Basic Results on Traced Monoidal Categories

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Workshop on GoI, Traced Cats and Implicit Computational Complexity
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Goal of This Talk

We will review some elements of traced monoidal categories of Joyal, Street and Verity, which are related to

- semantics of recursion (fixed-point operators) and
- Geometry of Interaction (the Int-construction),

using the graphical language for monoidal categories as a convenient tool.

In this talk, whenever possible, we consider general *balanced (braided)* monoidal categories, following the original development by JSV, rather than just *symmetric* monoidal categories — for better connection to math, especially low-dimensional topology; for prompting potential applications in CS; for likely new to many of you; and for just fun.

Part I: Traced Monoidal Categories

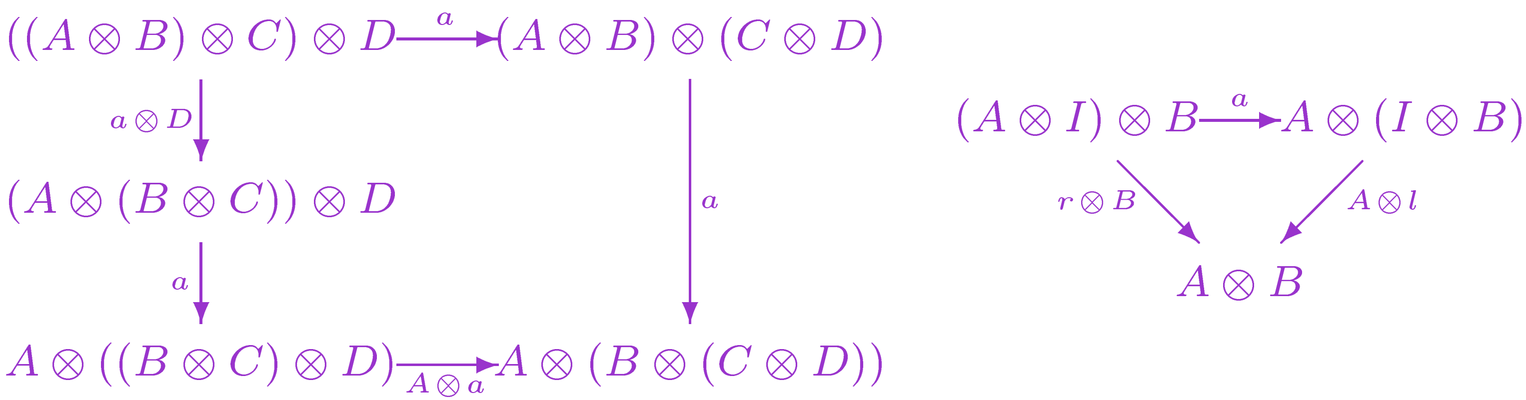
Part II: Geometry of Recursion

Part III: The Int-construction

(Part IV: On Closedness)

Preliminaries: Monoidal Categories

A *monoidal category* (*tensor category*) $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $l_A : I \otimes A \xrightarrow{\sim} A$ and $r_A : A \otimes I \xrightarrow{\sim} A$ such that the following two diagrams commute:



(In practice, often we can safely forget a, l, r, and identify $(A \otimes B) \otimes C$ with $A \otimes (B \otimes C)$ etc — thanks to the coherence theorem.)

Preliminaries: Braidings, Symmetries and Twists

A *braiding* is a natural isomorphism $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ such that both c and c^{-1} satisfy the "bilinearity" diagrams (the case for c^{-1} is omitted):

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{a} & A \otimes (B \otimes C) & \xrightarrow{c} & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle c \otimes C} & & & & \downarrow{\scriptstyle a} \\
(B \otimes A) \otimes C & \xrightarrow[a]{} & B \otimes (A \otimes C) & \xrightarrow[B \otimes c]{} & B \otimes (C \otimes A)
\end{array}$$

A *symmetry* is a braiding such that $c_{A,B} = c^{-1}_{B,A}$.

A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

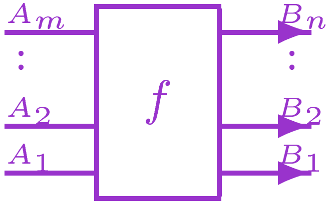
A *twist* or a *balance* for a braided monoidal category is a natural isomorphism $\theta_A : A \xrightarrow{\sim} A$ such that $\theta_I = id_I$ and $\theta_{A \otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$ hold.

A *balanced monoidal category* is a braided monoidal category with a twist.

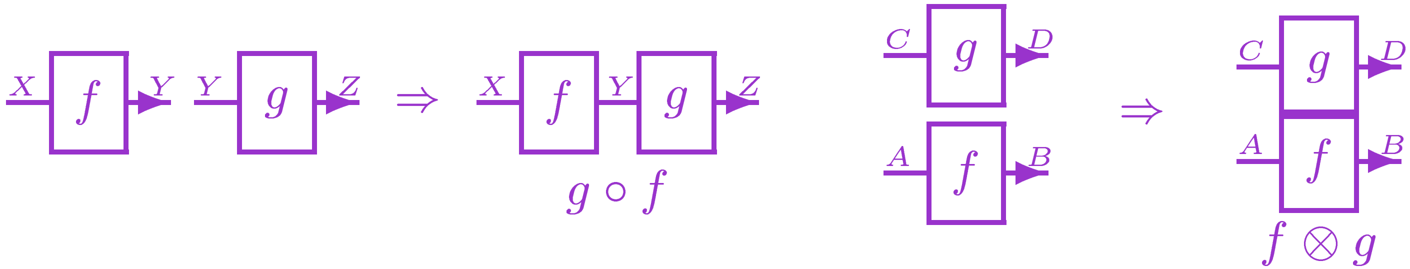
Geometry of Monoidal Categories

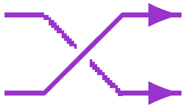
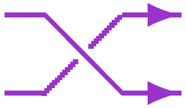
(Geometry of tensor calculus, Joyal and Street / string diagrams, Penrose)

A morphism $f : A_1 \otimes A_2 \otimes \ldots \otimes A_m \to B_1 \otimes B_2 \otimes \ldots \otimes B_n$ can be drawn as:



Morphisms can be composed, either sequentially or in parallel:

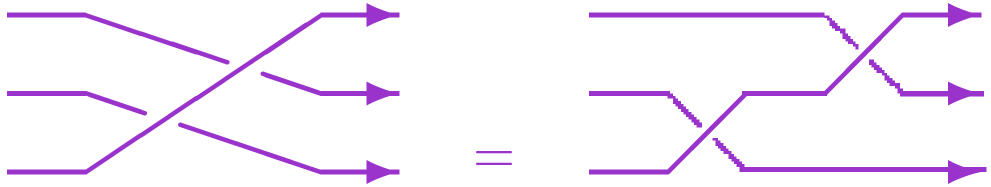


Braids: $c =$  $c^{-1} =$  Twists: $\theta =$  $\theta^{-1} =$ 

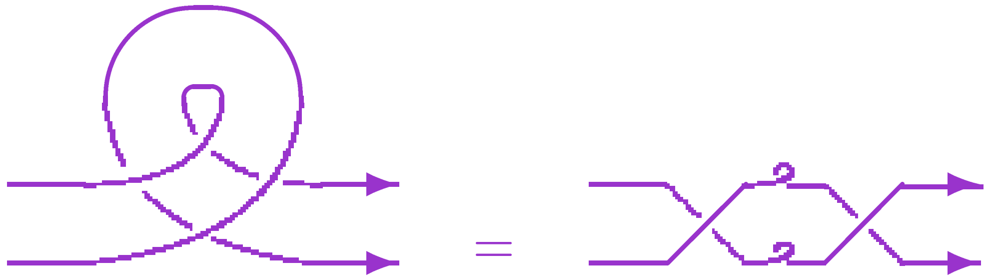
Geometry of Monoidal Categories (cont.)

The interpretation of these pictures is invariant under continuous deformation (Joyal and Street). Hence graphical reasoning can be used for establishing equalities on morphisms in monoidal categories.

Example: the bilinearity axiom for a braiding



Example: the axiom for twists $\theta_{A\otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$

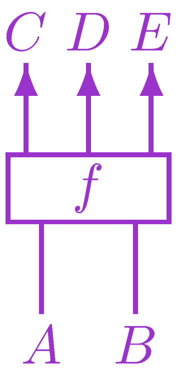
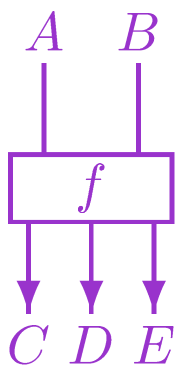
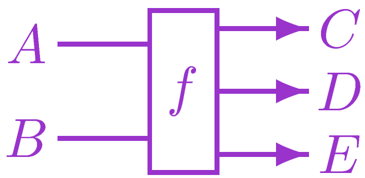
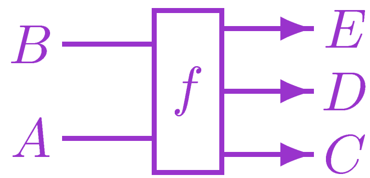


Note: Links in these pictures should be regarded as "ribbons" or "framed tangles".

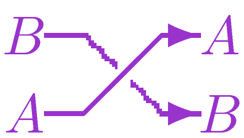
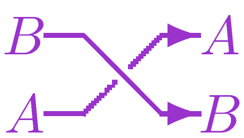
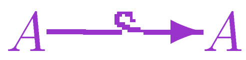
Geometry of Monoidal Categories (cont.)

Reference: There is a recent survey article by Peter Selinger on graphical languages for monoidal categories (available from his web page).

Warning: There are a few (equally good) styles for drawing a picture for a morphism like $f : A \otimes B \longrightarrow C \otimes D \otimes E$ in monoidal categories:

Joyal & Street	Freyd & Yetter	Haghverdi	Selinger, Hasegawa
			

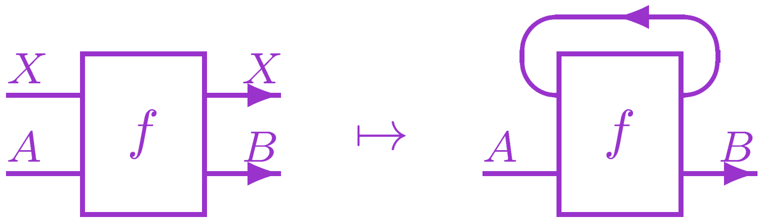
Unfortunately, Peter and I do not agree when drawing braids and twists;

Peter's  is my , and his  is my .

