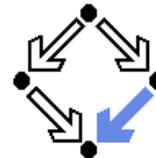A workshop on q-series, quantum modular forms
and representation theory
Kyoto University, October 6, Japan

RISC Packages for q-Series

Peter Paule

Johannes Kepler University Linz
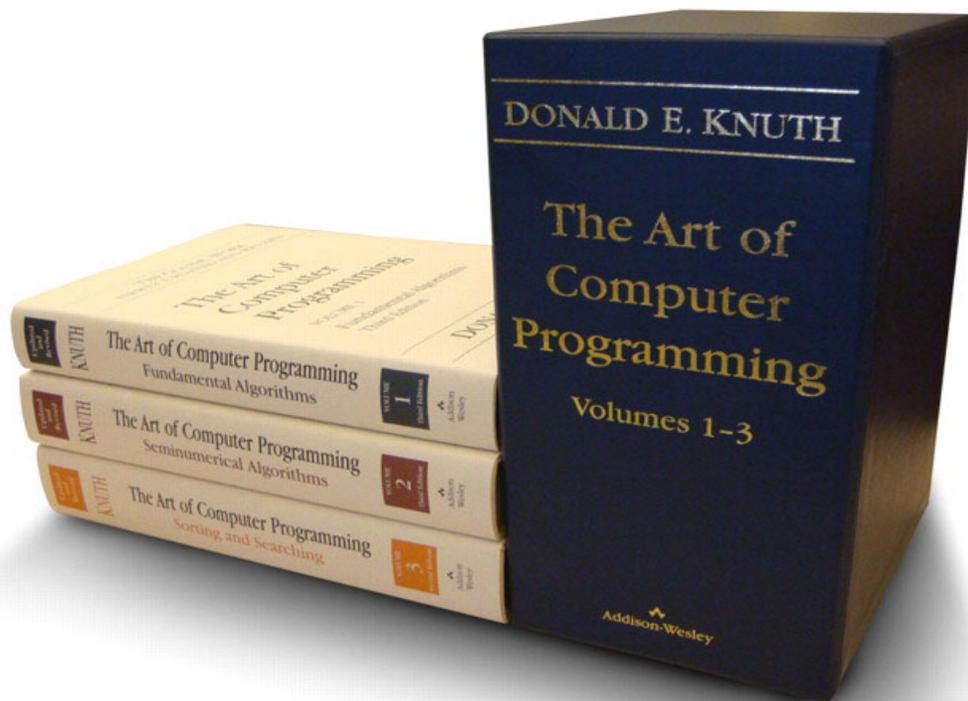Research Institute for Symbolic Computation (RISC)

A Bit of History

SYMBOLIC COMPUTATION with special functions & sequences

Number Theorists played a pioneering role; e.g.:

"Computers in Number Theory" (Oxford, 1969[1])
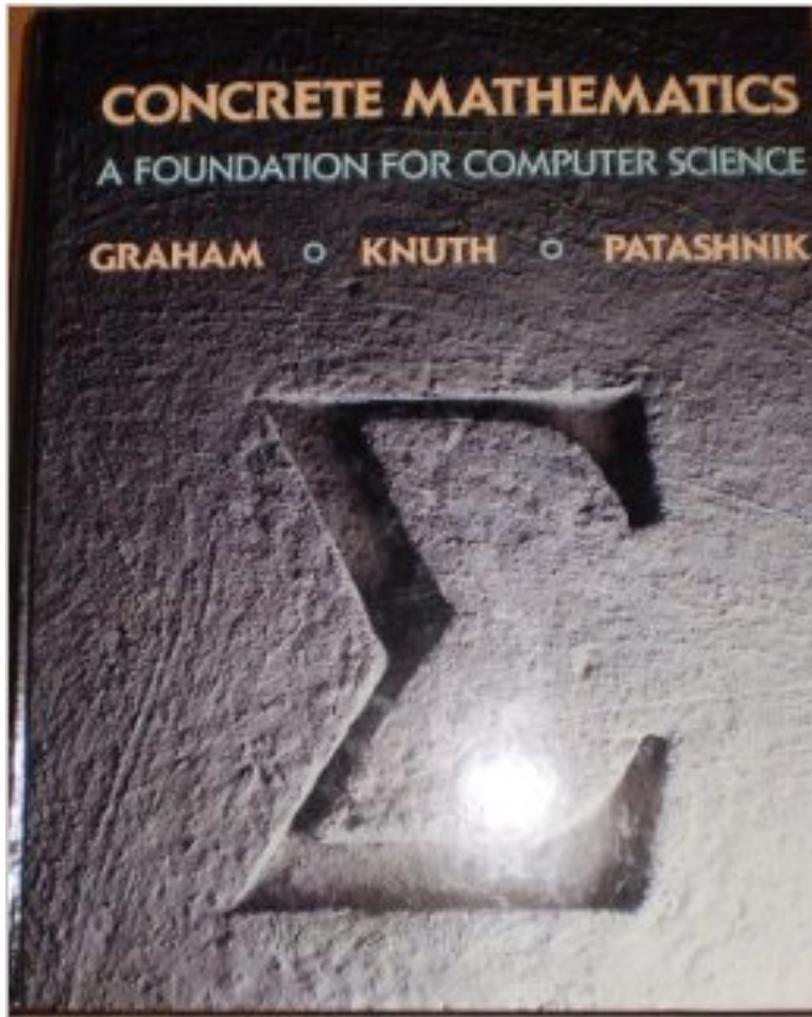

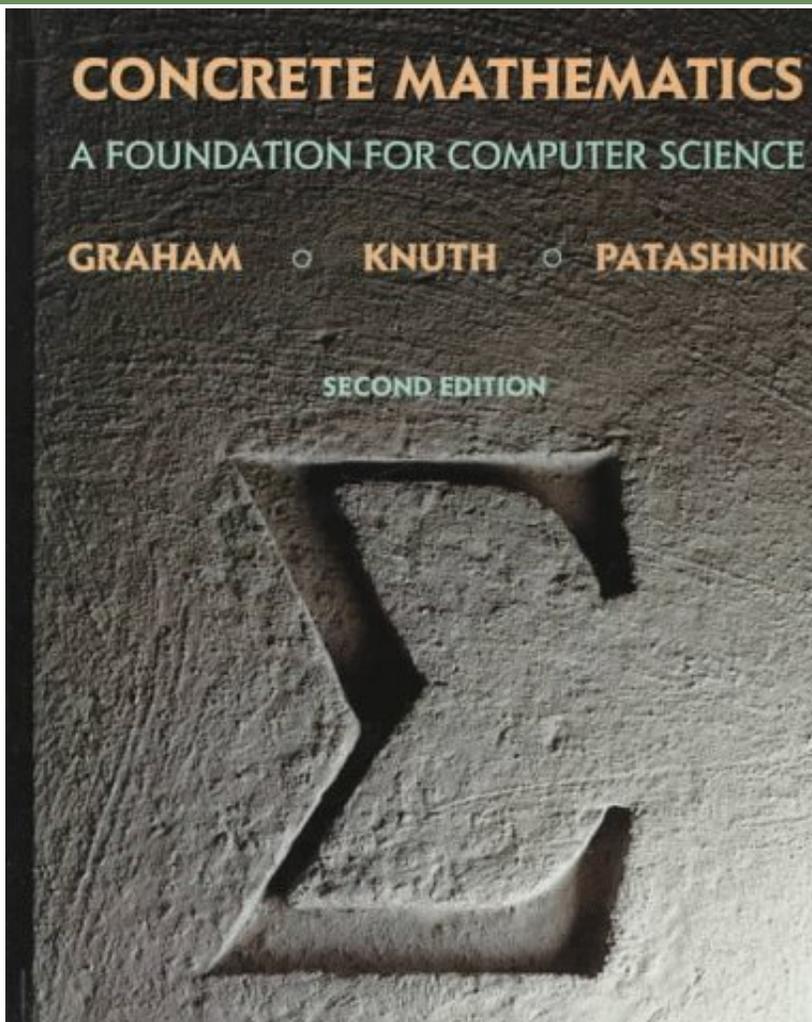[1] taken from John B. Cosgrove's home page

