Rogers–Ramanujan type identities

Andrew Sills

Georgia Southern University

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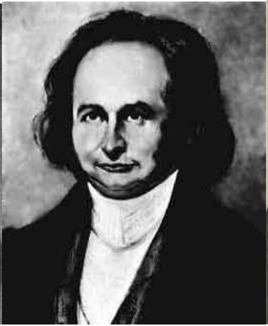
L. Euler (1707–1783)
C. G. J. Jacobi (1804–1851)
L. J. Rogers (1862–1933)

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(Rogers)

Rising q-factorial notation

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$$(a_1, a_2, \ldots a_r; q)_\infty := (a_1)_\infty(a_2)_\infty(a_3)_\infty\cdots(a_r)_\infty$$

S. Ramanujan (1887–1920)

Ramanujan's "theta" function

For $|ab| < 1$,

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Jacobi's triple product identity

$$f(a,b) = (a, b, ab; ab)_\infty.$$

Ramanujan's notation

$$f(-q) := f(-q, -q^2) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)/2} = (q)_\infty$$

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(Gauss's square numbers thm)

$$\psi(-q) := f(-q,-q^3) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(2n-1)} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

(Gauss's hexagonal numbers thm)

Rogers–Ramanujan identities

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2,-q^3)}{(q)_\infty}.$$

$$\sum_{n\geq 0} \frac{q^{n(n+1)}}{(q)_n} = \frac{f(-q,-q^4)}{(q)_\infty}.$$

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Over 50 are recorded in the lost notebook.

Bailey pairs, Bailey's lemma

If $(\alpha_n(a,q), \beta_n(a,q))$ satisfies

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

then (α_n, β_n) is called a *Bailey pair with respect to a*,

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then (α_n, β_n) is called a *Bailey pair with respect to a*, and $(\alpha'_n(a,q), \beta'_n(a,q))$ is also a Bailey pair, where

$$\alpha'_r(a,q) = \frac{(\rho_1)_r(\rho_2)_r}{(aq/\rho_1)_r(aq/\rho_2)_r} \left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r$$

and

$$\beta'_n(a,q) = \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}}{(aq/\rho_1)_n(aq/\rho_2)_n(q)_{n-j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j(a,q).$$

Limiting cases of Bailey's lemma

$$\sum_{n\geq 0} q^{n^2} \beta_n(1,q) = \frac{1}{(q)_\infty} \sum_{r\geq 0} q^{r^2} \alpha_r(1,q) \qquad \text{(PBL)}$$

$$\sum_{n\geq 0} q^{n^2} (-q;q^2)_n \beta_n(1,q^2) = \frac{1}{\psi(-q)} \sum_{r\geq 0} q^{r^2} \alpha_r(1,q^2) \qquad \text{(HBL)}$$

$$\sum_{n\geq 0} q^{n(n+1)/2} (-1)_n \beta_n(1,q) = \frac{2}{\varphi(-q)} \sum_{r\geq 0} \frac{q^{r(r+1)/2}}{1+q^r} \alpha_r(1,q) \qquad \text{(SBL)}$$

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Bailey, Dyson, and Slater

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- Lucy Slater found many Bailey pairs, and used them to generate a list of 130 RR type identities.

General Bailey pairs

For $d \mid n$, define

$$\alpha_n^{(d,e,k)}(a,q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1-a^{1/e})(q^{d/e};q^{d/e})_{n/d}}, \\ \times (1-a^{1/e}q^{2n/e})(a^{1/e};q^{d/e})_{n/d},$$

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$$\tilde{\alpha}_n^{(d,e,k)}(a,q) := q^{n(d-n)/2de} a^{-n/de} \frac{(-a^{1/e};q^{d/e})_{n/d}}{(-q^{d/e};q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a,q),$$

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$$\bar{\alpha}_n^{(d,e,k)}(a,q) := (-1)^{n/d} q^{n^2/2de} \frac{(q^{d/2e};q^{d/e})_{n/d}}{(a^{1/e}q^{d/2e};q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a,q).$$

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Let the corresponding $\beta_n^{(d,e,k)}(a,q)$, $\tilde{\beta}_n^{(d,e,k)}(a,q)$, and $\bar{\beta}_n^{(d,e,k)}(a,q)$ be determined by the Bailey pair relation.

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- For certain (d, e, k), the resulting expression for β is a very well-poised ${}_6\phi_5$, summable by a theorem of F. H. Jackson.
- Using only this, and an associated families of q-difference equations, one can recover the majority of Slater's list, as well as other identities.