Traced Monoidal Categories (Joyal, Street and Verity, 1996)

A *traced monoidal category* is a balanced monoidal category $\mathcal{C}$ equipped with a family of functions, called *trace* operator

$$Tr^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \longrightarrow \mathcal{C}(A, B)$$



subject to a few coherence axioms (slightly simpler than the original):

Tightening: $Tr^X_{A',B'}((k \otimes id_X) \circ f \circ (h \otimes id_X)) = k \circ Tr^X_{A,B}(f) \circ h$

Yanking: $Tr^X_{X,X}(c_{X,X}) \circ \theta^{-1}_X = id_X = Tr^X_{X,X}(c^{-1}_{X,X}) \circ \theta_X$

Superposing: $Tr^X_{C\otimes A,C\otimes B}(id_C \otimes f) = id_C \otimes Tr^X_{A,B}(f)$

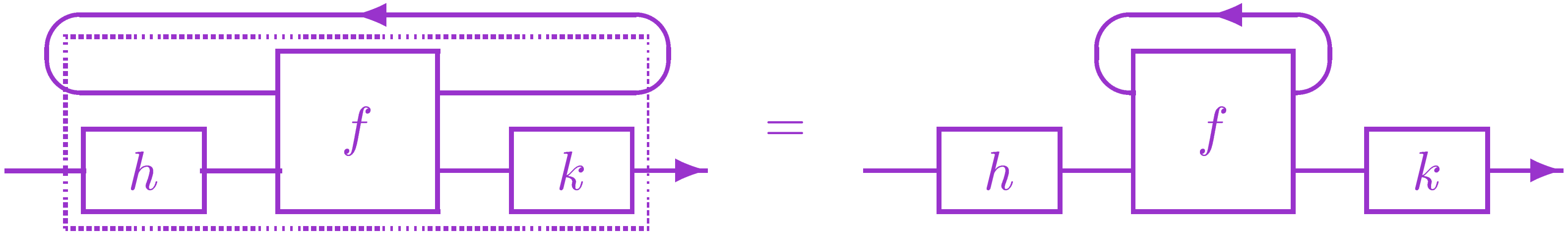
Exchange: $Tr^X_{A,B}(Tr^Y_{A\otimes X,B\otimes X}(f)) =$

$$Tr^Y_{A,B}(Tr^X_{A\otimes Y,B\otimes Y}((id_B \otimes c_{Y,X}) \circ f \circ (id_A \otimes c^{-1}_{Y,X})))$$

Axioms for Trace (1/2)

Tightening (Naturality)

$$Tr^X_{A',B'}((k \otimes id_X) \circ f \circ (h \otimes id_X)) = k \circ Tr^X_{A,B}(f) \circ h$$



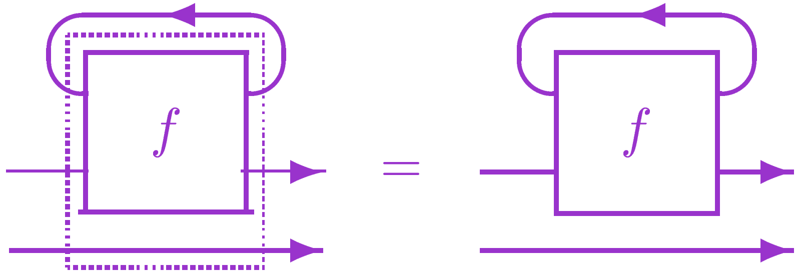
Yanking

$$Tr^X_{X,X}(c_{X,X}) \circ \theta_X^{-1} = id_X = Tr^X_{X,X}(c^{-1}_{X,X}) \circ \theta_X$$

Axioms for Trace (2/2)

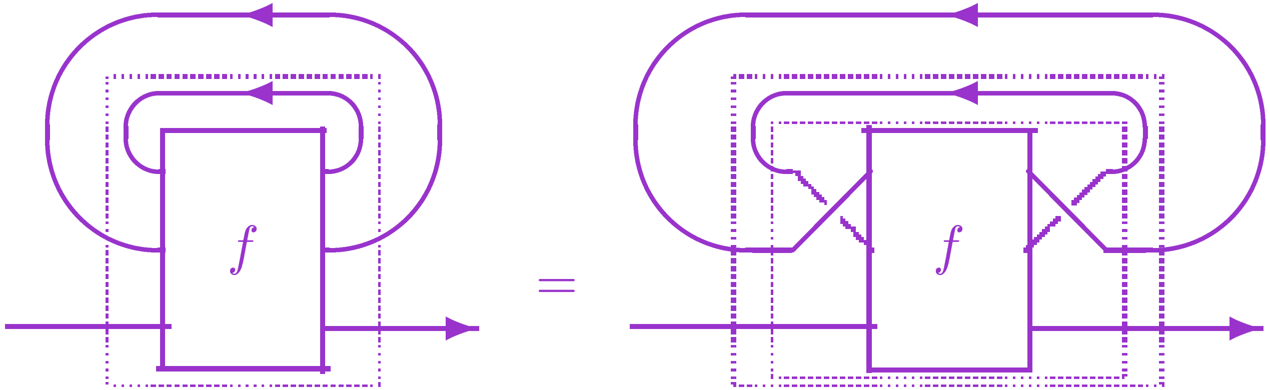
Superposing

$Tr^X_{C\otimes A, C\otimes B}(id_C \otimes f) = id_C \otimes Tr^X_{A,B}(f)$



Exchange

$Tr^X_{A,B}(Tr^Y_{A\otimes X, B\otimes X}(f)) =$
$Tr^Y_{A,B}(Tr^X_{A\otimes Y, B\otimes Y}((id_B \otimes c_{Y,X}) \circ f \circ (id_A \otimes c^{-1}_{Y,X})))$



On the Choice of Axioms (for those familiar with the original axiomatization)

If you are familiar with the original axiomatization by Joyal, Street and Verity, you should find no difficulty in seeing that the new Exchange axiom is derivable from the original axioms.

Conversely, original axioms are derivable from our axioms; we shall demonstrate a slightly non-trivial derivation of Sliding in the following slides. Vanising for tensor

$$Tr^{X \otimes Y}_{A,B}(f) = Tr^{X}_{A,B}(Tr^{Y}_{A \otimes X, B \otimes X}(f))$$

can be derived in the similar way, while Vanishing for unit

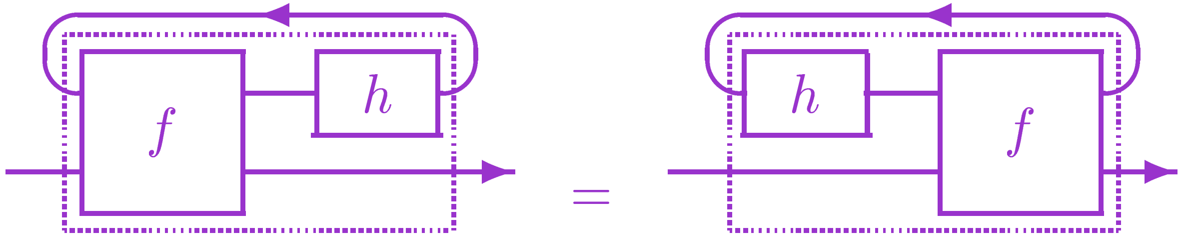
$$Tr^{I}_{A,B}(f) = f$$

is in fact redundant in the original axiomatization (cf. my MSCS paper).

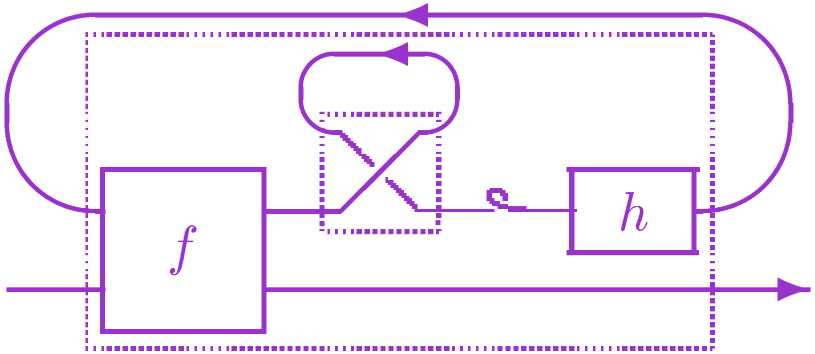
So our axioms are equivalent to the original axioms.

Exercise: Sliding (Dinaturality)

Proposition. The following equation is derivable.



Proof: By *Yanking*, LHS is equal to



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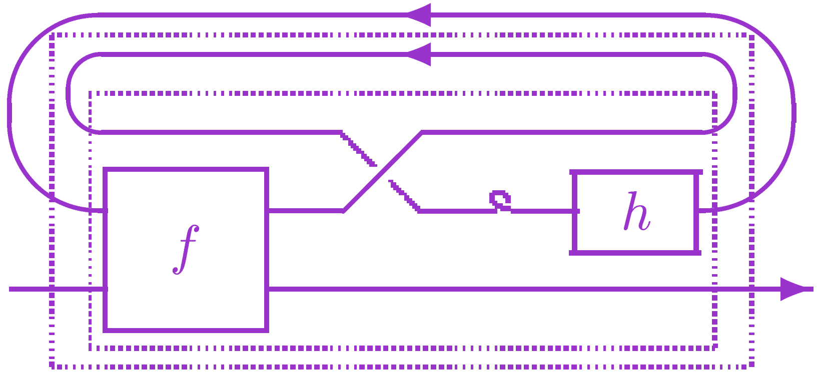
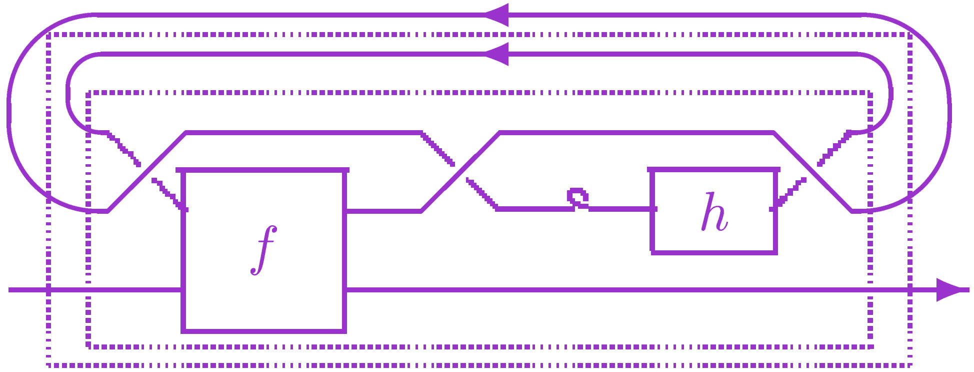
Using *Superposing* and *Tightening*, we have



Then we apply *Exchange*:



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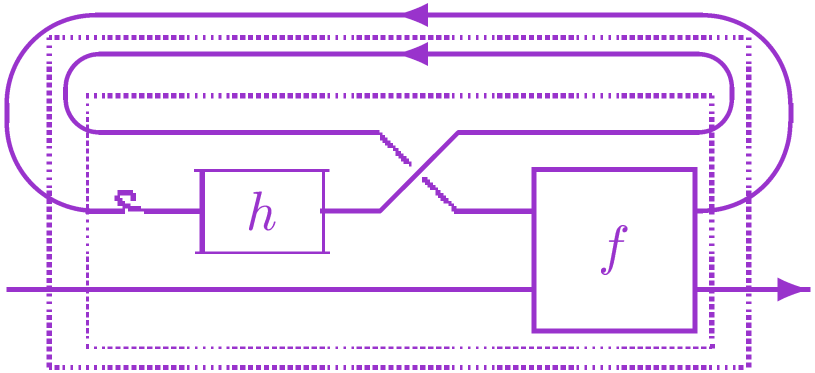
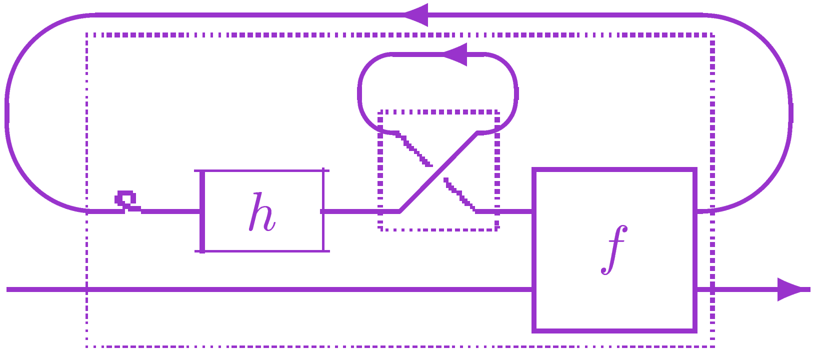
Thanks to the naturality of braidings, this is equal to the following.