Volume 1, 1st edition 1968, Exercise 1.2.6.63:
[50] Develop computer algebra programs for simplifying sums that involve binomial coefficients.
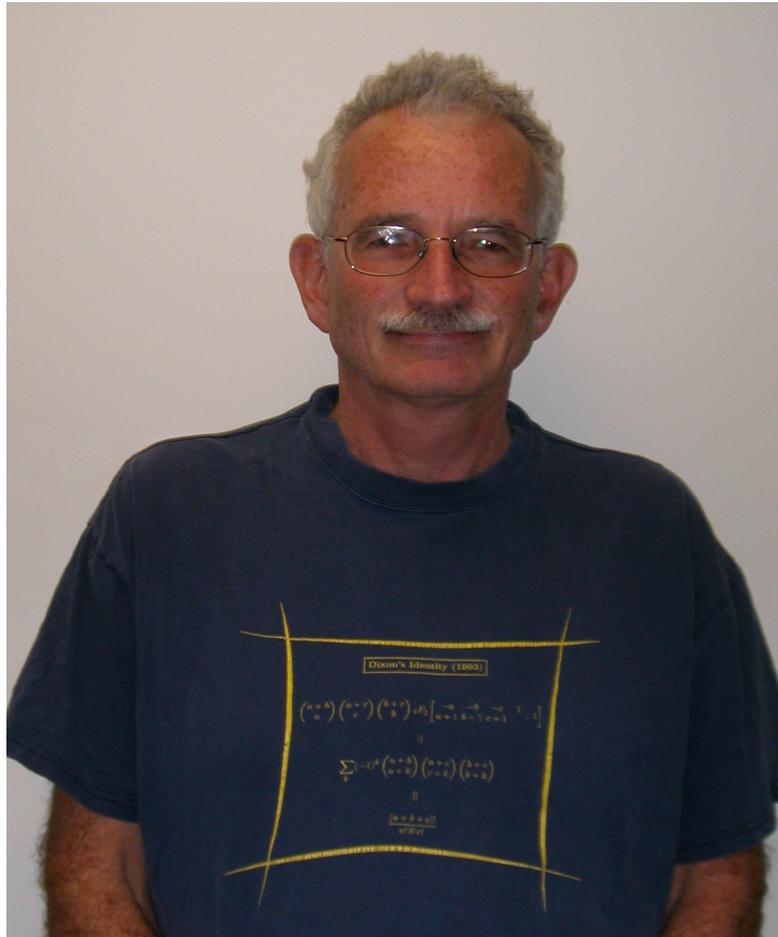
1st edition 1989; contains Gosper's algorithm (1978).

2nd edition 1994: What is the difference to the 1st edition?

Answer:

Gosper $\rightarrow$ Zeilberger:

174   IX. The eighteenth Century.

Fig. 37.   The Pascal triangle as given in Murai's
Sampō Dōshi-mon (1781).

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Web source: `https://archive.org/stream/historyofjapanes00smitiala`
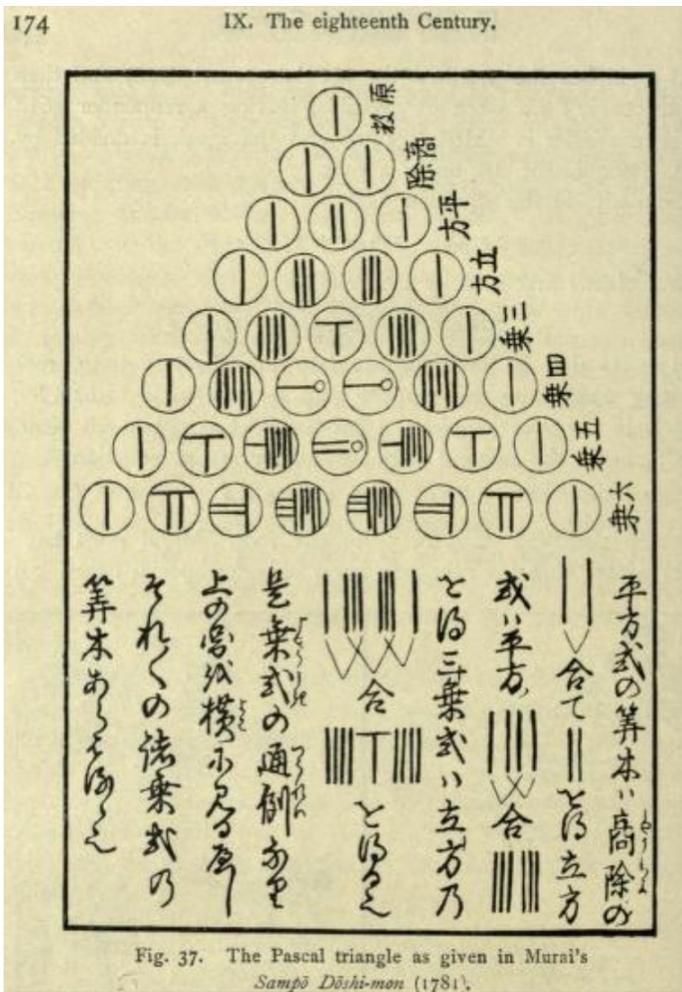
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Telescoping (Gosper):

$$(-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k} - (-1)^{k-1} \binom{n-1}{k-1},$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Telescoping (Gosper):

$$(-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k} - (-1)^{k-1} \binom{n-1}{k-1} = g(k) - g(k-1),$$

$$\sum_{k=0}^{\ell} (-1)^k \binom{n}{k} = (-1)^{\ell} \binom{n-1}{\ell} - (-1)^{-1} \binom{n-1}{-1} = g(\ell) - g(-1)$$

$$= (-1)^{\ell} \binom{n-1}{\ell}.$$

---

q-binomial coefficients:

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

$n = 0$:   1

$n = 1$:   1   1

$n = 2$:   1 $1 + q$ 1

$n = 3$: 1 $1 + q + q^2$ $1 + q + q^2$ 1

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

q-Telescoping (q-analogue of Gosper):

$$(-1)^k q^{\binom{k}{2}} \binom{n}{k}_q = (-1)^k q^{\binom{k}{2}+k} \binom{n-1}{k}_q - (-1)^{k-1} q^{\binom{k}{2}} \binom{n-1}{k-1}_q$$

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

q-Telescoping (q-analogue of Gosper):

$$(-1)^k q^{\binom{k}{2}} \binom{n}{k}_q = (-1)^k q^{\binom{k}{2}+k} \binom{n-1}{k}_q - (-1)^{k-1} q^{\binom{k}{2}} \binom{n-1}{k-1}_q$$

$$\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q = g(\ell) - g(-1)$$

$$= (-1)^\ell q^{\binom{\ell}{2}+\ell} \binom{n-1}{\ell}_q.$$

Back to $q = 1$: We know that there are identities like

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

but $\binom{n}{k}$ does not telescope! $\leadsto$ Creative Telescoping (Zeilberger):

$$\text{Recall:} \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Creative Telescoping (Zeilberger):

$$\binom{n+1}{k} - 2\binom{n}{k} = -\binom{n}{k} + \binom{n}{k-1} = g(k) - g(k-1)$$

where

$$g(k) = -\binom{n}{k}.$$

Recall: $\dbinom{n+1}{k} = \dbinom{n}{k} + \dbinom{n}{k-1}$

Creative Telescoping (Zeilberger):

$$\binom{n+1}{k} - 2\binom{n}{k} = -\binom{n}{k} + \binom{n}{k-1} = g(k) - g(k-1)$$

where

$$g(k) = -\binom{n}{k}.$$

$$\sum_{k=0}^{n+1}\left(\binom{n+1}{k} - 2\binom{n}{k}\right) = \sum_{k=0}^{n+1}\binom{n+1}{k} - 2\sum_{k=0}^{n+1}\binom{n}{k}$$

$$= g(n+1) - g(-1) = 0.$$

Recall: $\displaystyle\sum_{k=0}^{n+1}\left(\binom{n+1}{k}-2\binom{n}{k}\right)=\sum_{k=0}^{n+1}\binom{n+1}{k}-2\sum_{k=0}^{n+1}\binom{n}{k}$

$$= g(n+1) - g(-1) = 0.$$

Consequently, for

$$S(n) := \sum_{k=0}^{n}\binom{n}{k}$$

one has:

$$S(n+1) - 2\,S(n) = 0, \quad n \geq 0.$$

Alternatively,

Recall: $\displaystyle\sum_{k=0}^{n+1}\left(\binom{n+1}{k}-2\binom{n}{k}\right)=\sum_{k=0}^{n+1}\binom{n+1}{k}-2\sum_{k=0}^{n+1}\binom{n}{k}$

$$= g(n+1) - g(-1) = 0.$$

Consequently, for
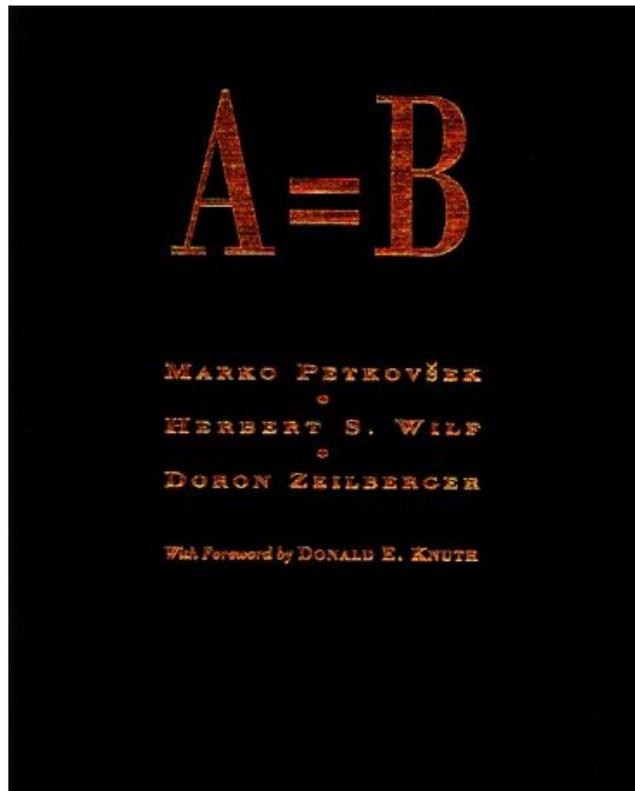
$$S(n) := \sum_{k=0}^{n}\binom{n}{k}$$

