{\alpha_1, \cdots, \alpha_r\}$$ be a multi-set of r positive integers for some $r \in \mathbb{Z}_{\geq 0}$. Then the quiver Δ_A of type A is defined by the following figure and data.

Table 14. $\Delta_A = (\Pi_A, E_A)$



where the set of vertices and the set of arrows are given as follows:

$$\begin{aligned} \Pi_A &:= \{v_0,\ v_1,\ \overline{v}_1\} \amalg \amalg_{i=1}^r \{v_{i,2},\ \cdots,v_{i,\alpha_i}\}, \\ E_A &:= \{(v_1,v_0;+),\ (\overline{v}_1,v_1;-)_1,\ (\overline{v}_1,v_1;-)_2\} \\ &\quad \amalg \amalg_{i=1}^r \{(v_{i,2},v_1;+),\ \cdots,(v_{i,\alpha_i},v_{i,\alpha_i-1};+),\ (\overline{v}_1,v_{i,2};+)\}. \end{aligned}$$

Remark 9. We have only two negatively signed arrows between the vertices $\bar{v}_1$ and v_1. They are distinguished by the subscripts 1 and 2 as $(\overline{v}_1,v_1;-)_1$ and $(\overline{v}_1,v_1;-)_2$. They shall later turn to relations in the path algebra.

Before we state the main theorem, we introduce one more numerical invariant: the *dual rank* ν_W for any regular weight system W. It is defined by using exponents $e_W(i)$ defined at Preface of the paper, as

$$\nu_W := -\sum_{j|h} j\cdot e_W(h/j). \tag{34}$$

It is introduced [Sa17] (7.2) as the rank of W^* (if it exists). Actually, we prove the formula (whose proof will appear elsewhere):

$$\nu_W = \sum_{i=1}^r(\alpha_i-1)+2(1-a_0)-\varepsilon_W$$

where $A(W)=\{\alpha_1,\cdots,\alpha_r\}$ is the signature set of W (see Footnote 31).

Remark 10. In this section, we have $\varepsilon_W=-1$ and $a_0=0$. So the formula reduces to

$$\nu_W = \sum_{i=1}^r(\alpha_i-1)+3.$$

Then, one observes that the first term of this formula coincides with the number of vertices on the r branches of the diagram Δ_A and the last term 3 coincides with the number of vertices on the central axis of the diagram Δ_A.

The same interpretation is possible for the case of the previous section §17, where one has $\varepsilon_W=1$ and $a_0=0$ so that one has $\nu_W = \sum_{i=1}^r(\alpha_i-1)+1$. Then this formula again describes the number of vertices in a Dynkin diagram. However, in the case when the weight systems are selfdual, rank and dual rank coincide with each other, and it is unnecessary to introduce such dual rank.

Theorem. *Let W be a regular system of weights with $\varepsilon_W=-1$ and $a_0=0$. We fix a polynomial f_W of type W. Let $\mathcal{T}_W$ (25) be triangulated category associated to f_W. Then, there exists a strongly exceptional collection $\mathcal{E}_{\Delta_A}=(E_1,\cdots,E_{\nu_W})$ of the category $\mathcal{T}_W$ satisfying the following properties.*

i) *The* $\mathcal{E}_{\Delta_A}$ *generate the triangulated category* $\mathcal{T}_W$ *.*

ii)) *The quiver associated with the collection* $\mathcal{E}_{\Delta_A}$ *is equal to* Δ_A (Table 14)*, where* A *is equal to the signature set* $A(W)$ *of* W*.*

iii) *If the* $(v_i, v_j; +)$ *is a real arrow of* Δ_A*, then* $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ *is a vector space of rank* $-C_{\mathcal{E},ij}(=1)$*. If, further, the arrow* $(v_i, v_j; +)$ *is on the branches of* Δ_A*, then* $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ *is spanned by an irreducible homomorphism.*

iv) *The assignments*

$$(v_i, v_j, +) \longmapsto f_{ij}$$

of a base f_{ij} *of* $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ *to each arrow* $(v_i, v_j; +)$ *of* $\Delta^+_{\mathcal{E}}$ *together with suitable choices, depending on* f_W *and* f_{ij}*, of the relations*