By applying *Tightening* and *Superposing*, we obtain



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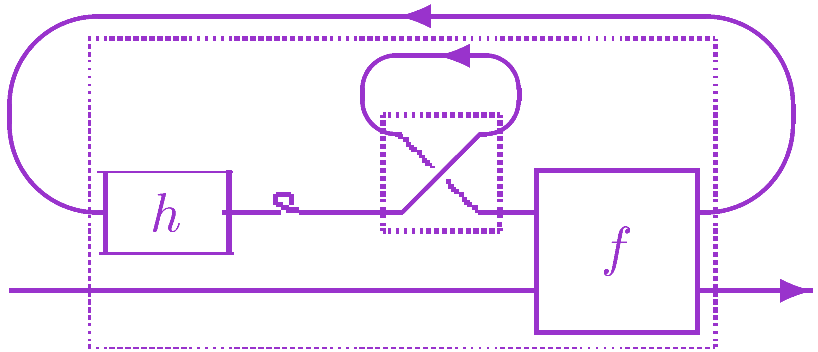
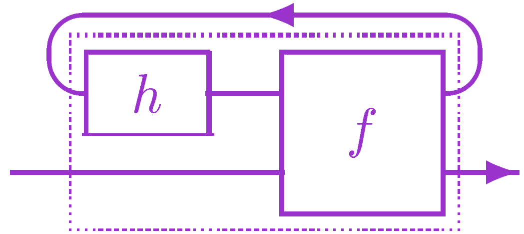
By the naturality of twisting



Now we have done, by *Yanking*.



(QED)

Examples of Traced Monoidal Categories (1)

Linear Algebra (the classical example)

The category $\mathbf{Vect}_K^{\mathbf{fin}}$ of fin. dim. vector spaces and linear maps over a field K. For a linear map $f : U \otimes_K W \to V \otimes_K W$, its trace $Tr^W_{U,V}(f) : U \to V$ is given by

$$(Tr^W_{U,V}(f))_{i,j} = \Sigma_k f_{i\otimes k, j\otimes k}$$

Quantum Invariants of Knots

The category of representations of a quasi-triangular Hopf algebra, which gives rise to knot (ribbon, tangle) invariants like Jones polynomials.

(These examples are not just traced but also enjoy certain self-duality. This issue will be addressed later.)

Examples of Traced Monoidal Categories (2)

Binary Relations (or Non-deterministic functions)

The category $\mathbf{Rel}$ of sets and binary relations have two traced monoidal structures, for each of $\times$ and $+$.

Multiplicative trace: For a binary relation $R : A \times X \to B \times X$, its trace $Tr^X_{A,B}(R) : A \to B$ is given by

$$a\, Tr^X_{A,B}(R)\; b \quad \Leftrightarrow \quad (a,x)\; R\; (b,x) \;\; \text{for some} \;\; x \in X$$

Additive trace: For $R : A + X \to B + X$, we have $Tr'^X_{A,B}(R) : A \to B$ by

$$a\, Tr'^X_{A,B}(R)\; b \quad \Leftrightarrow \quad a\, R\, x_1\, R\, x_2\, R\, \ldots\, R\, x_n\, R\, b \; \text{for some} \; x_1, \ldots, x_n \in X$$

Examples of Traced Monoidal Categories (3)

The category $\mathbf{Cpo}$ of pointed ω-complete partially ordered sets and continuous functions, with tensor given by the cartesian products.

For $f = \langle f_1, f_2 \rangle : A \times X \longrightarrow B \times X$ with
$f_1 : A \times X \longrightarrow B$ and $f_2 : A \times X \longrightarrow X$,
its trace $Tr^X_{A,B}(f) : A \longrightarrow B$ is given by

$$\begin{aligned} Tr^X_{A,B}(f)(a) &= f_1(a, \bigsqcup_i (\lambda x. f_2(a,x))^i(\bot)) \\ &= f_1(a, \mu x. f_2(a,x)) \end{aligned}$$

In fact, any category with finite products and a "well-behaved" fixed-point operator is traced; examples include almost all categories of domains used in domain theory. We shall look at this issue in the next pert of this talk.

Part I: Traced Monoidal Categories

Part II: Geometry of Recursion

Part III: The Int-construction

(Part IV: On Closedness)

Geometry of Recursive Programs (Fixed-point Operators)

Some laws on recursive programs (via recursive let-bindings)

mutual recursion (dinaturality):

letrec x=g(f(x)) in x = letrec y=f(g(y)) in g(y)

diagonal property:

letrec x={letrec y=h(x,y) in y} in x = letrec z = h(z,z) in z

simultaneous recursion (Bekič property):

letrec x=f(x,y), y=g(x,y) in x = letrec x=f(x,{letrec y=g(x,y) in y}) in x

(dinaturality + diagonal $\Longrightarrow$ Bekič)

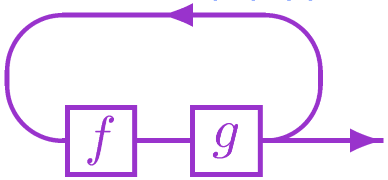
A Conway fixed-point operator is a fixed-point operator satisfying these equations. In particular, the least fixed-point operators used in domain theory are Conway operators.

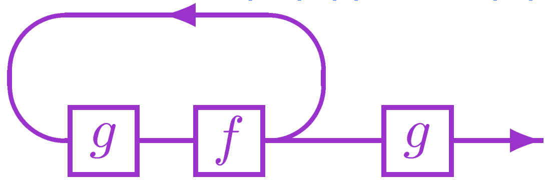
(More generally, canonically derived fixed-point operators in models of axiomatic domain theory are Conway; cf. Simpson and Plotkin, LICS2000)

These laws allow graphical interpretations:

Dinaturality:

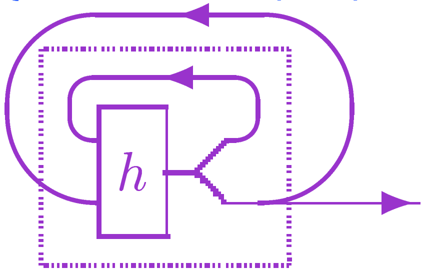
letrec x=g(f(x)) in x = letrec y=f(g(y)) in g(y)

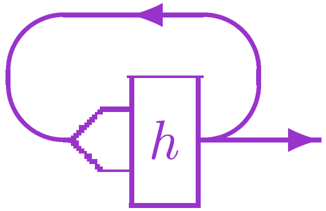




Diagonal property:

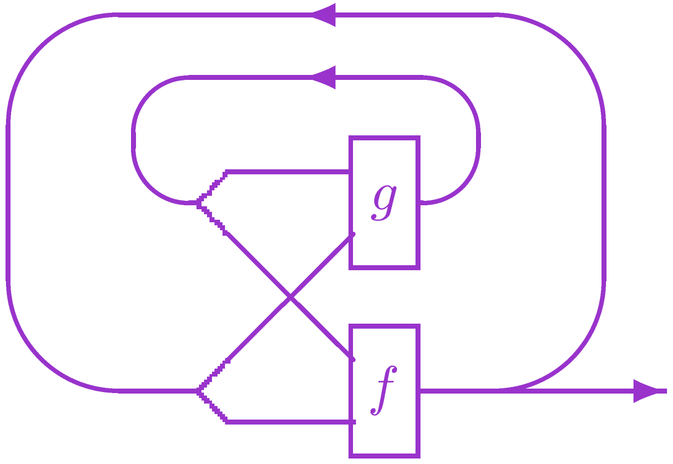
letrec x={letrec y=h(x,y) in y} in x = letrec z = h(z,z) in z

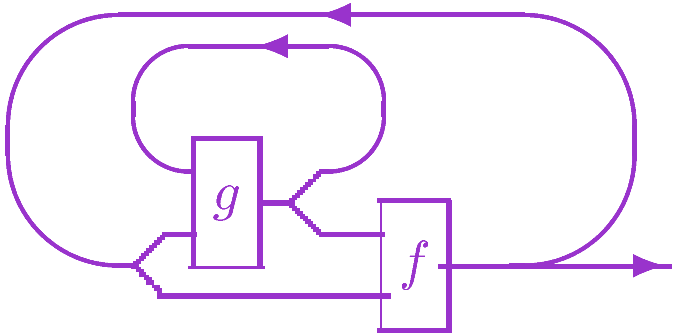




Bekič property:

letrec x=f(x,y), y=g(x,y) in x = letrec x=f(x,{letrec y=g(x,y) in y}) in x





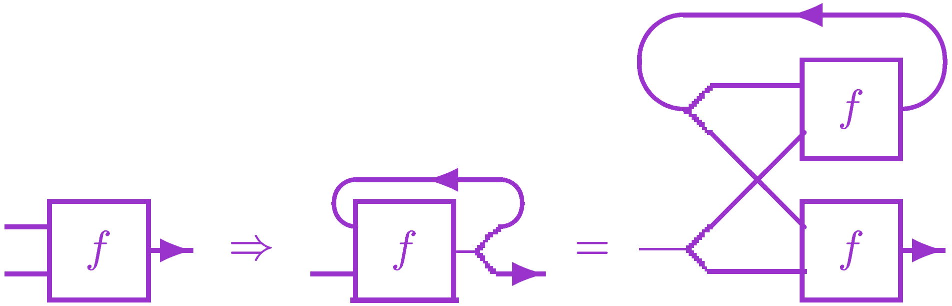
These are very similar to the pictures for traced monoidal categories

Recursion via Traces (1) The Trace-Fixpoint Correspondence

Theorem. (Hyland / Hasegawa 1997) For a category with finite products, to give a trace is to give a Conway fixed-point operator.

$$\frac{f : A \times X \to X}{f^\dagger = Tr^X_{A,X}(\Delta_X \circ f) : A \to X}$$

$$\frac{g : A \times X \to B \times X}{Tr^X_{A,B}(g) = \pi_{B,X} \circ (g \circ (id_A \times \pi'_{B,X}))^\dagger : A \to B}$$



Recursion via Traces (2) Recursion from Cyclic Sharing

In a setting where only values can be duplicated or discarded (while non-values are shared), the tensor is not cartesian. For such a situation, still we have:

Theorem. (Hasegawa 1997) Given a monoidal adjunction between a category with finite products and a traced symmetric monoidal category, there exists a dinatural fixed-point operator on the traced monoidal category.

Monoidal $\{$ Functors, Natural Transformations, Adjunctions,... $\}$

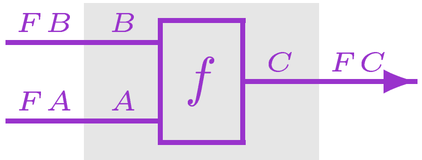
Here are some explanations on the technical terms in the previous slide.
Suppose that $\mathcal{C}$ and $\mathcal{D}$ below are all monoidal categories.