one has:

$$S(n+1) - 2\,S(n) = 0, \quad n \geq 0.$$
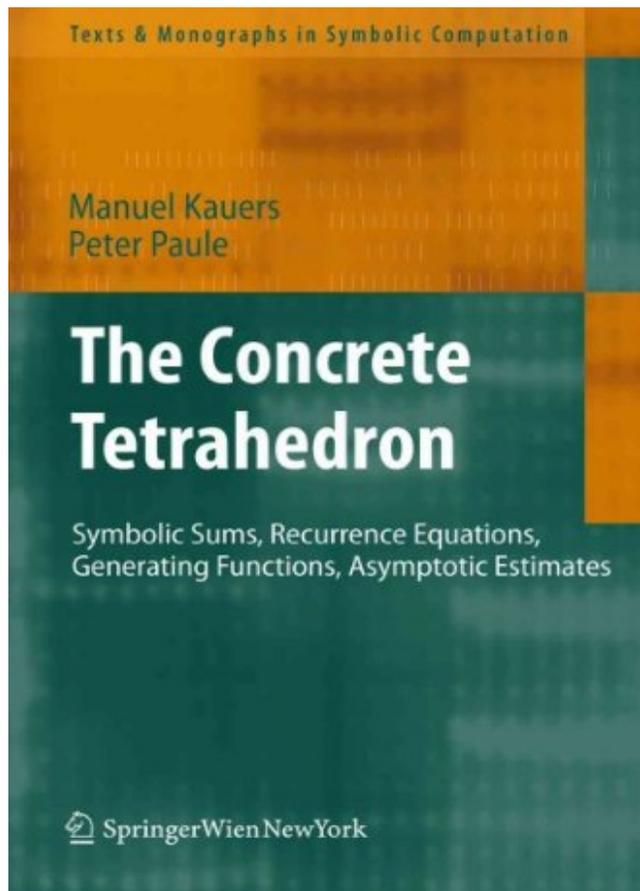
Alternatively,

$$S(n) = 2^n, \quad n \geq 0.$$

Zeilberger's algorithm solves Knuth's [50]-problem from 1968:
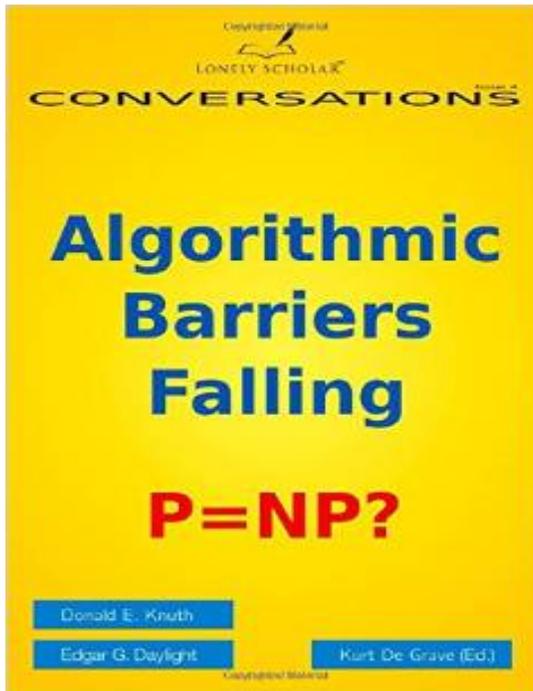

1st edition 1996

An algorithmic supplement to "Concrete Mathematics":

1st edition 2011

A conversation with Donald E. Knuth conducted
by Edgar G. Daylight (Paris, June 18, 2014):

*Knuth: Learning how to manipulate formulas fluently, and how to
see patterns in formulas instead of patterns in numbers — that's
what my book "Concrete Mathematics" is essentially about.*

A conversation with Donald E. Knuth conducted
by Edgar G. Daylight (Paris, June 18, 2014) [contd.]:
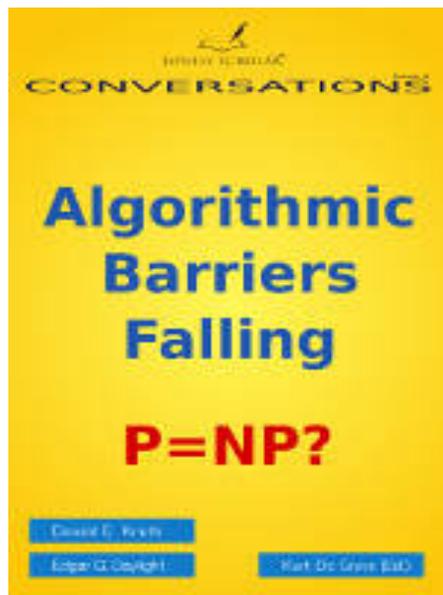
Edgar: Which was also the topic of Manuel Kauers this morning?

*Knuth: Right. In fact, he and Peter Paule in Austria recently
published a beautiful book called "The Concrete Tetrahedron",
which is sort of the sequel to "Concrete Mathematics".*

CONTENTS OF THE "CONCRETE TETRAHEDRON"

Contents of the "Concrete Tetrahedron"

Computer Algebra Session 1

In[1]:= `<< RISC`fastZeil``

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[1]:= `<< RISC`fastZeil``

Fast Zeilberger Package version 3.61

written by Peter Paule, Markus Schorn, and Axel Riese

Copyright Research Institute for Symbolic Computation (RISC),

Johannes Kepler University, Linz, Austria

In[4]:= $\binom{n\_}{k\_}_*$ `:= Binomial[n, k]`

In[7]:= $\left\{ \binom{n}{k}_* \middle/ \binom{n-1}{k}_* , \binom{n}{k}_* \middle/ \binom{n}{k-1}_* \right\}$ `// FullSimplify`

Out[7]= $\left\{ \dfrac{n}{-k+n}, \dfrac{1-k+n}{k} \right\}$

In[42]:=

$$\text{Gosper}\left[(-1)^k \binom{n}{k}_*, \{k, 0, l\}\right]$$

If `l' is a natural number and $n \neq 0$, then:

Out[42]=

$$\left\{\text{Sum}\left[(-1)^k \text{ Binomial}[n, k], \{k, 0, l\}\right] == \frac{(-1)^{1+l} (l - n) \text{ Binomial}[n, l]}{n}\right\}$$

NOTE. MMA procedure (i.e., comes with MMA):

In[42]:=

$$\text{Gosper}\left[(-1)^k \binom{n}{k}_*, \{k, 0, l\} \right]$$

If `l' is a natural number and $n \neq 0$, then:

Out[42]=

$$\left\{ \text{Sum}\left[(-1)^k \text{ Binomial}[n, k], \{k, 0, l\} \right] == \frac{(-1)^{1+l} (l - n) \text{ Binomial}[n, l]}{n} \right\}$$

NOTE. MMA procedure (i.e., comes with MMA):

In[43]:=

$$\sum_{k=0}^{l} (-1)^k \binom{n}{k}_*$$

Out[43]=

$$(-1)^l \text{ Binomial}[-1 + n, l]$$

Creative Telescoping:

In[44]:=

$$\text{Zb}\left[\binom{n}{k}_*, \{k, 0, n\}, n, 1\right]$$

If `n' is a natural number, then:

Out[44]=

$$\{2\,\text{SUM}[n] - \text{SUM}[1 + n] == 0\}$$

NOTE. Zeilberger's algorithm delivers a proof certificate:

Creative Telescoping:

In[44]:=

$$\text{Zb}\left[\binom{n}{k}_*, \{k, 0, n\}, n, 1\right]$$

If `n' is a natural number, then:

Out[44]=

$$\{2\, \text{SUM}[n] - \text{SUM}[1 + n] == 0\}$$

NOTE. Zeilberger's algorithm delivers a proof certificate:

In[45]:=

$$\text{Zb}\left[\binom{n}{k}_*, k, n, 1\right]$$

Out[45]=

$$\{2\, F[k, n] - F[k, 1 + n] == \triangle_k[F[k, n]\, R[k, n]]\}$$

In[47]:=

$$\text{show}[R]$$

Out[47]=

$$\frac{k}{1 - k + n}$$

q-Telescoping:

In[48]:=

```
<< RISC`qZeil`
```

Package q-Zeilberger version 4.50 written by Axel Riese

Copyright Research Institute for Symbolic Computation (RISC),

Johannes Kepler University, Linz, Austria

NOTE. The qZeil package implements objects like:

q-Telescoping:

In[48]:=

```
<< RISC`qZeil`
```

Package q-Zeilberger version 4.50 written by Axel Riese

Copyright Research Institute for Symbolic Computation (RISC),

Johannes Kepler University, Linz, Austria

NOTE. The qZeil package implements objects like:

In[49]:=

```
? qBinomial
```

qBinomial[n, k, q] represents the q–binomial coefficient of n and k in base q given by

$$\binom{n}{k}_q := \begin{cases} \dfrac{\left(q^{n-k+1};q\right)_k}{(q;q)_k} & \text{for } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In[49]:=

```
? qBinomial
```

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$$\binom{n}{k}_q := \begin{cases} \dfrac{(q^{n-k+1};q)_k}{(q;q)_k} & \text{for } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In[49]:=

