Building a (sort of) GoI from denotational semantics: an improvisation

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Denotational semantics vs. GoI

In synthesis:

- denotational semantics is *cut-as-composition*;
- the geometry of interaction is *cut-as-trace*.

We know how to go from the GoI view to the denotational semantics view: we use the Int construction.

The question we address here is: can we go the other way?

In other words, can we build a "cut-as-trace" interpretation of proofs starting from a more traditional, "cut-as-composition" interpretation?

One possible motivation: fix the mismatch between GoI execution and syntactical cut-elimination.

Previous work

We have illustrious predecessors: Abramsky and Jagadeesan followed a similar path in their "New Foundations" paper (1993).

Some comparison:

- Motivations and rationale: very similar.
- Methodology: quite different.
- Results: there is arguably some overlap, but also some important differences. . . ? (To be honest, I don't know exactly.)

Some background ideas

- Denotational semantics:
  - proofs are *vectors*;
  - a proof of $A^\perp, B$ is a vector of $A^* \otimes B$, i.e., a matrix;
  - cut is composition, i.e., matrix product.

- GoI:
  - proofs are *operators*;
  - a proof of $A^\perp, B$ is a linear operator on $A^* \otimes B$;
  - composition is trace.

- The two should be related in a "nice" way, e.g., *the denotational semantics should appear as a sum of eigenvectors of the GoI operator* (an extension of Regnier's conjecture).

Back to reality

It's going to be tough to make it work:

- negation must be involutive;
- at the same time, the exponential modalities force considering infinite-dimensional vector spaces;
- consequence: topological vector spaces are needed.
- That is far from trivial (Ehrhard 2005).
- Additional problem: the category is $*$-autonomous, not compact closed: what is the trace?

A low-profile setting

The category $\mathbf{Rel}$ of sets and relations.

- It hosts a model of linear logic: tensor is Cartesian product (not a categorical product in $\mathbf{Rel}$), the comonad is given by the free commutative monoid construction (finite multisets), negation is identity.

- A set X can be seen as the basis of a "free" vector space over... something which is not a field (or even a ring), but never mind. In fact, $(\wp(X), \cup, \emptyset)$ is a monoid (that's close enough to a vector space...).

- Given another set Y, it makes sense to define $\wp(X) \otimes \wp(Y) \cong \wp(X \times Y)$, and a monoid endomorphism can play the role of linear operators.

- $\mathbf{Rel}$ also hosts a model of differential interaction nets, which will turn out to be useful...

The Lafont double cover of a net

- A standard construction in topology (the orientable double cover of a non-orientable surface), specialized to a standard construction on graphs, the *bipartite double cover* of an undirected graph G, defined as $G \times K_2$.

- Applied for the first time by Lafont (1995) to nets of interaction combinators. We denote it by $(\cdot)^{\pm}$.

- It is the essence of the GoI!

- In the multiplicative case, it is easy; in the exponential case, one must define the Lafont double cover of a box. Girard's proposal unfortunately does not work perfectly.

Differential interaction nets and the Taylor expansion

- Twenty years after Girard's first proposal, and sixteen years after Abramsky and Jagadeesan work, we have "much newer foundations": *differential interaction nets* (Ehrhard-Regnier 2006).

- Exponential boxes of linear logic proof nets can be expressed in differential interaction nets by means of the *Taylor-Ehrhard expansion*, denoted by $\mathcal{T}(\cdot)$.

- In fact, differential interaction nets are an extremely useful bridge between syntax and denotational semantics.

- (Technical note: in what follows, to avoid treading on dangerous soil, we drop additive connectives, and we consider only atomic axioms.)

Entanglement

- Defining the Lafont double cover $\alpha^{\pm}$ of a differential interaction net α is trivial. Then, given a proof net π of conclusions $A_1, \ldots, A_n$, we have

$$[\![\mathcal{T}(\pi)^{\pm}]\!] \subseteq (A_1 \times \cdots \times A_n) \times (A_1 \times \cdots \times A_n),$$

  where $[\![\cdot]\!]$ denotes interpretation in $\mathbf{Rel}$. This is precisely a monoid endomorphism (i.e., an "operator") of $\wp(A_1) \otimes \cdots \otimes \wp(A_n)$. Perfect!

- Actually, not so perfect... It is easy to see that this is too naive, it won't model cut-elimination: "wrong" nets emerge in the simulation.

- Intriguingly, the solution requires handling a phenomen of entanglement. To deal with it, we formally do just as in quantum mechanics (the math is morally the same).

Entangled experiments

- Experiments are an extremely useful tool for concretely computing the interpretation of a proof net in "webbed" models (like $\mathbf{Rel}$).