- A monoidal functor is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation $m_{A,B} : FA \otimes FB \to F(A \otimes B)$ and a morphism $m_I : I \to FI$ subject to a few coherence conditions. It is called strong monoidal if $m_{A,B}$ and m_I are all isomorphisms.
- Given monoidal functors $(F, m^F_{A,B}, m^F_I)$ and $(G, m^G_{A,B}, m^G_I)$ from $\mathcal{C}$ to $\mathcal{D}$, a monoidal natural tansformation from $(F, m^F_{A,B}, m^F_I)$ to $(G, m^G_{A,B}, m^G_I)$ is a natural transformation from F to G which is compatible with m^F's and m^G's.
- A monoidal adjunction is an adjunction whose functors are monoidal and the unit and counit are monoidal natural transformations.

Geometry of Monoidal { Functors, Natural Transformations }

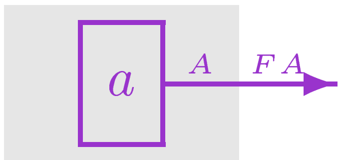
Monoidal functors allow concise graphical presentations, called functorial boxes (Cockett and Seely 1999; Melliès 2006).

Consider a monoidal functor $(F, m_{A,B}, m_I) : \mathcal{C} \to \mathcal{D}$. Given $f : A \otimes B \to C$ in $\mathcal{C}$, we may draw a picture with "box"



which represents $FA \otimes FB \stackrel{m_{A,B}}{\to} F(A \otimes B) \stackrel{Ff}{\to} FC$.

Similarly, given $a : I \to A$, the picture



represents $I \stackrel{m_I}{\to} FI \stackrel{Fa}{\to} FA$. The coherence conditions ensure that this notation works well for general $f : A_1 \otimes \ldots \otimes A_n \to B$.

Monoidal natural transformations also allow nice presentations using this box notation. For instance, one of the conditions for monoidal natural transformations can be shown as follows:

where φ is a monoidal natural transformation from F to G.

Recursion from Cyclic Sharing ≈ Linear Fixed-Points

Spelling out the claim of the last theorem:

Let $\mathcal{C}$ be a cat with finite products, $\mathcal{D}$ a traced symmetric monoidal cat, with a monoidal left adjoint functor $F : \mathcal{C} \to \mathcal{D}$.
Given a morphism $f : FA \otimes X \longrightarrow X$ in $\mathcal{D}$, there exists $f^{\dagger} : FA \longrightarrow X$ such that the fixed-point equation

$$f^{\dagger} = f \circ (1_{FA} \otimes f^{\dagger}) \circ m^{F^{-1}} \circ F\Delta_A$$

holds (where $m^F : FA \otimes FA \stackrel{\sim}{\to} F(A \times A)$ and $\Delta : A \to A \times A$).

In terms of linear logic: there is a *linear* fixed-point operator $\mathsf{Y} : !(X \multimap X) \multimap X$ such that

$$\mathsf{Y}\,(!f) \;=\; f\,(\mathsf{Y}\,(!f)) \qquad (f : X \multimap X)$$

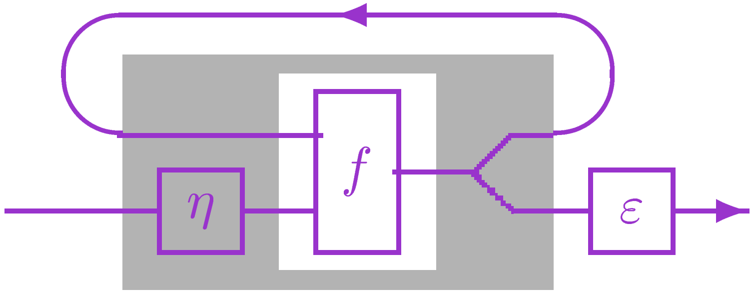
(Note: not $!(!X \multimap X) \multimap X$)

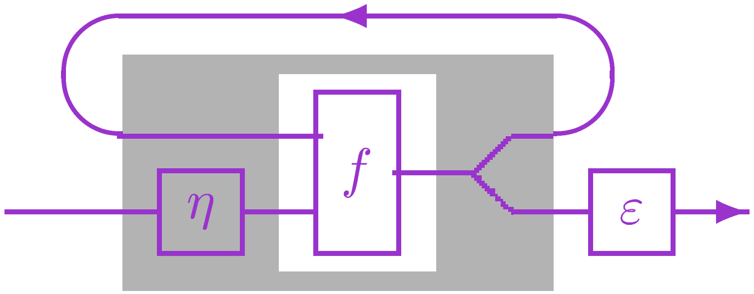
Geometry of Linear Fixed-Points

Explicitly, this $f^\dagger$ is given by

$$f^\dagger = \varepsilon \circ Tr^{FUX}(m^{F^{-1}} \circ F(\Delta \circ Uf \circ m^U \circ (\eta \times id)) \circ m^F)$$

Using functorial boxes, it can be expressed as follows.





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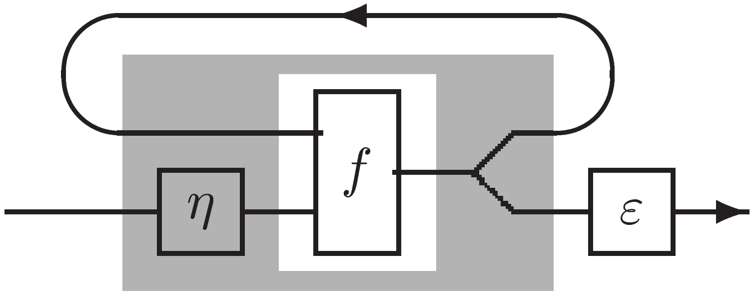

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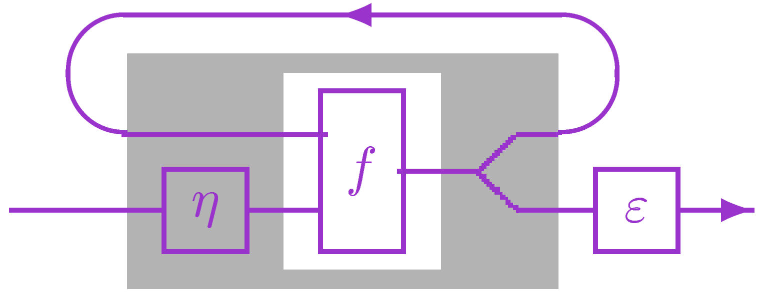

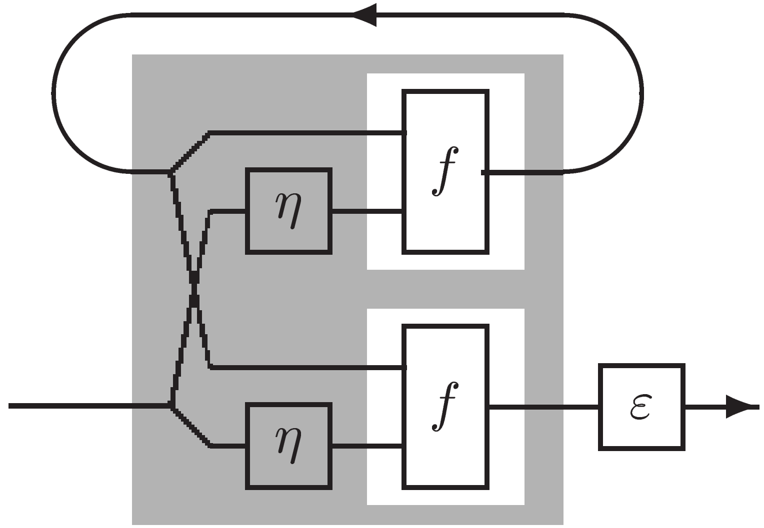

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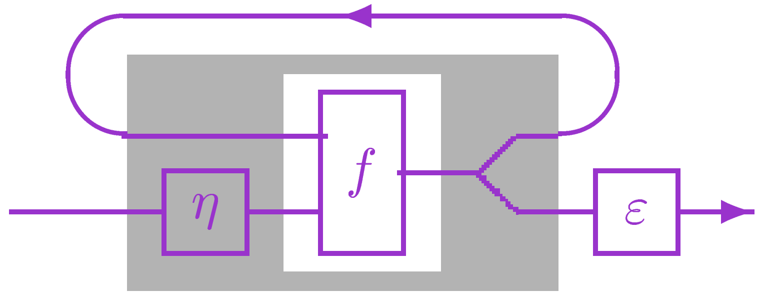

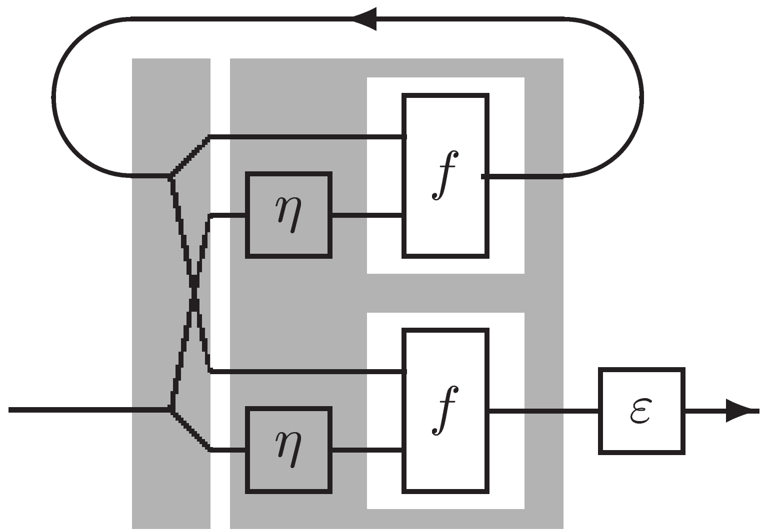

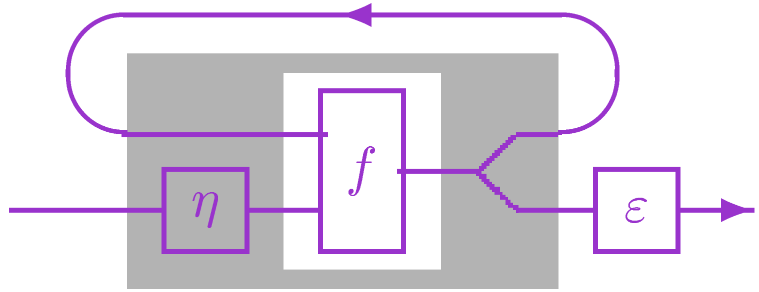
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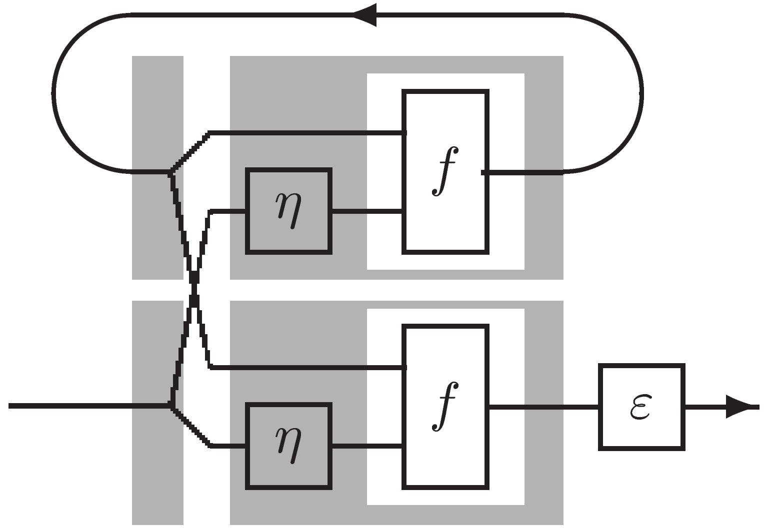
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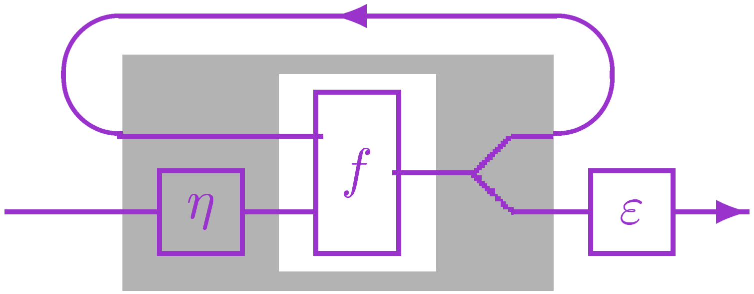