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$$\binom{n}{k}_q := \begin{cases} \dfrac{\left(q^{n-k+1}; q\right)_k}{(q; q)_k} & \text{for } k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In[50]:=

? qPochhammer

qPochhammer[a, q] denotes the limit of qPochhammer[a, q, k] for k approaching (positive) infinity.

qPochhammer[a, q, k] represents the q–shifted factorial of a in base q with index k given by

$$(a; q)_k := \begin{cases} (1-a)(1-a\,q)\cdots(1-a\,q^k), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ \dfrac{1}{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)\cdots\left(1-\frac{a}{q^k}\right)}, & \text{if } k < 0. \end{cases}$$

In[51]:=

```
qP = qPochhammer;
```

In[52]:=

$$\binom{n\_}{k\_}_{q\_} \text{ := qBinomial}[n, k, q]$$

In[51]:=

```
qP = qPochhammer;
```

In[52]:=

$$\binom{n\_}{k\_}_{q\_} := \text{qBinomial}[n, k, q]$$

In[54]:=

```
<< RISC`qSimplify`
```

In[55]:=

$$\binom{n}{k}_q \Big/ \frac{\text{qP}[q, q, n]}{\text{qP}[q, q, k]\ \text{qP}[q, q, n-k]} \ \text{// qSimplify}$$

Out[55]=

```
1
```

In[56]:=

$$\left\{\binom{n}{k}_q \Big/ \binom{n-1}{k}_q,\ \binom{n}{k}_q \Big/ \binom{n}{k-1}_q\right\} \text{// qSimplify}$$

Out[56]=

$$\left\{\frac{1 - q^n}{1 - q^{-k+n}},\ \frac{1 - q^{1-k+n}}{1 - q^k}\right\}$$

In[58]:=

$$\texttt{qTelescope}\left[(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_q q^{\texttt{Binomial}[k,2]}, \{k, 0, l\}, \{n\}\right]$$

Out[58]=

$$\left\{\texttt{Sum}\left[(-1)^k q^{-\frac{k}{2}+\frac{k^2}{2}} \texttt{qBinomial}[n, k, q], \{k, 0, l\}\right] ==\right.$$

$$\left.(-1)^l q^{\frac{l}{2}+\frac{l^2}{2}} \texttt{qBinomial}[-1+n, l, q], \{n \neq 0\}\right\}$$

In[58]:=

$$\texttt{qTelescope}\left[(\texttt{-1})^{\texttt{k}}\begin{pmatrix}\texttt{n}\\\texttt{k}\end{pmatrix}_{\texttt{q}}\texttt{q}^{\texttt{Binomial[k,2]}}, \{\texttt{k, 0, l}\}, \{\texttt{n}\}\right]$$

Out[58]=

$$\left\{\texttt{Sum}\left[(\texttt{-1})^{\texttt{k}}\texttt{q}^{-\frac{\texttt{k}}{2}+\frac{\texttt{k}^2}{2}}\texttt{qBinomial[n, k, q]}, \{\texttt{k, 0, l}\}\right] == \right.$$

$$\left. (\texttt{-1})^{\texttt{l}}\texttt{q}^{\frac{\texttt{l}}{2}+\frac{\texttt{l}^2}{2}}\texttt{qBinomial[-1 + n, l, q]}, \{\texttt{n} \neq \texttt{0}\}\right\}$$

Recall:

$$\sum_{k=0}^{\ell}(-1)^k q^{\binom{k}{2}}\binom{n}{k}_q = (-1)^{\ell}q^{\binom{\ell}{2}+\ell}\binom{n-1}{\ell}_q.$$

NOTE. MMA has no built-in q-hypergeometric summation.

In[17]:=

$$\text{qZeil}\left[\begin{pmatrix} n \\ k \end{pmatrix}_q q^{\text{Binomial}[k,2]}, \{k, 0, n\}, n, 1\right]$$

Out[17]=

$$\text{SUM}[n] == \left(1 + q^{-1+n}\right) \text{SUM}[-1 + n]$$

In[17]:=

$$\texttt{qZeil}\left[\binom{n}{k}_q q^{\texttt{Binomial[k,2]}}, \{k, 0, n\}, n, 1\right]$$

Out[17]=

$$\texttt{SUM[n]} == \left(1 + q^{-1+n}\right) \texttt{SUM[-1 + n]}$$

For

$$\mathrm{SUM}(n) := \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}$$

this implies

$$\mathrm{SUM}(n) = (1 + q^{n-1})(1 + q^{n-2})\dots(1 + q)\mathrm{SUM}(0)$$
$$= (q; q)_n.$$

$\rightsquigarrow$ Algorithmically this special case of the q-binomial theorem can be found as follows:

In[89]:=

? FindAlphaBeta

FindAlphaBeta is an option for qZeil and qTelescope. With

FindAlphaBeta –> True, qZeil and qTelescope suggest all possible choices of alpha

and beta such that there possibly exists a solution for

F' := F * q^(alpha Binomial[k,2] + beta k).

In[89]:=

```
? FindAlphaBeta
```

FindAlphaBeta is an option for qZeil and qTelescope. With

FindAlphaBeta –> True, qZeil and qTelescope suggest all possible choices of alpha

and beta such that there possibly exists a solution for

F' := F * q^(alpha Binomial[k,2] + beta k).

In[87]:=

$$\text{qZeil}\left[\binom{n}{k}_q, \{k, 0, n\}, n, 1, \text{FindAlphaBeta -> True}\right]$$

```
α     β
---   ---
1     Interval[{-∞, ∞}]
```

In[89]:=

    ? FindAlphaBeta

> FindAlphaBeta is an option for qZeil and qTelescope. With
>
> FindAlphaBeta -> True, qZeil and qTelescope suggest all possible choices of alpha
>
> and beta such that there possibly exists a solution for
>
> F' := F * q^(alpha Binomial[k,2] + beta k).

In[87]:=

$$\text{qZeil}\left[\binom{n}{k}_q,\ \{k,\ 0,\ n\},\ n,\ 1,\ \text{FindAlphaBeta}\ \text{->}\ \text{True}\right]$$

$$
\begin{array}{cc}
\alpha & \beta \\
--- & --- \\
1 & \text{Interval}[\{-\infty,\ \infty\}]
\end{array}
$$

In[95]:=

$$\text{qZeil}\left[\binom{n}{k}_q\ q^{\text{Binomial}[k,2]}\ q^{bk},\ \{k,\ 0,\ n\},\ n,\ 1,\ \{b\}\right]$$

Out[95]=

$$\text{SUM}[n]\ ==\ \left(1 + q^{-1+b+n}\right)\ \text{SUM}[-1+n]$$

In[95]:=

$$\text{qZeil}\left[\binom{n}{k}_q q^{\text{Binomial}[k,2]} q^{bk}, \{k, 0, n\}, n, 1, \{b\}\right]$$

Out[95]=

$$\text{SUM}[n] == \left(1 + q^{-1+b+n}\right) \text{SUM}[-1 + n]$$

In[95]:=

$$\mathtt{qZeil}\left[\binom{n}{k}_q q^{\mathtt{Binomial[k,2]}}\, q^{bk},\ \{k,\ 0,\ n\},\ n,\ 1,\ \{b\}\right]$$

Out[95]=

$$\mathtt{SUM[n]} \;=\; \left(1 + q^{-1+b+n}\right) \mathtt{SUM[-1+n]}$$

In[96]:=

$$\mathtt{qZeil}\left[\binom{n}{k}_q q^{\mathtt{Binomial[k,2]}}\, z^k,\ \{k,\ 0,\ n\},\ n,\ 1\right]$$

Out[96]=

$$\mathtt{SUM[n]} \;=\; \left(1 + q^{-1+n}\, z\right) \mathtt{SUM[-1+n]}$$

In[95]:=
$$\texttt{qZeil}\left[\begin{pmatrix} n \\ k \end{pmatrix}_q q^{\texttt{Binomial[k,2]}} q^{bk}, \{k, 0, n\}, n, 1, \{b\}\right]$$

Out[95]=
$$\texttt{SUM[n]} == \left(1 + q^{-1+b+n}\right) \texttt{SUM[-1+n]}$$

In[96]:=
$$\texttt{qZeil}\left[\begin{pmatrix} n \\ k \end{pmatrix}_q q^{\texttt{Binomial[k,2]}} z^k, \{k, 0, n\}, n, 1\right]$$

Out[96]=
$$\texttt{SUM[n]} == \left(1 + q^{-1+n} z\right) \texttt{SUM[-1+n]}$$

Also the q-version of Zeilberger's algorithm delivers a **proof certificate**:

In[96]:=

$$\texttt{qZeil}\left[\binom{n}{k}_q q^{\texttt{Binomial[k,2]}} z^k, \{k, 0, n\}, n, 1\right]$$

Out[96]=

$$\texttt{SUM[n]} == \left(1 + q^{-1+n} z\right) \texttt{SUM[-1 + n]}$$

In[96]:=

$$\text{qZeil}\left[\binom{n}{k}_q q^{\text{Binomial}[k,2]} z^k, \{k, 0, n\}, n, 1\right]$$