The Bailey pair that arises from

$$\left(\alpha_n^{(1,1,2)}(a,q), \beta_n^{(1,1,2)}(a,q)\right) = \left(\frac{(-1)^n a^n q^{n(3n-1)/2}(1-aq^{2n})(a)_n}{(1-a)(q)_n}, \frac{1}{(q)_n}\right)$$

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- $\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{f(-q^3,-q^5)}{\psi(-q)}$ upon insertion into (HBL).

New identities arising from this framework (S.)

$$\sum_{n,r\geq 0} \frac{q^{n^2+2nr+2r^2}(-q;q^2)_r}{(q)_{2r}(q)_n} = \frac{f(-q^{10},-q^{10})}{(q)_\infty}$$

by insertion of $(\tilde{\alpha}_n^{(2,1,5)}(1,q), \tilde{\beta}_n^{(2,1,5)}(1,q))$ into (PBL).

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$$\sum_{n,r\geq 0} \frac{q^{4n^2+8nr+8r^2}(-q;q^2)_{2r}}{(q^4;q^4)_{2r}(q^4;q^4)_n} = \frac{f(q^9,q^{11})}{(q^4;q^4)_\infty}$$

by insertion of $(\bar{\alpha}_n^{(1,2,4)}(1,q), \bar{\beta}_n^{(1,2,4)}(1,q))$ into (PBL).

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A family of mod 24 identities

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{f(-q,-q^{11})f(-q^{10},-q^{14})}{\psi(-q)(q^{24};q^{24})_\infty}$$

(McLaughlin.-S.)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \frac{f(-q^2,-q^{10})f(-q^8,-q^{16})}{\psi(-q)(q^{24};q^{24})_\infty}$$ (Ramanujan)

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{f(-q^3,-q^9)f(-q^6,-q^{18})}{\psi(-q)(q^{24};q^{24})_\infty}$$ (M.-S.)

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q^3;q^6)_n}{(q^2;q^2)_{2n}(1-q^{2n+1})} = \frac{f(-q^4,-q^8)f(-q^4,-q^{20})}{\psi(-q)(q^{24};q^{24})_\infty}$$ (M.-S.)

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(q^4;q^4)_n(q^{2n+4};q^2)_{n+1}} = \frac{f(-q^5,-q^7)f(-q^2,-q^{22})}{\psi(-q)(q^{24};q^{24})_\infty}$$

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Combinatorial considerations

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A *partition* λ of n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the *parts* of λ) that sum to n.

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$$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1).$$

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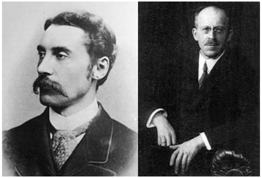
$$9, 81, 72, 63, 621, 54, 531, 432$$

Combinatorial Rogers–Ramanujan (due to MacMahon and Schur)



The number of partitions of n into parts that mutually differ by at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod{5}$.

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Note: The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.

G. Andrews' analytic counterpart to Gordon's theorem

$$\sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+n_{i+1}+\cdots+n_{k-1}}}{(q)_{n_1}(q)_{n_2-n_1}(q)_{n_3-n_2}\cdots(q)_{n_{k-1}-n_{k-2}}} = \frac{f(-q^i,-q^{2k+1-i})}{(q)_\infty}.$$

Combinatorial interpretations of these "(d, e, k)" identities (S.)

Let $d \in \mathbb{N}$ and let $1 \leq i \leq k$.
Let $G_{d,k,i}(n)$ denote the number of partitions π of n such that

$$m_d(\pi) \leq i - 1 \text{ and } m_{dj}(\pi) + m_{dj+d}(\pi) \leq k - 1$$

for any $j \in \mathbb{N}$.

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Let $H_{d,k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm di \pmod{2d(k+1)}$.

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Then $G_{d,k,i}(n) = H_{d,k,i}(n)$ for all integers n.
This is a combinatorial interpretation of of the identity obtained by inserting the Bailey pair $(\alpha_n^{(d,1,k)}(1,q), \beta_n^{(d,1,k)}(1,q))$ into (PBL) (along with associated systems of q-difference equations).

WHO

(outside the partitions and q-series community)

CARES?

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- In the 1980's J. Lepowsky and R. Wilson showed that the principally specialized characters of standard modules for the odd levels of $A_1^{(1)}$ are given by the The Andrews–Gordon identity.
- The two Rogers–Ramanujan identities occur at level 3.
- The even levels of $A_1^{(1)}$ correspond to D. Bressoud's even modulus analog of Andrews–Gordon.