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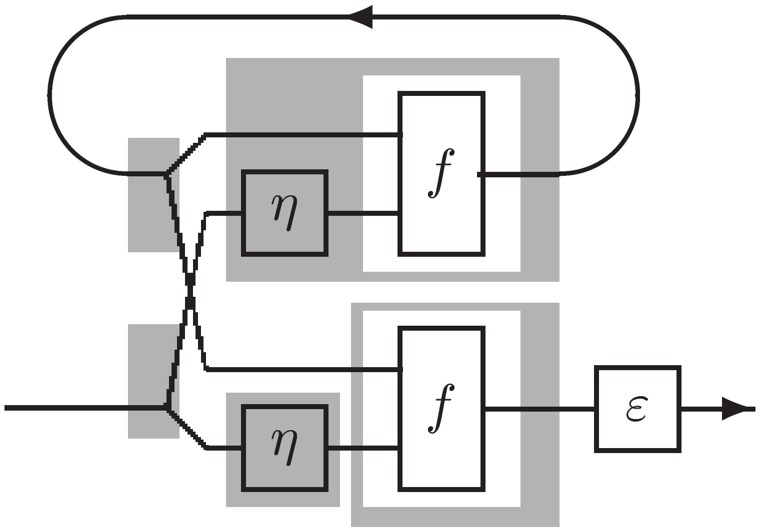

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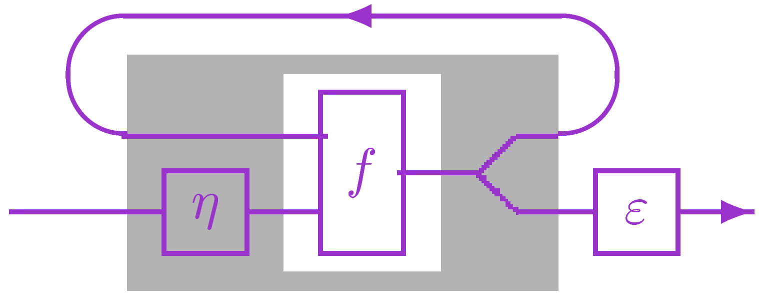

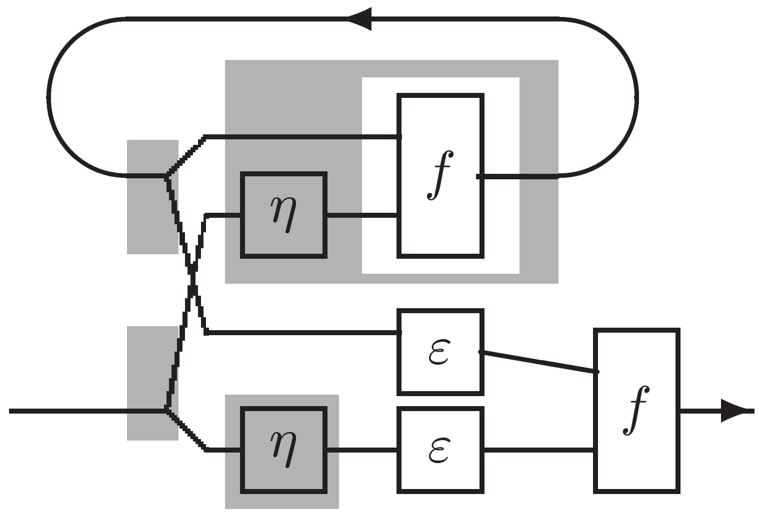

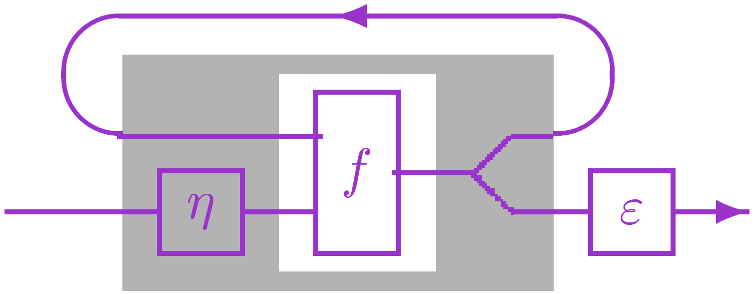
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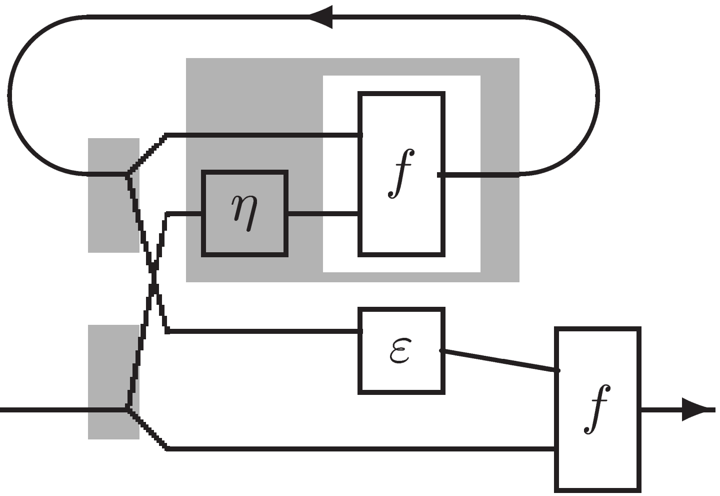
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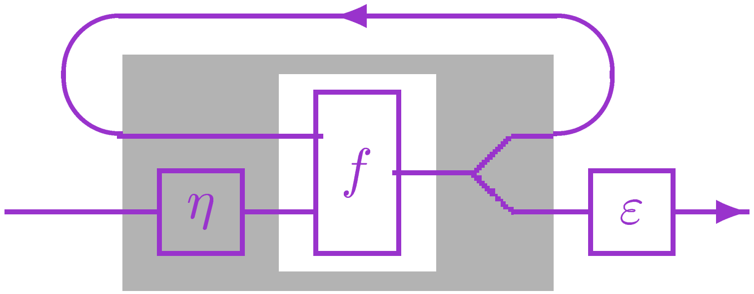

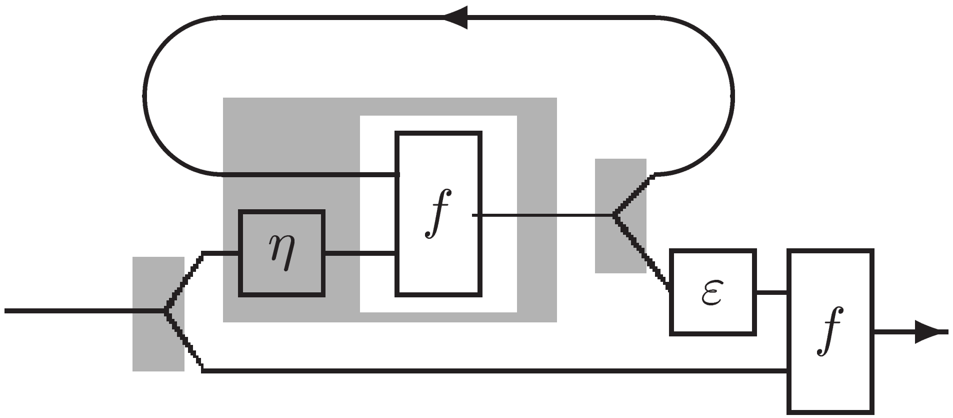

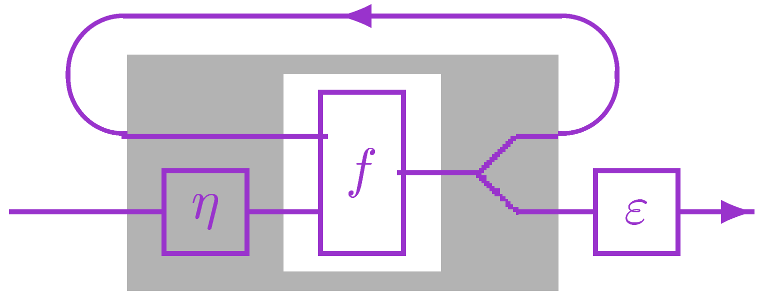

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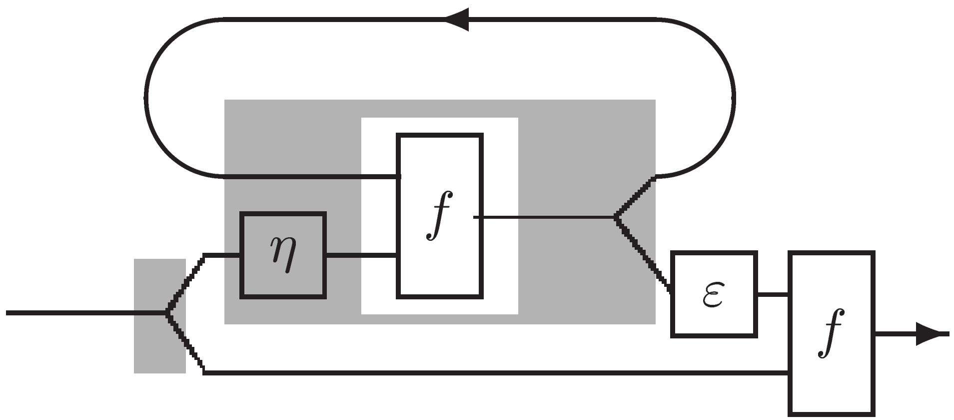

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Geometry of Recursive Programs in Traced Monoidal Categories

Geometry of recursive programs, built upon traced monoidal categories and their geometric interpretation, captures not only the traditional fixed-point semantics but also a more general form of recursion created from cyclic sharing.

Indeed, geometry of recursive programs prompted various studies on recursive computation with shared resources, and with non-trivial side-effects ("monadic effects" in functional programming languages) for the last decade, e.g. traced premonoidal categories of Benton and Hyland.

(The final story, however, is still yet to be told . . .)