- Let α be a differential interaction net. Given a port p of $\alpha^{\pm}$, we can always define its *twin* $\overline{p}$.

- An experiment e of $\alpha^{\pm}$ is *strongly entangled* iff, for all ports p, q of $\alpha^{\pm}$, $e(p) = e(q)$ implies $e(\overline{p}) = e(\overline{q})$.

Lemma 1. *An experiment is strongly entangled iff the above condition is verified by all* atomic *ports of* $\alpha^{\pm}$.

- If an experiment satisfies the above condition only on the premises of exponential cells, we call it *weakly entangled*, or simply *entangled*.

The GoI interpretation

- If α is a differential interaction net, we denote by $(\!|\alpha^{\pm}|\!)$ (resp. $(\!|\alpha^{\pm}|\!)_s$) the set of the results of all entangled (resp. strongly entangled) experiments on $\alpha^{\pm}$.

- We denote by $\alpha_{\bullet}$ the "cut-free" version of α. We define the GoI interpretation of a proof net π as

$$\mathrm{GoI}\,\pi = \bigcup_{\alpha \in \mathcal{T}(\pi)} (\!|\alpha_{\bullet}^{\pm}|\!) \qquad \text{(and } \mathrm{GoI}_s\pi = \bigcup_{\alpha \in \mathcal{T}(\pi)} (\!|\alpha_{\bullet}^{\pm}|\!)_s\text{).}$$

- Cut-elimination is modeled by the usual trace in $\mathbf{Rel}$. In particular, thanks to the definition of experiment, we have

Lemma 2. $\mathrm{Tr}(\mathrm{GoI}\,\alpha) = (\!|\alpha^{\pm}|\!)$, *and hence* $\mathrm{Tr}(\mathrm{GoI}\,\pi) = \bigcup_{\alpha \in \mathcal{T}(\pi)} (\!|\alpha^{\pm}|\!)$.

Soundness

- We have the following fundamental result:

  Lemma 3. $\alpha \to \beta$ *implies* $(\!|\alpha^{\pm}|\!) = (\!|\beta^{\pm}|\!)$.

- Then, thanks to the soundness of the Taylor-Ehrhard expansion (i.e., $\pi \to \pi'$ implies $\mathcal{T}(\pi) \to^{*} \mathcal{T}(\pi')$), and to Lemma 2 and Lemma 3, we have

  Theorem 4. [Soundness] $\pi \to \pi'$ *implies* $\mathrm{Tr}(\mathrm{GoI}\,\pi) = \mathrm{Tr}(\mathrm{GoI}\,\pi')$.

- Note that, just like in "New Foundations" GoI, there is no restriction on the validity of soundness.

- All of the above also hold when we replace entangled semantics with strongly entangled semantics.

Retrieving denotational semantics?

Remember that denotational semantics should appear as a sort of "sum of eigenvectors". This is the closest approximation we get in our framework:

Lemma 5. *Let α be a cut-free differential interaction net. Then,*

$$\mathrm{GoI}_s\alpha([\![\alpha]\!]) = [\![\alpha]\!].$$

(Probably $[\![\alpha]\!]$ is the biggest set with such property, we don't know. . .).

If α_1, α_2 are different summands of the Taylor-Ehrhard expansion of a cut-free proof net π of conclusion A, then $\mathrm{GoI}_s\alpha_1$ and $\mathrm{GoI}_s\alpha_2$ should have "disjoint domains", i.e., there exist disjoint subsets A_1, A_2 of A such that the only sets not in the "kernel" of $\mathrm{GoI}_s\alpha_i$ are included in A_i.

Then, the union $\bigcup_{\alpha\in\mathcal{T}(\pi)} \mathrm{GoI}_s\alpha$ is actually a "direct sum", which should be enough to guarantee the following

Conjecture 6. *Let π be a proof net. Then,* $\mathrm{GoI}_s\pi([\![\pi]\!]) = [\![\pi]\!]$.

To do. . .

- Strong entanglement is. . . too strong. Fortunately, weak entanglement is enough for soundness; we keep hoping that it is also enough to get Conjecture 6.

- Speaking of Conjecture 6, note that this fails in general: if α, β are arbitray differential interaction nets, $[\![\alpha + \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$ will not in general be a fixpoint of $\mathrm{GoI}_s\alpha \cup \mathrm{GoI}_s\beta$. This suggests that there are perhaps two sums/unions of nets: one "uniform", and one "non-uniform", maybe in analogy with *pure states* and *mixed-states*?

- What about paths? Clearly this is not "particle-style" GoI, but maybe "wave-style", or better, particles moving according to quantum mechanical "trajectories"?

- This is a bit *ad hoc*. Can one find a more abstract formulation?