Out[96]=

$$\text{SUM}[n] == \left(1 + q^{-1+n} z\right) \text{SUM}[-1 + n]$$

In[97]:=

$$\text{Cert}[]$$

Out[97]=

$$-\frac{q^{-1-k+n}\left(-q^k + q^n\right) z}{-1 + q^n}$$

In[96]:=

$$\texttt{qZeil}\left[\binom{n}{k}_q q^{\texttt{Binomial[k,2]}} z^k, \{k, 0, n\}, n, 1\right]$$

Out[96]=

$$\texttt{SUM[n]} \;==\; \left(1 + q^{-1+n}\, z\right)\, \texttt{SUM[-1+n]}$$

In[97]:=

$$\texttt{Cert[]}$$

Out[97]=

$$-\frac{q^{-1-k+n}\left(-q^k + q^n\right) z}{-1 + q^n}$$

$$f_{n,k} - (1 + q^{n-1}) f_{n-1,k} = g_{n,k} - g_{n,k-1}$$

and

$$g_{n,k} = \mathrm{cert}(n, k)\, f_{n,k}.$$

Details on Cert[]: P. and Riese (1997)

Computer Algebra Session 2

In[6]:= `<< RISC`GeneratingFunctions` `

> Package GeneratingFunctions version 0.8 written by Christian Mallinger
> Copyright Research Institute for Symbolic Computation (RISC),
> Johannes Kepler University, Linz, Austria

In[7]:= $\text{LF}[n\_] := \sum_{k=0}^{n} \binom{2\,n - k}{k}_{*}$

In[8]:= `Table[LF[n], {n, 0, 5}]`

Out[8]= `{1, 2, 5, 13, 34, 89}`

In[9]:= `GuessRE[%, a[n]]`

Out[9]= `{{a[n] - 3 a[1 + n] + a[2 + n] == 0, a[0] == 1, a[1] == 2}, ogf}`

In[10]:= `Table[Fibonacci[n], {n, 0, 11}]`

Out[10]= `{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}`

In[11]:= `Table[Fibonacci[2 n], {n, 0, 5}]`

Out[11]= `{0, 1, 3, 8, 21, 55}`

In[12]:= `GuessRE[%, a[n]]`

Out[12]= `{{a[n] - 3 a[1 + n] + a[2 + n] == 0, a[0] == 0, a[1] == 1}, ogf}`

To PROVE our guess, we compute the recurrence of the
Fibonacci subsequence in question :

In[13]:= **? *RESub***

RecurrenceEquationSubsequence[re,a[n],m*n+k] gives a recurrence that
is satisfied by a subsequence of the form a[m*n+k] of every solution a[n] of
the input recurrence re.

Alias: RESubsequence

See also: REInfo, REInterlace

In[14]:= **RESubsequence[{-a[n] - a[1 + n] + a[2 + n] == 0, a[0] == 0, a[1] == 1},
a[n], 2 n]**

Out[14]= $\{a[n] - 3\,a[1 + n] + a[2 + n] == 0,\ a[0] == 0,\ a[1] == 1\}$

Next we find a q-analogue to

$$LF[n\_] := \sum_{k=0}^{n} \binom{2\,n - k}{k}_*$$

In[15]:= `qZeil[(binom{2 n - k}{k})_q, {k, 0, n}, n, 2, FindAlphaBeta → True]`

```
α      β
---    ---
-3     Interval[{-∞, ∞}]
-2     Interval[{-∞, ∞}]
-1     Interval[{-∞, ∞}]
0      Interval[{-∞, ∞}]
1      Interval[{-∞, ∞}]
2      Interval[{-∞, ∞}]
3      Interval[{-∞, ∞}]
4      Interval[{-∞, ∞}]
5      Interval[{-∞, ∞}]
6      Interval[{-∞, ∞}]
7      Interval[{-∞, ∞}]
```

In[16]:= `qZeil[(binom{2 n - k}{k})_q q^{(-3) binom{k}{2}_* + b k}, {k, 0, n}, n, 2, {b}]`

Out[16]= { }

In[17]:= `qZeil[(binom{2 n - k}{k})_q q^{1 binom{k}{2}_* + b k}, {k, 0, n}, n, 2, {b}]`

Out[17]= { }

In[18]:= `qZeil[ `$\binom{2\,n-k}{k}_q \, q^{2\binom{k}{2}_*+b\,k}$ `, {k, 0, n}, n, 2, {b}]`

Out[18]= $\text{SUM}[n] == -q^{-7+2\,b+4\,n}\,\text{SUM}[-2+n] - \dfrac{\left(-q^3 - q^{b+2\,n} - q^{1+b+2\,n}\right)\,\text{SUM}[-1+n]}{q^3}$

In[19]:= `qZeil[ `$\binom{2\,n-k}{k}_q \, q^{3\binom{k}{2}_*+b\,k}$ `, {k, 0, n}, n, 2, {b}]`

Out[19]= { }

In[20]:= `qZeil[ `$\binom{2\,n-k}{k}_q \, q^{7\binom{k}{2}_*+b\,k}$ `, {k, 0, n}, n, 2, {b}]`

We choose b = 1 :

In[21]:= `qZeil[ `$\binom{2\,n-k}{k}_q \, q^{2\binom{k}{2}_*+k}$ `, {k, 0, n}, n, 2, {b}]`

Out[21]= $\text{SUM}[n] == -q^{-5+4\,n}\,\text{SUM}[-2+n] - \dfrac{\left(-q^2 - q^{2\,n} - q^{1+2\,n}\right)\,\text{SUM}[-1+n]}{q^2}$

In[23]:= `Cert[]`

Out[23]= $-\big(\big(q^{-5-4\,k+2\,n}\,\left(-q^k + q^n\right)\,\left(q^k + q^n\right)\,\left(-q^{1+2\,k} + q^{2\,n}\right)$
$\left(-q^{8+4\,k} - q^{9+4\,k} + q^{6\,n} + q^{6+2\,k+2\,n} + q^{5+3\,k+2\,n} + q^{6+3\,k+2\,n} + q^{7+3\,k+2\,n} +\right.$
$\left. q^{5+4\,k+2\,n} - q^{3+k+4\,n} - q^{2+2\,k+4\,n} - q^{3+2\,k+4\,n} - q^{4+2\,k+4\,n}\right)\big) /$
$\big(\left(q^k - q^{2\,n}\right)\,\left(-q^{1+k} + q^{2\,n}\right)\,\left(-q^{2+k} + q^{2\,n}\right)\,\left(-q^{3+k} + q^{2\,n}\right)\big)\big)$

Another sum for the Fibonacci sequence in question :

In[22]:= $\mathsf{RF[}n\mathsf{\_]}$:= $\displaystyle\sum_{k=-n}^{n} \binom{2\,n+2}{n-5\,k}_{*} \frac{10\,k+2}{2\,n+2}$

In[24]:= $\mathsf{Table[RF[n], \{n, 0, 5\}]}$

Out[24]= $\{1, 2, 5, 13, 34, 89\}$

Indeed, with Zeilberger's algorithm one instantly proves that for n≥0:

$$\sum_{k=0}^{n} \binom{2\,n-k}{k}_{*} = \sum_{k=-n}^{n} \binom{2\,n+2}{n-5\,k}_{*} \frac{10\,k+2}{2\,n+2} \ .$$

TASK : Find a q-analogue of this identity.

In[25]:= ? qBrackets

qBrackets[n, q] represents the basic number n in base q given by $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$.

In[26]:= $\mathsf{qB = qBrackets;}$

In[27]:= $\mathsf{qZeil\left[\dbinom{2\,n+2}{n-5\,k}_q \dfrac{qB[10\,k+2, q]}{qB[2\,n+2, q]} , \{k, -n, n\}, n, 2, FindAlphaBeta \to True \right]}$

qBrackets[n, q] represents the basic number n in base q given by $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$.

In[26]:= `qB = qBrackets;`

In[27]:= $\text{qZeil}\left[\dbinom{2\,n+2}{n-5\,k}_q \dfrac{\text{qB}[10\,k+2,\,q]}{\text{qB}[2\,n+2,\,q]} , \{k,\, -n,\, n\},\, n,\, 2,\, \text{FindAlphaBeta} \to \text{True}\right]$

```
α     β
---   ---
20    Interval[{-∞, ∞}]
21    Interval[{-∞, ∞}]
22    Interval[{-∞, ∞}]
23    Interval[{-∞, ∞}]
24    Interval[{-∞, ∞}]
25    Interval[{-∞, ∞}]
26    Interval[{-∞, ∞}]
27    Interval[{-∞, ∞}]
28    Interval[{-∞, ∞}]
29    Interval[{-∞, ∞}]
30    Interval[{-∞, ∞}]
```

In[28]:= $\text{qZeil}\left[\dbinom{2\,n+2}{n-5\,k}_q \dfrac{\text{qB}[10\,k+2,\,q]}{\text{qB}[2\,n+2,\,q]}\, q^{20\binom{k}{2}_*+8\,k} ,\, \{k,\, -n,\, n\},\, n,\, 2\right]$

Out[28]= $\{\}$

In[29]:= $\text{qZeil}\left[\dbinom{2\,n+2}{n-5\,k}_q \dfrac{\text{qB}[10\,k+2,\,q]}{\text{qB}[2\,n+2,\,q]}\, q^{20\binom{k}{2}_*+9\,k} ,\, \{k,\, -n,\, n\},\, n,\, 2\right]$