$$(35) \qquad \begin{array}{lcl} R((\overline{v}_1, v_1; -)_1) & = & \sum_{i=1}^r \lambda_{1,i}\ (\overline{v}_1, v_{i,2}; +) \circ (v_{i,2}, v_1; +), \\ R((\overline{v}_1, v_1; -)_2) & = & \sum_{i=1}^r \lambda_{2,i}\ (\overline{v}_1, v_{i,2}; +) \circ (v_{i,2}, v_1; +), \end{array}$$

induce an isomorphism:

$$(36) \qquad \mathbb{C}(\Delta_{A(W)}, R) \ \simeq\ \mathrm{Hom}_{\mathcal{T}_W}(\mathcal{E}, \mathcal{E})$$

between the path algebra (32) *and the hom-algebra (recall* §16 **4.** Theorem*).*

Combining the isomorphism (36) with the theorem of Bondal-Kaplanov (see §16 **4.**) on the enhanced category $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$, we obtain:

Corollary. *We have the equivalence between the triangulated categories:*

$$D^b(\mathrm{mod}\text{-}\mathbb{C}(\Delta_{A(W)}, R)) \ \simeq\ \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W).$$

Recall that the signature set $A(W)$ for the 14 weight systems coincides with the set of Dolgachev numbers (§13), and that it is equal to the set of Gabrielov numbers (recall Table 12, 13) for the $*$-dual weight system W^* (§14, Fact 7.).

Recall the basis f_i (or g_i) defined by the formula (28) of the K-group of the category $D^b(\mathrm{mod}\text{-}\mathbb{C}(\Delta_{A(W)}, R))$. In view of the definition (27) of the bilinear form on the K-group and the intesection number (29) among the basis elements, we see that the K-group, as a lattice, coincides with the lattice associated with the Gabrielov diagram (Table 12) for the dual weight system W^*. That is, we have the isomorphism of lattices equipped with symmetric bilinear forms:

$$(37) \qquad \mathrm{K}_0(\mathcal{T}_W) \ \simeq\ \mathrm{H}_2(X_{W^*,1}, \mathbb{Z})$$

of the K-group of the category for the weight system W and the middle homology group of the Milnor fiber (see Footnote 35) of the dual weight system W^*. In this sense, mirror symmetry at the homology group level is confirmed. However, the characterization of the subset in the LHS corresponding to the set of vanishing cycles R_{W^*} in the RHS is unknown. We ask whether it is the set of images of indecomposable exceptional objects in $\mathcal{T}_W$ or not (see Footnote 48).

As was discussed in the Preface, there are three Lie algebras associated to the 14 regular weight systems W (which admit a $*$-dual weight system W^*):

i) the algebra $\mathfrak{g}_{W^*}$ defined by the Chevalley generators and generalized Serre relations [S-Y] (4.1.1) for the Cartan matrix associated to the diagram $\Delta_{A(W)}$,

ii) the algebra $\mathfrak{g}'_{W*}$ generated by the vertex operators e^{α} for roots $\alpha \in R_{W^*}$ in the Lie algebra $V_{K_0(\mathcal{T}_W)}/DV_{K_0(\mathcal{T}_W)}$ ([Bo1], [S-Y](3.2.1)) for the lattice $K_0(\mathcal{T}_W)$,

iii) the algebra $\mathfrak{g}''_{W^*}$ constructed by Ringel-Hall construction ([To], [P-X], [X-X-Z]) for the derived category $D^b(\text{mod-}\mathbb{C}(\Delta_{A(W)}, R))$ of the path algebra $\mathbb{C}(\Delta_{A(W)}, R)$.

The following question is the last question of the present paper.

Problem. Clarify the relationship among these three Lie algebras. Are they isomorphic to each other? Do any of (or the covering of) these algebras satisfies the requirements posed by Question in §12 and by Addition to Question in §14?