Part I: Traced Monoidal Categories

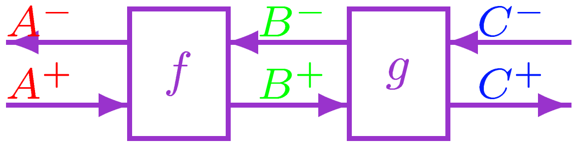
Part II: Geometry of Recursion

Part III: The Int-construction

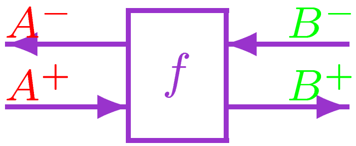
(Part IV: On Closedness)

Modelling Bi-directional Communication

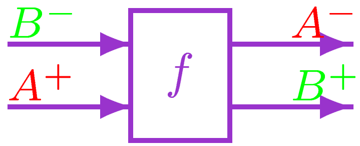
Want to model interactive (bi-directional) communication:

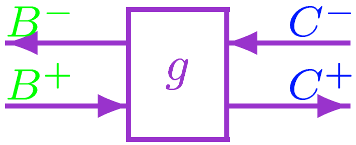


In models of one-way "directed" computation:

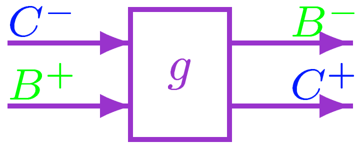


can be implemented as





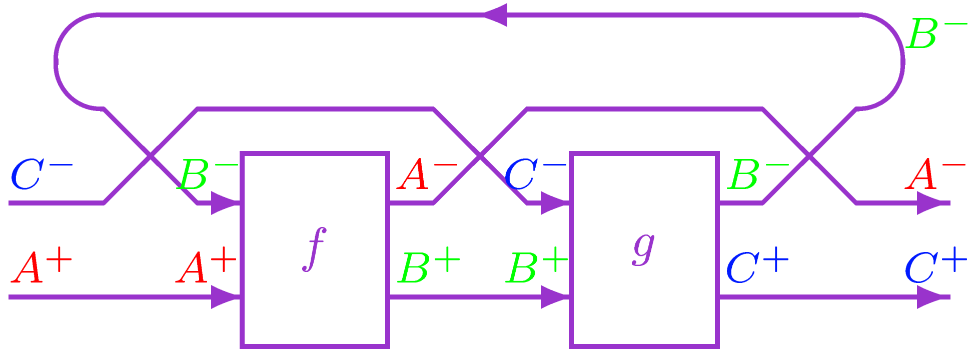
can be implemented as



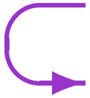
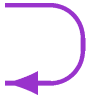
But how can we compose them?

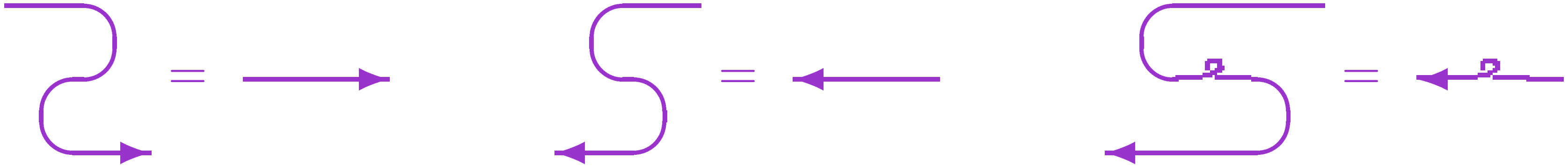
Geometry of Interaction (GoI) (Girard/Abramsky)

Solution: use a feedback, or trace!



Tortile Monoidal Categories

A *tortile monoidal category*[a] is a balanced monoidal category with an A^* for each object A, unit $I \to A \otimes A^*$ and counit $A^* \otimes A \to I$ drawn as  and  respectively, such that



$A \mapsto A^*$ extends to a contravariant equivalence which is an involution: $A^{**} \simeq A$. The functor $(-) \otimes A$ is left (and right) adjoint to $(-) \otimes A^*$.

(Note: tortile symmetric monoidal categories = compact closed categories)

[a] other names: *ribbon category*, *braided compact closed category*, ...

Examples of Tortile Monoidal Categories

- $\mathbf{Vect}_K^{\mathbf{fin}}$ with V^* the dual space.
- $\mathbf{Rel}$ with $X^* = X$.
- Representations of quantum groups.

Tortile Monoidal Categories in Knot Theory

The following result is important for applications in knot theory:

Theorem (M.-C. Shum). The tortile category freely generated by a single object is equivalent to the category of framed tangles.

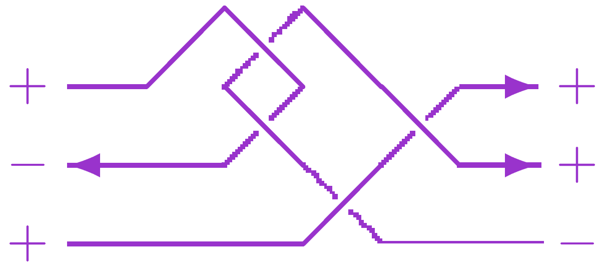
Therefore, tortile categories give rise to invariants for tangles, for example the quantum invariants.

Cf. The free cartesian closed category is equivalent to (the term model of) the typed λ-calculus. Hence cartesian closed categories give rise to models of the typed λ-calculus. *"Abstract is concrete" (Hyland)*

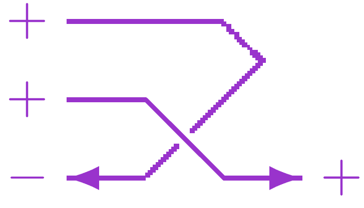
The Category of Tangles

Objects are finite lists of $+$ and $-$.

Morphisms are tangles with suitable source and target.



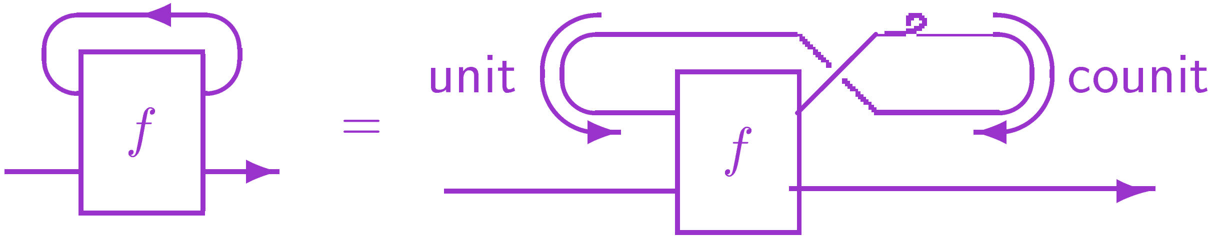
$(+,-,+) \longrightarrow (-,+,+)$



$(-,+,+) \longrightarrow (+)$

Traced Categories vs Tortile Categories

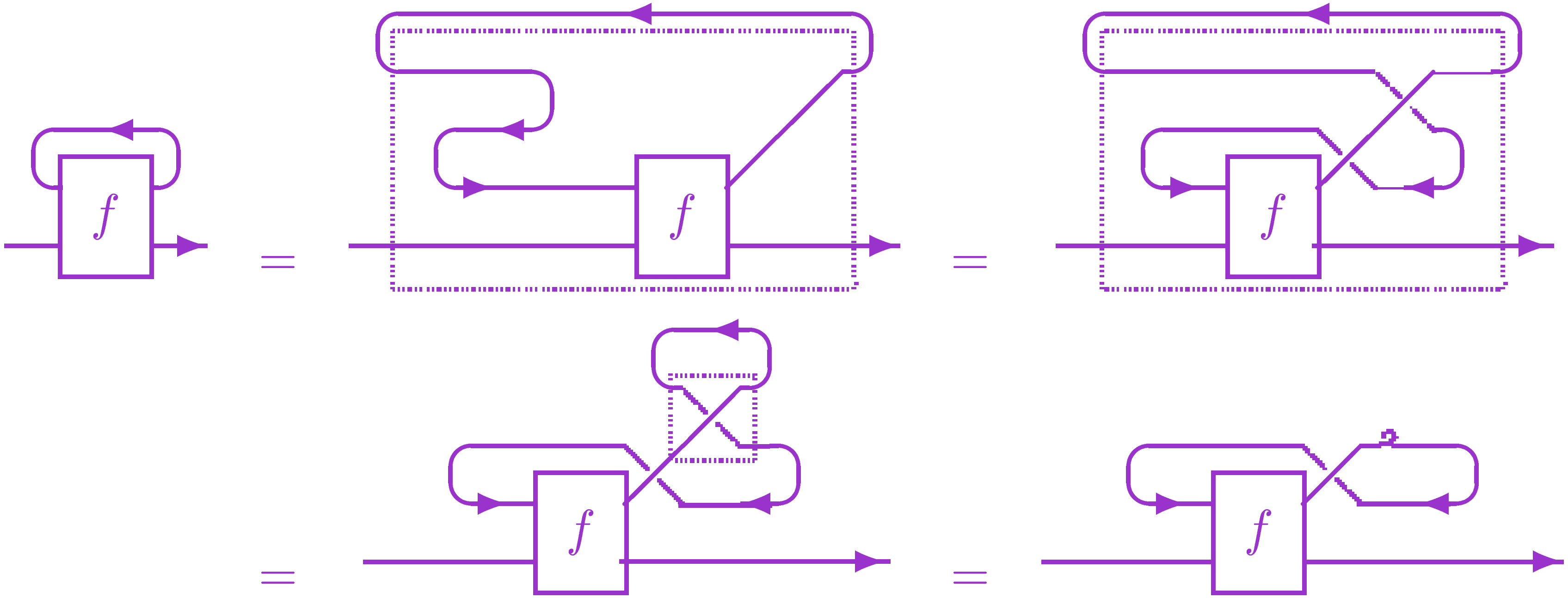
Any tortile category has a unique trace (called *canonical trace*), hence is also a traced monoidal category.



It follows that a monoidal full subcategory of a tortile category is traced.

In fact, every traced monoidal category arises in this way: given a traced monoidal category $\mathcal{C}$, we can construct a tortile category $\mathbf{Int}\,\mathcal{C}$ to which $\mathcal{C}$ fully faithfully embeds, via the Int-construction of Joyal, Street and Verity — an abstract version of GoI.

(Digression) Uniqueness of Trace on Tortile Monoidal Categories



The Int Construction (Joyal, Street and Verity)

Given a traced monoidal cat $\mathcal{C}$, we construct a category $\mathbf{Int}\,\mathcal{C}$ as follows.

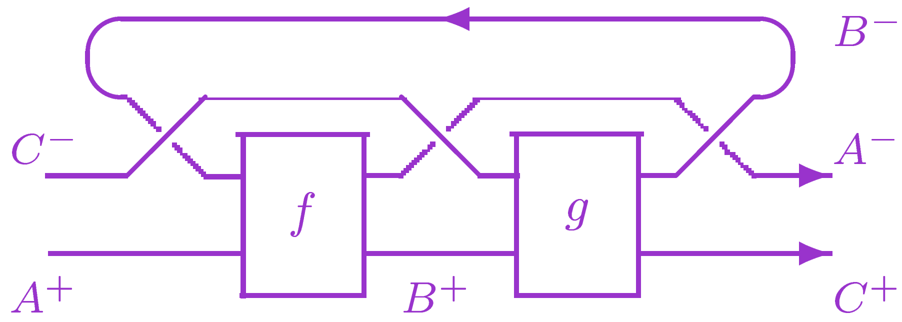
(cf. definition of integers as $\mathbf{N} \times \mathbf{N}/\sim$ where $(x, x') \sim (y, y')$ iff $x + y' = y + x'$)

Objects: pairs of objects of $\mathcal{C}$

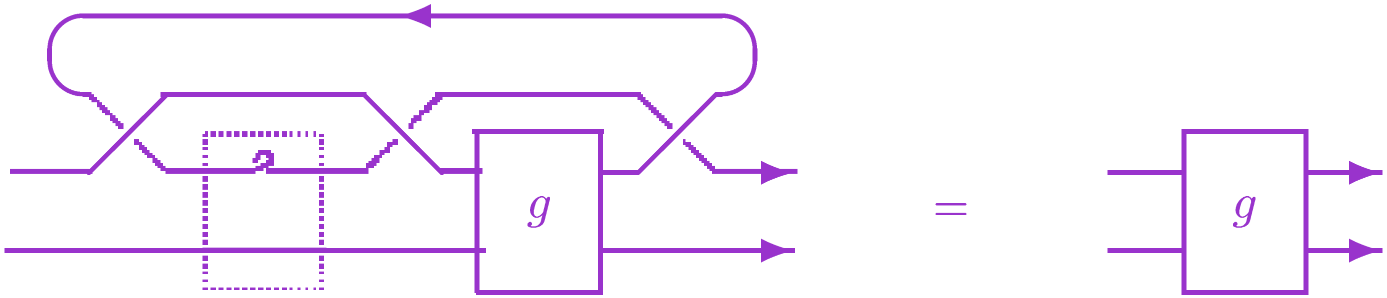
Arrows: $\mathbf{Int}\,\mathcal{C}((A^+, A^-), (B^+, B^-)) = \mathcal{C}(A^+ \otimes B^-, B^+ \otimes A^-)$

The identity on (A^+, A^-) is $id_{A^+} \otimes \theta^{-1}_{A^-} \in \mathcal{C}(A^+ \otimes A^-, A^+ \otimes A^-)$.