Out[29]= $\text{SUM}[n] == -q^{-5+4\,n}\, \text{SUM}[-2+n] - \dfrac{\left(-q^2 - q^{2\,n} - q^{1+2\,n}\right)\, \text{SUM}[-1+n]}{q^2}$

In[30]:= `Cert[]`

Out[30]= $-\left(\left(q^{-3-10\,k+n}\left(-q^{5\,k}+q^{n}\right)\left(-q^{1+5\,k}+q^{n}\right)\left(-q^{2+5\,k}+q^{n}\right)\left(-q^{3+5\,k}+q^{n}\right)\left(-q^{4+5\,k}+q^{n}\right)\right)\,/\right.$
$\left.\left(\left(-1+q^{1+5\,k}\right)\left(1+q^{1+5\,k}\right)\left(-1+q^{n}\right)\left(1+q^{n}\right)\right.\right.$
$\left.\left.\left(-q+q^{n}\right)\left(q+q^{n}\right)\left(-q+q^{2\,n}\right)\left(-1+q^{1+2\,n}\right)\right)\right)$

The choice $b=9$ can be justified as follows :

In[31]:= $\text{qLF}[n\_] := \sum_{k=0}^{n}\binom{2\,n-k}{k}_{q}q^{2\binom{k}{2}_{*}+k}$

In[32]:= `Table[qLF[n] // qSimplify // Factor, {n, 0, 3}]`

Out[32]= $\{1,\ 1+q,\ 1+q+q^{2}+q^{3}+q^{4},\ 1+q+q^{2}+q^{3}+2\,q^{4}+2\,q^{5}+2\,q^{6}+q^{7}+q^{8}+q^{9}\}$

In[33]:= $\text{qRF}[n\_] := \sum_{k=-n}^{n}\binom{2\,n+2}{n-5\,k}_{q}\frac{\text{qB}[10\,k+2,\,q]}{\text{qB}[2\,n+2,\,q]}q^{20\binom{k}{2}_{*}+b\,k}$

In[34]:= `Table[qRF[n] // qSimplify // Factor // Expand, {n, 0, 3}]`

Out[34]= $\{1,\ 1+q,\ 1+q+q^{2}+q^{3}+q^{4},$
$1+q+q^{2}+2\,q^{3}+2\,q^{4}+2\,q^{5}+2\,q^{6}+q^{7}+q^{8}+q^{9}-q^{12-b}\}$

As a consequence, $b=9$ is necessary for equality!

Summary : we rediscovered and proved the Andrews-Schur identity for n⩾0,

$$\sum_{k=0}^{n} \binom{2n-k}{k}_{q} q^{k^2} = \sum_{k=-n}^{n} \binom{2n+2}{n-5k}_{q} \frac{1-q^{10k+2}}{1-q^{2n+2}} q^{10k^2-k} .$$

Summary : we rediscovered and proved the Andrews-Schur identity for n≥0,

$$\sum_{k=0}^{n} \binom{2n-k}{k}_q q^{k^2} = \sum_{k=-n}^{n} \binom{2n+2}{n-5k}_q \frac{1-q^{10k+2}}{1-q^{2n+2}} q^{10k^2-k}.$$

Note:
$$\binom{2n-k}{k}_q = \frac{(q;q)_{2n-k}}{(q;q)_k (q;q)_{2n-2k}} \rightarrow \frac{1}{(q;q)_k} \text{ for } n \rightarrow \infty,$$

and
$$\binom{2n+2}{n-5k}_q = \frac{(q;q)_{2n+2}}{(q;q)_{n-5k}(q;q)_{n+5k+2}} \rightarrow \frac{1}{(q;q)_\infty} \text{ for } n \rightarrow \infty;$$
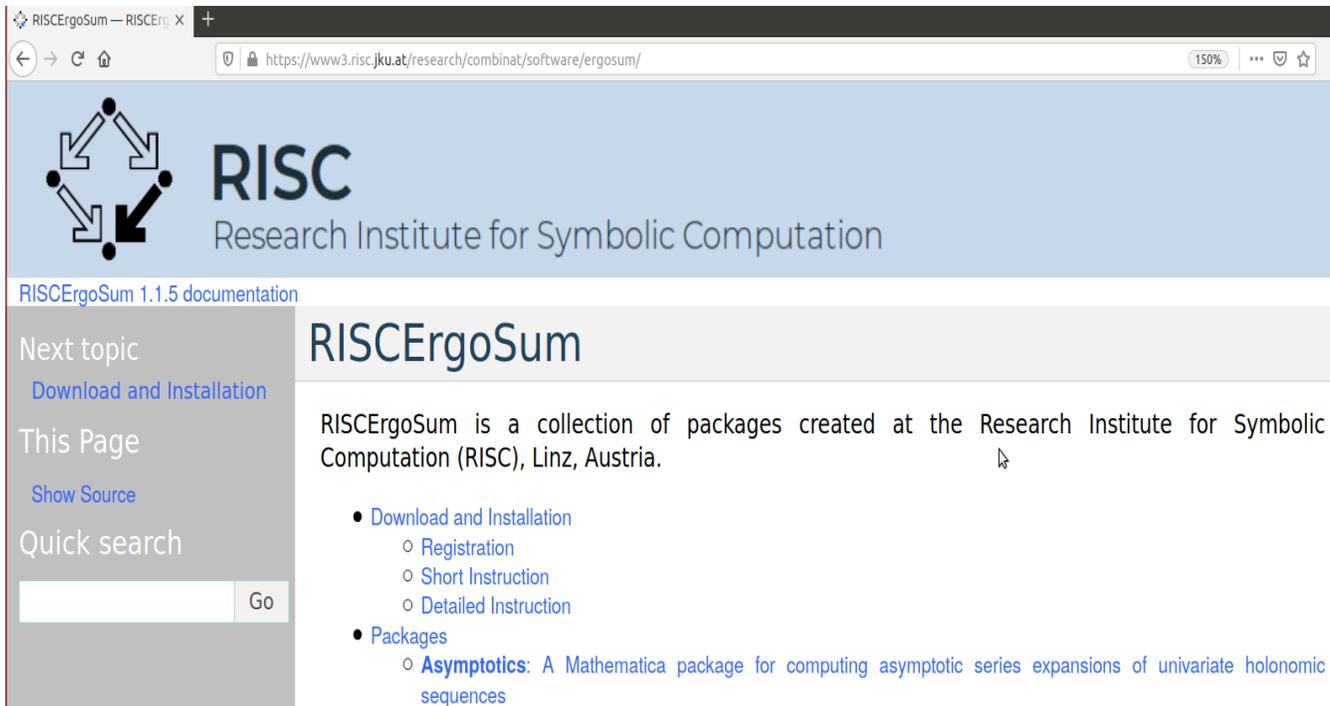
hence, by Jacobi's triple product identity, Andrews-Schur turns into

Summary : we rediscovered and proved the Andrews-Schur identity for n⩾0,

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Note: $\binom{2n-k}{k}_q = \dfrac{(q;q)_{2n-k}}{(q;q)_k (q;q)_{2n-2k}} \to \dfrac{1}{(q;q)_k}$ for $n \to \infty$,

and

$$\binom{2n+2}{n-5k}_q = \frac{(q;q)_{2n+2}}{(q;q)_{n-5k}(q;q)_{n+5k+2}} \to \frac{1}{(q;q)_\infty} \text{ for } n \to \infty;$$

hence, by Jacobi's triple product identity, Andrews-Schur turns into

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} \quad \text{[1st Rogers-Ramanujan id.]}$$

How to download the RISC packages?

How to download the RISC packages?

Step 1: go to

- Download and Installation
    - ○ Registration
    - ○ Short Instruction
    - ○ Detailed Instruction
- Packages
    - ○ **Asymptotics**: A Mathematica package for computing asymptotic series expansions of univariate holonomic sequences
    - ○ **Dependencies**: A Mathematica package for computing algebraic relations of C-finite sequences and multi-sequences
    - ○ **Engel**: A Mathematica implementation of q-Engel Expansion
    - ○ **fastZeil**: The Paule/Schorn Implementation of Gosper's and Zeilberger's Algorithms
    - ○ **GeneratingFunctions**: A Mathematica package for manipulations of univariate holonomic functions and sequences
    - ○ **GenOmega**: A Mathematica implementation of Guo-Niu Han's general Algorithm for MacMahon's Partition Analysis
    - ○ **Guess**: A Mathematica package for guessing multivariate recurrence equations
    - ○ **HolonomicFunctions**: A Mathematica package for dealing with multivariate holonomic functions, including closure properties, summation, and integration
    - ○ **MultiSum**: A Mathematica package for proving hypergeometric multi-sum identities
    - ○ **Omega**: A Mathematica implementation of Partition Analysis
    - ○ **OreSys**: A Mathematica implementation of several algorithms for uncoupling systems of linear Ore operator equations
    - ○ **pqTelescope**: A Mathematica implementation of a generalization of Gosper's algorithm to bibasic hypergeometric summation
    - ○ **qGeneratingFunctions**: A Mathematica package for manipulations of univariate q-holonomic functions and sequences
    - ○ **qMultiSum**: A Mathematica package for proving q-hypergeometric multi-sum identities
    - ○ **qZeil**: A Mathematica implementation of q-analogues of Gosper's and Zeilberger's algorithm
    - ○ **qSimplify**: A Mathematica package for the simplification of q-expressions
    - ○ **Stirling**: A Mathematica package for computing recurrence equations of sums involving Stirling numbers or Eulerian numbers
    - ○ **SumCracker**: A Mathematica implementation of several algorithms for identities and inequalities of special sequences, including summation problems
- Notes for Authors
    - ○ Getting the Sources
    - ○ Development

Step 2: email to me and download the RISCErgosum package

Registration

The download is password protected. To get the password, send an email to Peter Paule . It will be given for free to all researchers and non-commercial users.