Remark. For the 14 exceptional weight systems W, the (conjectural) period domain for the period map for the primitive form of type W is introduced [Lo6], [Sa22] (c.f. [Ao]) as

$$\mathcal{B}_V \ := \ \{ \ \varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \ker(\varphi) < 0\}$$

where $V := (Q_W \otimes \mathbb{R}, I)$ is the real vector space equipped with a quadratic form I of the signature $(l+2, 0, 2)$, and $\ker(\varphi) < 0$ means that the restriction of I to the subspace $\ker(\varphi)$ is negative definite. It is interesting to clarify the relationship of the period domain $\mathcal{B}_V$ for W with the space of stability conditions (Bridgeland [Bri 1,2,3],[H-M-S]) for the category $\mathcal{T}_{W^*}$ through the identification $\mathrm{K}_0(\mathcal{T}_{W^*}) \simeq (Q_W, -I)$ due to the above Theorem. The ring of "automorphic forms" (in suitable sense, c.f. [Ao]) on $\mathcal{B}_V$ with respect to the group W_W is expected to carry the flat structure (c.f. §12 Question vi)). For some recent developments on the geometry of the modular varieties for the orthogonal groups $O(2, n)$, we refer to [Bo1, Bo2, Bo3], [G-H-S 1, G-H-S 2, G-H-S 3] and [Gr].

§19. Appendix. McKay correspondence and its Inverse.

1. McKay correspondence (1979) [Mc].

We recall McKay correspondence in its original form [Mc]. For its further understanding from a categorical view point, see [B-K-R].

Let ρ be the faithful representation of the Kleinean group $\tilde{G}$ into $SU(2)$. Let $\{\rho_0 = 1, \rho_2, \ldots, \rho_n\}$ be the set of isomorphism classes of all irreducible representations of $\tilde{G}$. Consider the decomposition

$$\rho \otimes \rho_j \quad = \quad \sum_{i=\phi}^{n} n_{ij} \rho_i \qquad (j = 0, \ldots, n)$$

for some nonnegative integers $n_{ij} \in \mathbb{Z}_{\geq 0}$. Then, one has:

0) $n_{ij} \in \{0, 1\}$
i) $n_{ii} = 0 \quad (i = 0, \ldots, n)$
ii) $n_{ij} = n_{ji}$
iii) $\tilde{C} := 2I_{n+1} - (n_{ij})_{i,j=0}^{n}$ is negative semi-definite with 1-dimensional kernel.

Actually, from these properties, it is not hard see that $\tilde{C}$ is an affine Cartan matrix of one of types $\tilde{A}_l, \tilde{D}_l$ or $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$, and that the matrix C obtained by deleting column and low for the trivial representation is a Cartan matrix of one of types A_l, D_l, E_6, E_7 or E_8 (see Table 3.). The correspondence:

(MC): $\tilde{G} \longmapsto \Gamma :=$ the graph associated to C

induces the bijection, called the McKay correspondence:

$\{$Kleinean groups$\}$ $\xrightarrow{\sim}$ $\{$Simply laced Coxeter-Dynkin graphs of finite type$\}$

McKay wrote [Mc] "Would not the Greeks appreciate the result that the simple Lie algebras may be derived from Platonic solids?".

2. Gonzalez-Verdier interpretation of McKay correspondence

The work by Gonzalez-Verdier [G-V] says that the representations ρ_i are interpreted as vector bundles $\tilde{V}_i$ on the resolution $\tilde{X}_0$ of the singularity X_0. Then, the 1-st Chern classes $c_1(\tilde{V}_i)$ of the vector bundles form the dual basis to the homology classes of the exceptional curves $[E_i]$ in $\tilde{X}_0$. That is: $c_1(\tilde{V}_i)$ form the fundamental weight for the simple root system.