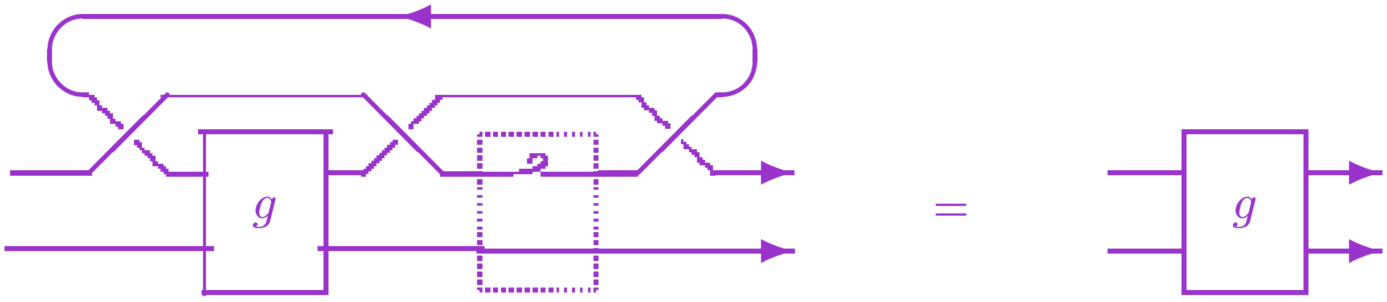
The composition of
$f \in \mathbf{Int}\,\mathcal{C}((A^+, A^-), (B^+, B^-)) = \mathcal{C}(A^+ \otimes B^-, B^+ \otimes A^-)$ and
$g \in \mathbf{Int}\,\mathcal{C}((B^+, B^-), (C^+, C^-)) = \mathcal{C}(B^+ \otimes C^-, C^+ \otimes B^-)$ is given by



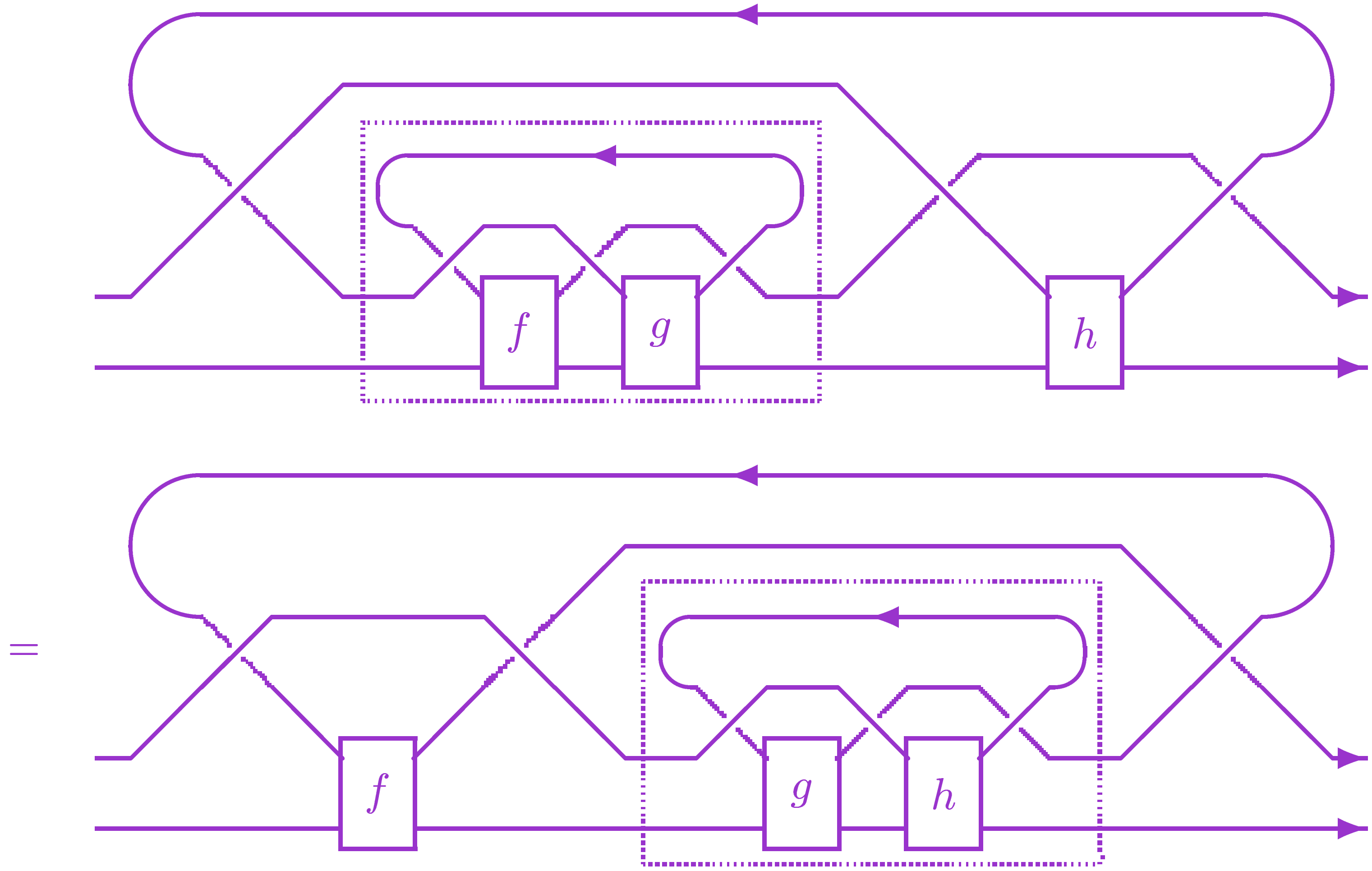
$g \circ id = g$:



$id \circ g = g$:

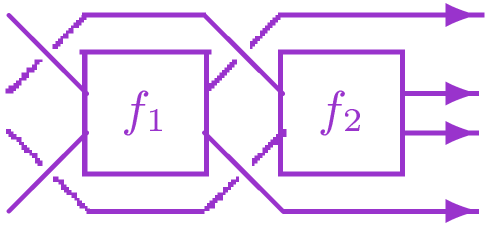


$h \circ (g \circ f) = (h \circ g) \circ f$:

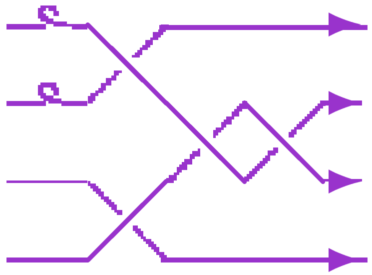


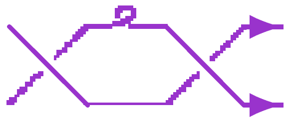
The Int Construction (cont.)

Monoidal structure: we define tensor and unit by $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, B^- \otimes A^-)$ and $I = (I, I)$.
For $f_1 : (A_1^+, A_1^-) \to (B_1^+, B_1^-)$ and $f_2 : (A_2^+, A_2^-) \to (B_2^+, B_2^-)$,

$f_1 \otimes f_2 \quad =$ 

Braids and twists (not quite obvious for non-symmetric case):

$c =$ 

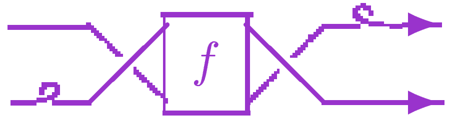
$\theta =$ 

(c^{-1} and θ^{-1} are far more complicated; can you guess?)

The Int Construction (cont.)

Duality: $(A^+, A^-)^* = (A^-, A^+)$

For $f : (A^+, A^-) \to (B^+, B^-)$ define $f^* : (B^+, B^-)^* \to (A^+, A^-)^*$ as



The unit $I \to (A^+, A^-) \otimes (A^+, A^-)^*$ is given by $id_{A^+} \otimes \theta^{-1}_{A^-}$.

The counit $(A^+, A^-)^* \otimes (A^+, A^-) \to I$ is $id_{A^-} \otimes \theta_{A^+}$.

Theorem (Joyal, Street and Verity).

These data determine a tortile monoidal structure on $\mathbf{Int}\,\mathcal{C}$.

Moreover, the functor $\mathcal{N} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ sending A to (A, I) strongly preserves the traced monoidal structure, and is full faithful.

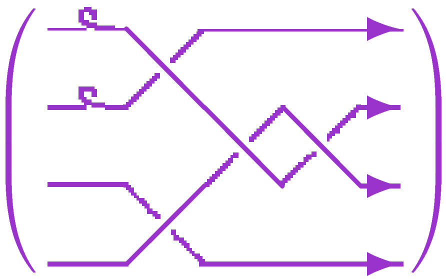
Corollary. Traced monoidal categories are monoidal full subcategories of tortile monoidal categories.

Int Construction as a Universal Construction

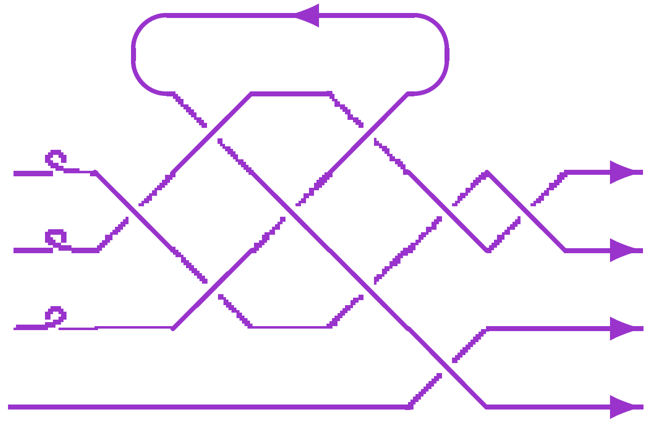
In fact, Int-construction is *universal*, as shown by JSV: it gives a left biadjoint to the forgetful 2-functor from the 2-category of tortile categories to that of traced monoidal categories.