Capparelli's identities (1988)

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Performing an analogous analysis of the level 3 modules of $A_2^{(2)}$, S. Capparelli discovered:

The number of partitions of n into parts $\equiv \pm 2, \pm 3 \pmod{12}$ equals the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of n where

- $\lambda_i - \lambda_{i+1} \geq 2$,
- $\lambda_i - \lambda_{i+1} = 2 \implies \lambda_i \equiv 1 \pmod 3$,
- $\lambda_i - \lambda_{i+1} = 3 \implies \lambda_i \equiv 0 \pmod 3$

$$1+\sum_{\substack{n,j,r\geq 0\\(n,j,r)\neq(0,0,0)}} \frac{q^{3n^2+\frac{9}{2}r^2+3j^2+6nj+6nr+6rj-\frac{5}{2}r-j}(1+q^{2r+2j})(1-q^{6r+6j})}{(q^3;q^3)_n(q^3;q^3)_r(q^3;q^3)_j(-1;q^3)_{j+1}(q^3;q^3)_{n+2r+2j}}$$

$$=\frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}$$

Analytic versions of Capparelli's identity (S.)

$$1 + \sum_{\substack{n,j,r \geq 0 \\ (n,j,r) \neq (0,0,0)}} \frac{q^{3n^2 + \frac{9}{2}r^2 + 3j^2 + 6nj + 6nr + 6rj - \frac{5}{2}r - j}(1 + q^{2r+2j})(1 - q^{6r+6j})}{(q^3;q^3)_n (q^3;q^3)_r (q^3;q^3)_j (-1;q^3)_{j+1} (q^3;q^3)_{n+2r+2j}}$$

$$= \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}$$

$$\sum_{n,j \geq 0} \frac{q^{n^2} \binom{\frac{n-j+1}{3}}{}}{(q)_{2n-j}(q)_j} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}.$$

$A_2^{(2)}$ level 4 identities

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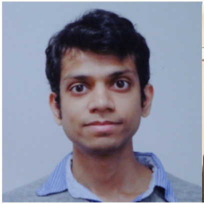
One of these identities is:

The number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n where

- $\lambda_i - \lambda_{i+1} \geq 2$
- $\lambda_i - \lambda_{i+2} \geq 3$
- $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,
- $\lambda_i - \lambda_{i+2} = 3$ and $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$.
- $\lambda_i - \lambda_{i+2} = 4$ and $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$,
- Consider the first differences $\Delta\lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{l-1} - \lambda_l)$. None of the following subwords are permitted in $\Delta\lambda$: $(3,3,0), (3,2,3,0), (3,2,2,3,0), \ldots, (3,2,2,2,2,\ldots,2,3,0)$.

Shashank Kanade and Matthew Russell (2014)



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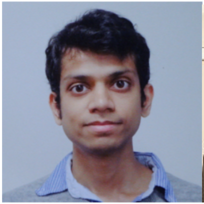

Related to level 3 standard modules of $D_4^{(3)}$, Kandade and Russell conjectured several partition identities, including:

The number of partitions of n into parts $\equiv \pm 1, \pm 3 \pmod 9$ equals the number of partitions λ of n such that

- $\lambda_j - \lambda_{j+2} \geq 3$,
- $\lambda_j - \lambda_{j+1} \leq 1 \implies 3 \mid (\lambda_j + \lambda_{j+1})$.

Kanade–Russell conjectures

Kanade and Russell have released a steady stream of q-series and partition identity conjectures over the past six years.

Kanade–Russell conjectures

Kanade and Russell have released a steady stream of q-series and partition identity conjectures over the past six years. Many have been proved by

- Katherin Bringmann, Chris Jennings-Shaffer, and Karl Mahlburg;
- Kagan Kurşungöz;
- Hjalmar Rosengren;
- Kanade and Russell themselves.

WHO ELSE CARES?

Polynomial RR identities

$$D_0(q) = D_1(q) = 1$$
$$D_n(q) = D_{n-1}(q) + q^{n-1} D_{n-2} \text{ if } n \geqq 2$$

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$$= \sum_{k \in \mathbb{Z}} \left(q^{k(10k+1)} \tau_0(n, 5k; q) - q^{(5k+3)(2k+1)} \tau_0(n, 5k+3; q) \right)$$

(Andrews)

We can prove these polynomial identities via recurrences, and then the original series–infinite product identity follows via asymptotics of q-bi/trinomial coëfficients, and the triple product identity.

q-binomial and q-trinomial coëfficients

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := (q)_A (q)_B^{-1} (q)_{A-B}^{-1} \text{ if } 0 \leqq B \leqq A; \text{ 0 o/w}$$

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$$\mathrm{T}_0(L, A; q) := \sum_{r=0}^{L} (-1)^r \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q$$

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linear combinations of q-trinomial coëfficients

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and

$$\mathrm{V}(L, A; q) := \mathrm{T}_1(L-1, A; q) + q^{L-A}\mathrm{T}_0(L-1, A-1; q).$$

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$$\lim_{n \to \infty} D_n(q) = G(q)$$

- I “algorithmitized” and generalized Andrews’ heuristic, and implemented it in Maple.