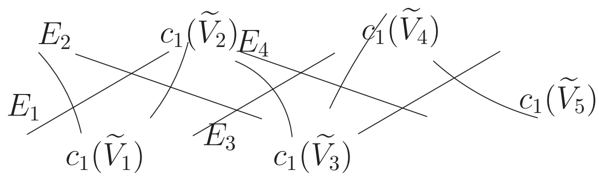
Let $\rho_i : G \to GL(V_i)$ be an irred. repr. of G. So G acts on $\mathbb{C}^2 \times V_i$ diagonally. Then the diagram (not precise)

$$\begin{array}{ccc} (\mathbb{C}^2 \times V_i)/G & \leftarrow & \widetilde{V}_i \\ \downarrow & & \downarrow \\ \mathbb{C}^2/G \simeq X_0 & \leftarrow & \widetilde{X}_0 \end{array}$$

defines an irreducible vector bundle $\widetilde{V}_i$ on $\widetilde{X}_0$

Theorem (Gonzalez-Verdier 1984)**.** *The first Chern class* $c_1(\widetilde{V}_i)$ *of* $\widetilde{V}_i$ *defines a divisor (a smooth curve) in* $\widetilde{X}_0$*, which is transversal to exactly one irreducible component, say* E_i*, of* $E = \pi^{-1}(0)$*. That is:* $c_1(\widetilde{V}_1), \cdots, c_1(\widetilde{V}_l) \in \mathrm{H}^2(\widetilde{X}_0, \mathbb{C})$ *forms the dual basis of* $[E_1], \ldots, [E_l] \in \mathrm{H}_2(\widetilde{X}_0, \mathbb{Z}) \simeq Q := \bigoplus_{i=1}^{l} \mathbb{Z}\rho_i^v$.

Table 15. The first Chern classes of irreducible vector bundles over $\tilde{X}_{A_4,0}$.



3. The inverse of McKay correspondence: $\Gamma \mapsto W_\Gamma \mapsto \langle A(W_\Gamma)\rangle$.

Let us construct conceptually the inverse of the McKay correspondence (MC) (through regular systems of weights) without using the classification.

Let a simply-laced Dynkin diagram Γ (or, equivalently a Cartan matrix C of finite type) be given. The data determine the Coxeter-Killing transformation $\mathbf{c}$ and using its eigenvalues, as we did in §8, we obtain the system of exponents $m_1, \cdots, m_\mu$. Then, as was discussed in §8, the generating function (1) of the exponents decomposes as (2) so that we obtain a simple weight system $W = W_\Gamma$.

How to recover the Kleinean group $\tilde{G}$ *from a simple weight system* W*?*

Let W be a simple weight system (i.e. $\varepsilon_W > 0$, see §8 Fact 1). Let f_W be the simple polynomial of the type W (9), and let us consider the associated hypersurface $X_{W,0}$ (11) (the simple singularity). Due to Fact 2 in §8 and Theorem in §1, the fundamental group of $X_{W,0} \setminus \{0\}$ is nothing but the isomorphic to the Kleinean group to define the simple singularity. on the other hand, we can determine the fundamental group purely arithmetically as follows.

Fact 12. *Let* W *be either a simple weight system or one of the 14 non-degenerate weight systems with* $\varepsilon_W = -1$ *and* $a_0 = 0$*. Then, we have the following isomorphism:*

$$\pi_1(X_{W,0} \setminus \{0\}, *) \;\simeq\; \langle A(W)\rangle,$$

where we recall that $A(W)$ is the signature set of W (see §11 a) Fact 5.*) and that $\langle\{p,q,r\}\rangle := \langle p,q,r\rangle$ denotes the group defined in* §1*.*

Proof. Combining a result of Mumford, which we quote below, and the description of the singularity $X_{W,0}$ in §11 a) Fact 6. (see also its following Example), we get the result. □

Theorem (Mumford 1961)**.** *Let X_0 be a two dimensional normal singularity, and let $\tilde{X}_0 \to X_0$ be a resolution of the singularity such that the exceptional set $E := \pi^{-1}(0)$ is a union of $\mathbb{P}^1$ such that the intersection diagram is a tree. Then, by the use of the data of the tree (details are omitted), one can write down $\pi_1(X_0 \backslash \{0\}, *)$ by suitable generators and relations, explicitly.*

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