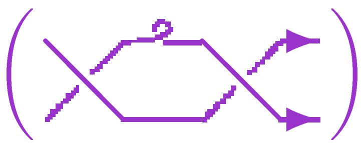
As a corollary, if $\mathcal{C}$ is a free traced monoidal category, $\mathbf{Int}\,\mathcal{C}$ is equivalent to a free tortile monoidal category, hence is equivalent to the category of tangles. Thus traced monoidal categories are formally related to knot theory.

(The choice of 2-cells of the 2-category of traced monoidal categories in JSV paper was wrong, and their proof was not complete. This error is pointed out and corrected in a recent paper by Hasegawa and Katsumata — it took me 13 years to spot it!)

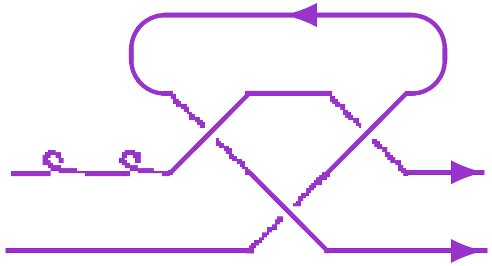


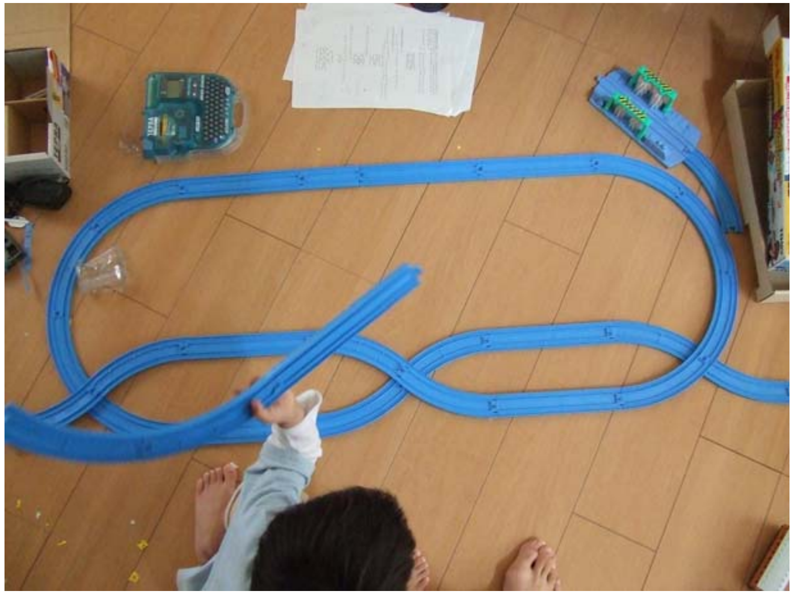
$^{-1} \quad =$





$^{-1} \quad =$





Trying to realize Int-construction

Part I: Traced Monoidal Categories

Part II: Geometry of Recursion

Part III: The Int-construction

(Part IV: On Closedness)

Closed Structure, or Higher Types

Recall that a monoidal category $\mathcal{C}$ is *closed* if $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A \multimap -$:

$$\mathcal{C}(X \otimes A, Y) \simeq \mathcal{C}(X, A \multimap Y)$$

In particular, tortile categories are closed, with $A \multimap B = B \otimes A^*$.

In the context of linear logic, being symmetric monoidal closed means that we can interpret the intuitionistic multiplicative fragment (tensor $\otimes$, unit $\mathbf{1}$, and linear implication $\multimap$) in $\mathcal{C}$.

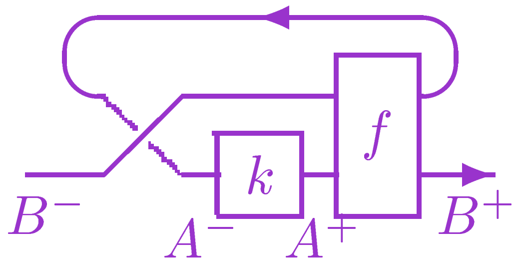
Here is a very simple observation on closedness — it took me 10 years to notice, however.

Theorem.
Let $\mathcal{C}$ be a traced monoidal category, and $\mathcal{N} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ be the canonical inclusion from $\mathcal{C}$ into $\mathbf{Int}\,\mathcal{C}$ (i.e. $\mathcal{N}(A) = (A, I)$).
Then $\mathcal{C}$ is closed if and only if $\mathcal{N}$ has a right adjoint.

Closed Structure, or Higher Types (cont.)

Proof Outline: If $\mathcal{N}$ has a right adjoint $\mathcal{U}$, let $A \multimap B$ be $\mathcal{U}(B, A)$.

Conversely, if $\mathcal{C}$ is closed, we define $\mathcal{U}(A^+, A^-) = A^- \multimap A^+$.
For $f : (A^+, A^-) \to (B^+, B^-)$, let $\mathcal{U}(f) : (A^- \multimap A^+) \to (B^- \multimap B^+)$
send $k : A^- \to A^+$ to $Tr^{A^-}_{B^-,B^+}(f \circ (k \otimes B^-) \circ c_{B^-,A^-}) : B^- \to B^+$.



(QED)

There are a number of implications of this result on traced models of LL and (linear) fixed-points. In particular, any CCC with a Conway operator arises from a traced model of IMELL, as discussed in my MSCS paper.

Examples of Traced Monoidal Closed Categories

- Tortile monoidal categories (compact closed categories)
  $\{\text{tortile cats}\} \subset \{\text{traced closed cats}\} \subset \{\text{traced cats}\}$
- Traced cartesian closed categories (e.g. **Cpo**)
- (Negative) Conway Games: the category $\mathcal{Y}^-$ of negative Conway games (Melliès, 2004) is a symmetric monoidal full subcategory of the compact closed category $\mathcal{Y}$ of Conway games (Joyal 1977). The inclusion from $\mathcal{Y}^-$ to $\mathcal{Y}$ has a right adjoint, and $\mathcal{Y}^-$ is a *traced symmetric monoidal closed category*.

$\mathcal{Y}^-$ is one of very few interesting traced symmetric monoidal closed categories which are nether cartesian closed nor compact closed.

Application: Program Transformations

Katsumata and Nishimura (2006) introduced a program transformation technique called *(semantic) higher-order removal*.

Roughly, their technique transforms a higher-order functional

$$g : (A^- \Rightarrow A^+) \Rightarrow (B^- \Rightarrow B^+)$$

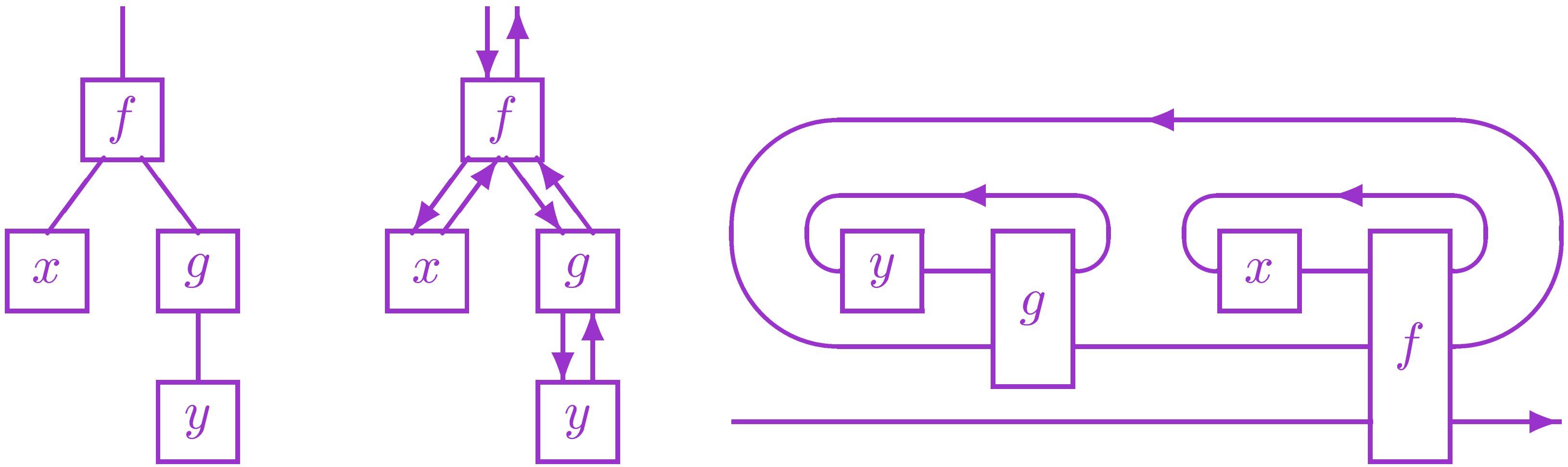
(created in the process of dealing with bi-directional information flow) to a first-order function $f : A^+ \times B^- \Rightarrow B^+ \times A^-$ such that $\mathcal{U}(f) = g$, where $\mathcal{U}$ is right adjoint to $\mathcal{N}$.

They give a syntactic condition which ensures that g is in the image of $\mathcal{U}$ in their semantic models, and presented a procedure for identifying f such that $\mathcal{U}(f) = g$.

Application: Attribute Grammars

Katsumata (2008) has shown that a substantial part of the theory of attribute grammars (Knuth 1968) can be carried out very cleanly in terms of traced monoidal categories and Int-construction. Roughly, it works as the pictures below (for the term $f(x, g(y))$):

term tree attribute grammar interpretation via trace



In Katsumata's work, the adjuntion $\mathcal{N} \dashv \mathcal{U}$ provides the equivalence between attribute grammars and synthesized attribute grammars.

Some Thoughts

Traditionally, research on GoI has focused on bi-directional information flow upon exchanges of rather primitive atomic data, or tokens — witnessed by GoI-based token machines (or context semantics) a la Gonthier-Abadi-Levy. In this reading, GoI decomposes higher-order computation into local interactions of atomic data.

However, our observation on traced monoidal closed categories suggests that a serious look at exchanges of higher-order data does add a new dimension to the world of GoI. Our rather simple theorem provides surprisingly many new models of linear logic and fixed-point computation.
The work by Katsumata reminds us that such higher-order information flow are ubiquitous, and suggests that our theorem is just an abstract account for many known (and yet to be known) situations.

So, why not *Geometry of Higher-Order Interaction*?

Conclusion

Geometry of Recursive Computation, Geometry of Interaction

We looked at some elements of traced monoidal categories,
and observed that they are intimately related to recursive computation
and interactive computation.

I think that traced monoidal *closed* categories will provide some further insights and applications, and I expect that *Geometry of Higher-Order Interaction* can be developped along this line.

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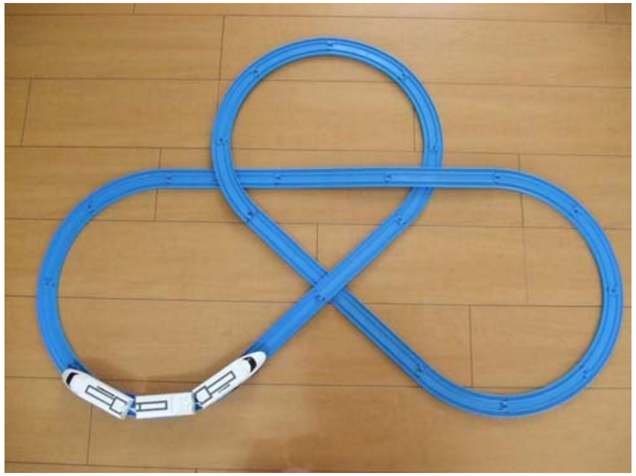

End of Slides – Thank You.