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- “Finitized” all 130 identities in Slater’s list of RR type identities.

Ramanujan's Lost Notebook, p. 35 of Narosa Edition

$$\sum_{j \geqq 0} \frac{q^{j(j+1)/2}(-q^2;q^2)_j}{(q)_j(q;q^2)_{j+1}} = \frac{\psi(-q^2)}{\varphi(-q)}.$$

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For fixed n,

$$\sum_{i,j,k\geqq 0} q^{j(j+1)/2+i^2+i+k} \begin{bmatrix} j \\ i \end{bmatrix}_{q^2} \begin{bmatrix} j+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2i-2k \\ j \end{bmatrix}_q$$
$$= \sum_{j\in\mathbb{Z}} (-1)^j q^{2j(2j+1)} \mathrm{V}(n+1, 4j+1; \sqrt{q}).$$

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q-Pell numbers: $P_0 = 1, \quad P_1 = q+1, \quad P_2 = q^3 + q^2 + 2q + 1$

$$P_n = (1+q^n)P_{n-1} + qP_{n-2} + (q^n - q)P_{n-3}.$$

Bowman–McLaughlin–S.

$$\sum_{j \geqq 0} \frac{q^{j(j+1)}(-q^3;q^3)_j}{(-q)_j(q)_{2j+1}} = \frac{f(-q^3,-q^6)f(-q^3,-q^{15})}{(q)_\infty(q^{18};q^{18})_\infty}$$

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For fixed n,

$$\sum_{i,j,k,l,m\geqq 0} (-1)^{k+m} q^{j^2+2j+3i(i+1)/2+k+l+m} \begin{bmatrix} j \\ i \end{bmatrix}_{q^3} \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q \times \begin{bmatrix} j+l \\ l \end{bmatrix}_{q^2} \begin{bmatrix} j+m-1 \\ m \end{bmatrix}_q \begin{bmatrix} n-3i-j-k-2l-m \\ j \end{bmatrix}_q$$

$$= \sum_{k\in\mathbb{Z}} q^{9k(3k+1)/2} \begin{bmatrix} n+1 \\ \lfloor \frac{n+9k+3}{2} \rfloor \end{bmatrix}_q - q^3 \sum_{k\in\mathbb{Z}} q^{27k(k+1)/2} \begin{bmatrix} n+1 \\ \lfloor \frac{n+9k+6}{2} \rfloor \end{bmatrix}_q$$

Berkovich–Uncu

$$\sum_{j \geqq 0} \frac{q^{3j^2}(-q;q^2)_{3j}}{(q^6;q^6)_{2j}} = \frac{f(q^4,q^8)}{\psi(-q^3)}$$

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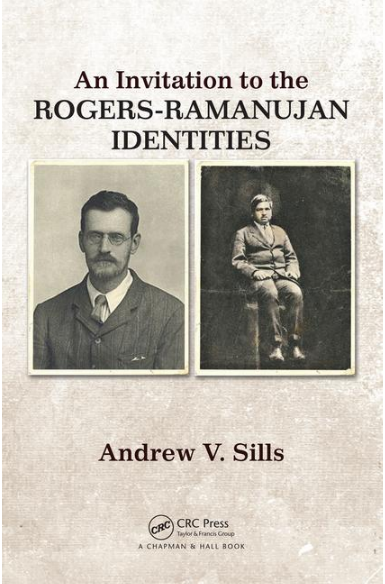
$$\sum_{j\geqq 0} \frac{q^{3j^2}(-q;q^2)_{3j}}{(q^6;q^6)_{2j}} = \frac{f(q^4,q^8)}{\psi(-q^3)}$$

For fixed n,

$$\sum_{i,j,k\geqq 0} (-1)^k q^{3j^2+i^2+3k} \begin{bmatrix} 3j \\ i \end{bmatrix}_{q^2} \begin{bmatrix} 2j+k-1 \\ k \end{bmatrix}_{q^3} \begin{bmatrix} n+j-i-k \\ 2j \end{bmatrix}_{q^3}$$
$$= \sum_{j\in\mathbb{Z}} q^{6j^2+2j} \left(\mathrm{T}_0(n,2j;q^3) + \mathrm{T}_0(n-1,2j;q^3) \right).$$

Japanese translation in preparation!



THANK YOU!