Note that redistribution of RISCErgoSum is not allowed.

Short Instruction

Get one of the files riscergosum-1.1.5.tgz or riscergosum-1.1.5.zip from http://www.risc.jku.at /research/combinat/software/source/mathematica/ergosum and extract it in the `Applications` subdirectory of your local Mathematica `$UserBaseDirectory`.

Detailed Instruction

Start Mathematica and type:

```
$UserBaseDirectory
```

at the command prompt.

Suppose this value is `/home/hemmecke/.Mathematica`. Exit Mathematica and extract the .tgz file into the `Applications` subdirectory.

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Under Linux (substitute appropriately):

```
tar xvzf riscergosum-x.y.z.tgz -C /home/hemmecke/.Mathematica/Applications
```

Under MacOS the output from `$UserBaseDirectory` will most probably look similar to `/Users/hemmecke/Library/Mathematica`. Thus, in a terminal type:

```
tar xvzf riscergosum-x.y.z.tgz -C /Users/hemmecke/Library/Mathematica/Applications
```

Note that the `Library` folder is usually hidden in the file browser.

Under Windows (substitute appropriately) unzip the `riscergosum-x.y.z.zip` and move the files (and subdirectories) to `$UserBaseDirectory/Applications`.

After extracting the files into the `Applications` subdirectory, you should find there the subdirectories

```
Applications/RISC
Applications/InvEulerPhi
```

and in particular the file

```
Applications/RISC/MultiSum.m
```

After extracting the files into the `Applications` subdirectory, you should find there the subdirectories

```
Applications/RISC
Applications/InvEulerPhi
```

and in particular the file

```
Applications/RISC/MultiSum.m
```

If you don't see `RISC` and `InvEulerPhi` directly under `Applications`, then something went wrong and the following will not work.
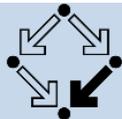
Start Mathematica and type:

```
<< RISC`MultiSum`
```

You should then see something like:

```
Package MultiSum version 2.3 written by Kurt Wegschaider
enhanced by Axel Riese and Burkhard Zimmermann
Copyright 1995-2009, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria
```

NOTE. For each of the packages some extra information is given; for example, for qZeil:

Go

qZeil: A Mathematica implementation of q-analogues of Gosper's and Zeilberger's algorithm

This package is part of the RISCErgoSum bundle. See Download and Installation.

Short Description

`qZeil` is a Mathematica implementation of q-analogues of Gosper's and Zeilberger's algorithm for proving and finding indefinite and definite q-hypergeometric summation identities.

The package has been developed by Axel Riese, a former member of the RISC Combinatorics group.

Author

- Axel Riese

Accompanying Files

- WhatsNew.txt an ASCII-documentation of all changes and new features since version 1.4
- qZeilExamples.nb a Mathematica notebook consisting of about 500 examples

Literature

Information on the installation and use of the package (together with its theoretical background) can be found in the paper

- P. Paule and A. Riese, **A Mathematica q-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping**, in Special Functions, q-Series and Related Topics, Fields Inst. Commun., Vol. 14, pp. 179-210, 1997. [pdf]

Moreover, an up-to-date manual for the package is presented in Riese's Ph.D. thesis

- A. Riese, **Contributions to Symbolic q-Hypergeometric Summation**, Ph.D. Thesis, RISC, J. Kepler

Not all RISC combinatorics packages are bundled in RISCErgoSum:

Not all RISC combinatorics packages are bundled in RISCErgoSum:

RISC
Research Institute for Symbolic Computation

JYU
JOHANNES KEPLER
UNIVERSITÄT LINZ

The summation package Sigma

Short Description

Sigma is a Mathematica package that can handle multisums in terms of indefinite nested sums and products. The summation principles of Sigma are: telescoping, creative telescoping and recurrence solving. The underlying machinery of Sigma is based on difference field theory. The package has been developed by Carsten Schneider, a member of the RISC Combinatorics group.

Registration and Legal Notices

The source code for this package is password protected. To get the password send an email to Peter Paule. It will be given for free to all researchers and non-commercial users.

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A Note on Encoded Files

This package contains one or more Mathematica input files which are encoded. Those files cannot be read or modified directly as plain text, but can be loaded into Mathematica just like any normal input file (i.e., with `<<"file"` or `Get["file"]`). There is no need (and also no way) to decode them by using additional software or a special key.

If loading an encoded file causes a syntax error, open it with a text editor and remove any blank lines at the beginning (for some reason your Mac could have inserted them silently...).

The Package

The *Sigma* package consists of the file

- Sigma.m (encoded)

For a demo see the built in help of Sigma.

Literature

For a detailed description and a collection of non-trivial examples we refer, e.g., to the article

C. Schneider, **Symbolic Summation Assists Combinatorics**, Sem.Lothar.Combin. 56, pp.1-36. 2007. Article B56b. [pdf]

For further literature click [here].

Symbolic Summation in Quantum Field Theory

JKU Collaboration with DESY (Berlin–Zeuthen)
(Deutsches Elektronen–Synchrotron)

Project leader: Carsten Schneider (RISC)
Partners: Johannes Blümlein (DESY)
                  Peter Paule (RISC)

Evaluation of Feynman diagrams
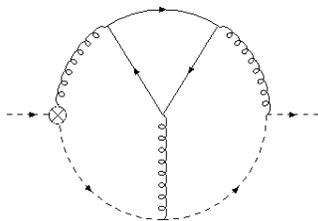
Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman
integrals
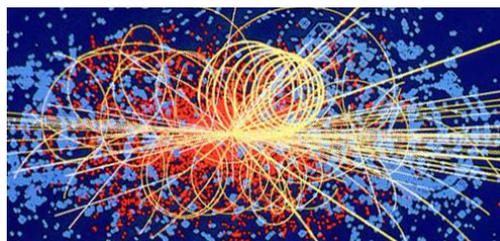
Evaluations required for the
LHC experiment at CERN

DESY

processable by physicists

RISC
(symbolic summation)

simple sum expressions

$$\sum f(N,\epsilon,k)$$

multi-sums
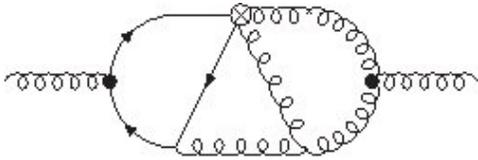
$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$||$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times \frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\left[4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k)\right.$$

$$- (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)$$

$$\left. + 2S_1(s-1) - 2S_1(r+s)\right] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} = \quad \text{(using \texttt{Sigma.m}, \texttt{EvaluateMultiSums.m} and J. Ablinger's \texttt{HarmonicSums.m} package)}$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2$$

$$+ \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\right)S_2(N) + \left(\frac{29}{3} - (-1)^N\right)S_3(N)\right.$$

$$+ \left(2 + 2(-1)^N\right)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\right)S_1(N) + \left(\frac{3}{4} + (-1)^N\right)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\left(\frac{2(3N-5)}{N(N+1)} + \left(26 + 4(-1)^N\right)S_1(N) + \frac{4(-1)^N}{N+1}\right)$$

$$+ \left(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\right)S_2(N) + S_{-2}(N)\left(10S_1(N)^2 + \left(\frac{8(-1)^N(2N+1)}{N(N+1)}\right.\right.$$

$$+ \frac{4(3N-1)}{N(N+1)}\right)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \left(-22 + 6(-1)^N\right)S_2(N) - \frac{16}{N(N+1)}\right)$$

$$+ \left(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\right)S_3(N) + \left(\frac{19}{2} - 2(-1)^N\right)S_4(N) + \left(-6 + 5(-1)^N\right)S_{-4}(N)$$

$$+ \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\right)S_{2,1}(N) + \left(20 + 2(-1)^N\right)S_{2,-2}(N) + \left(-17 + 13(-1)^N\right)S_{3,}$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \left(24 + 4(-1)^N\right)S_{-3,1}(N) + \left(3 - 5(-1)^N\right)S_{2,1}$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$\boxed{F_0(N)} =$ (using Sigma.m, EvaluateMultiSums.m and J. Ablinger's HarmonicSums.m package)

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \left(\frac{35N^2 - 2N - 5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2$$

$$+ \left(-\frac{4(13N+5)}{N^2(N+1)^2} + \left(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\right)S_2(N) + \left(\frac{29}{3} - (-1)^N\right)S_3(N)\right.$$

$$+ \left(2 + 2(-1)^N\right)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \left(\frac{3}{4} + (-1)^N\right)S_2(N)^2$$

$$- 2(-1) \cdots \qquad (-1)^N)S_1(N) + \frac{4(-1)^N}{\cdots}\Big)$$

$$+ \left(\frac{(-1}{2N} \cdots \right) \cdots N)^2 + \left(\frac{8(-1)^N}{N(\cdots} \cdots \right)$$

$$+ \frac{4(3N \cdots}{N(N} \cdots \qquad 6(-1)^N)S_2(N) - \frac{\cdots}{N(N+1)}\right)$$

$$+ \left(\frac{(-1)^N}{N(N+1)} - \frac{\cdots}{3N}\right)S_3(N) + \left(\frac{\cdots}{2} - 2(-1)^N\right)S_4(N) + \left(-6 + 5(-1)^N\right)S_{-4}(N)$$

$$+ \left(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\right)S_{2,1}(N) + \left(20 + 2(-1)^N\right)S_{2,-2}(N) + \left(-17 + 13(-1)^N\right)S_{3,\cdots}$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \left(24 + 4(-1)^N\right)S_{-3,1}(N) + \left(3 - 5(-1)^N\right)S_{2,1,\cdots}$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Challenges of the project

About 1000 difficult Feynman diagrams have been treated so far

(some took 50 days of calculation time)

$\downarrow$

About a million multi-sums have been simplified

(most were double and triple sums)

Resources

▶ up to 9 full time employed researchers at RISC/DESY

▶ 4 up-to-date mainframe DESY computers at RISC
  + exploiting DESY's computer farms

▶ New computer algebra/special functions technologies
  (new/tuned algorithms, efficient implementations,...)

Not all RISC combinatorics packages are bundled in RISCErgoSum:
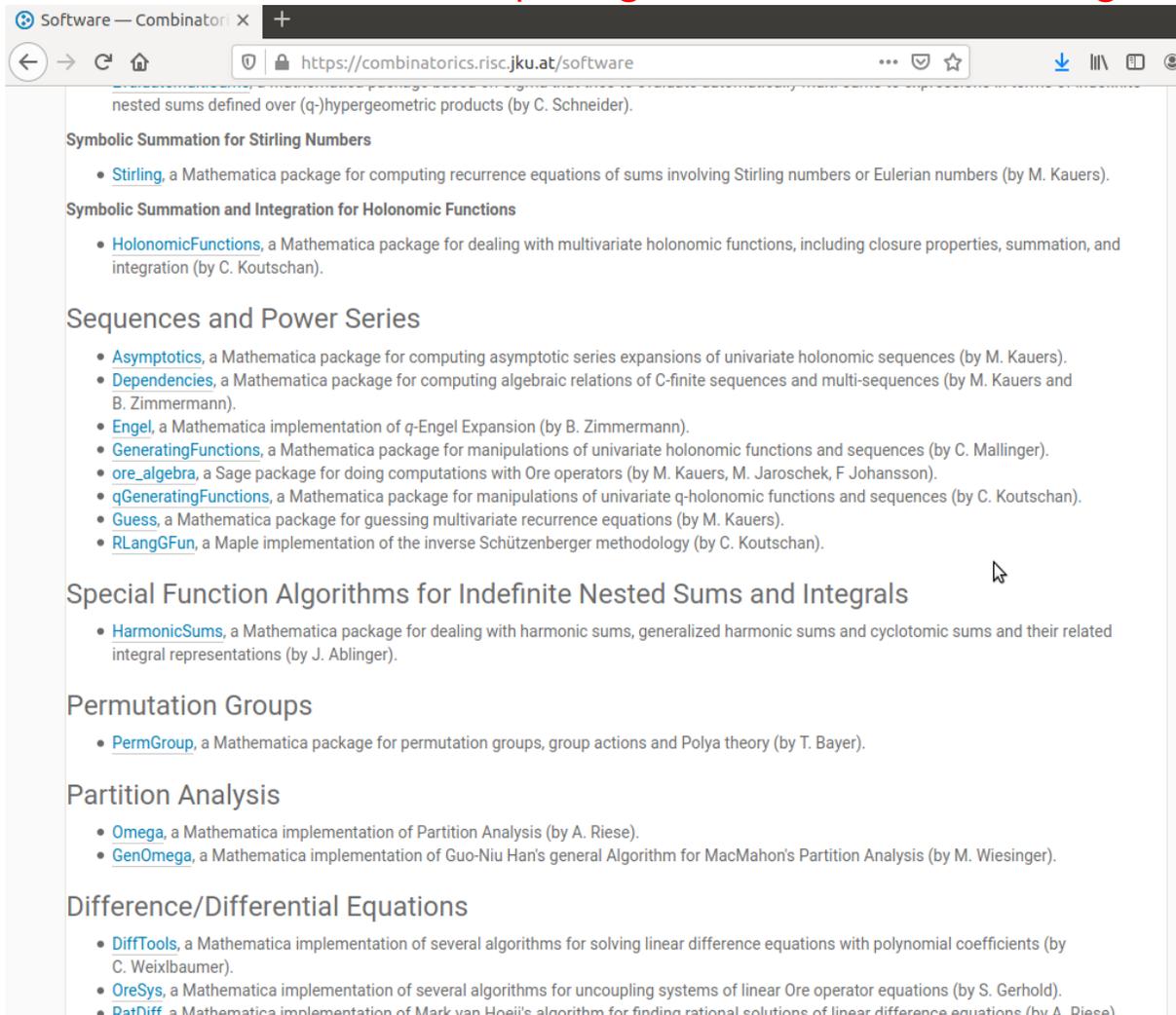
Software — Combinatori ✕ +

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nested sums defined over (q-)hypergeometric products (by C. Schneider).

Symbolic Summation for Stirling Numbers

- Stirling, a Mathematica package for computing recurrence equations of sums involving Stirling numbers or Eulerian numbers (by M. Kauers).

Symbolic Summation and Integration for Holonomic Functions

- HolonomicFunctions, a Mathematica package for dealing with multivariate holonomic functions, including closure properties, summation, and integration (by C. Koutschan).

Sequences and Power Series

- Asymptotics, a Mathematica package for computing asymptotic series expansions of univariate holonomic sequences (by M. Kauers).
- Dependencies, a Mathematica package for computing algebraic relations of C-finite sequences and multi-sequences (by M. Kauers and B. Zimmermann).
- Engel, a Mathematica implementation of q-Engel Expansion (by B. Zimmermann).
- GeneratingFunctions, a Mathematica package for manipulations of univariate holonomic functions and sequences (by C. Mallinger).
- ore_algebra, a Sage package for doing computations with Ore operators (by M. Kauers, M. Jaroschek, F Johansson).
- qGeneratingFunctions, a Mathematica package for manipulations of univariate q-holonomic functions and sequences (by C. Koutschan).
- Guess, a Mathematica package for guessing multivariate recurrence equations (by M. Kauers).
- RLangGFun, a Maple implementation of the inverse Schützenberger methodology (by C. Koutschan).

Special Function Algorithms for Indefinite Nested Sums and Integrals

- HarmonicSums, a Mathematica package for dealing with harmonic sums, generalized harmonic sums and cyclotomic sums and their related integral representations (by J. Ablinger).

Permutation Groups

- PermGroup, a Mathematica package for permutation groups, group actions and Polya theory (by T. Bayer).

Partition Analysis

- Omega, a Mathematica implementation of Partition Analysis (by A. Riese).
- GenOmega, a Mathematica implementation of Guo-Niu Han's general Algorithm for MacMahon's Partition Analysis (by M. Wiesinger).

Difference/Differential Equations

- DiffTools, a Mathematica implementation of several algorithms for solving linear difference equations with polynomial coefficients (by C. Weixlbaumer).
- OreSys, a Mathematica implementation of several algorithms for uncoupling systems of linear Ore operator equations (by S. Gerhold).
- RatDiff, a Mathematica implementation of Mark van Hoeij's algorithm for finding rational solutions of linear difference equations (by A. Riese).

but "HolonomicFunctions" is!

Not all RISC combinatorics packages are bundled in RISCErgoSum:

🔒 https://www3.risc.jku.at/research/combinat/software/ergosum/RISC/Holo

RISC
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Table of Contents

Previous topic

Next topic

This Page

Show Source

Quick search

[_____] Go

HolonomicFunctions: A Mathematica package for dealing with multivariate holonomic functions, including closure properties, summation, and integration

This package is part of the RISCErgoSum bundle. See Download and Installation.

Short Description

The HolonomicFunctions package allows to deal with multivariate holonomic functions and sequences in an algorithmic fashion. For this purpose the package can compute annihilating ideals and execute closure properties (addition, multiplication, substitutions) for such functions. An annihilating ideal represents the set of linear differential equations, linear recurrences, q-difference equations, and mixed linear equations that a given function satisfies. Summation and integration of multivariate holonomic functions can be performed via creative telescoping. As subtasks, the following functionalities have been implemented in HolonomicFunctions: computations in Ore algebras (noncommutative polynomial arithmetic with mixed difference-differential operators), noncommutative Gröbner bases, and solving of coupled linear systems of differential or difference equations.

Author

- Christoph Koutschan

Accompanying Files

Not all RISC combinatorics packages are bundled in RISCErgoSum:

(handwritten annotation) "But Holonomic Functions" is !

for Mathematica version 6

- ExamplesV5.2.nb
  for Mathematica version 5.2

Right now you are using Version 1.7.3 released on March 17, 2017. This version is compatible with Mathematica versions from 5.2 to 11.0. Please report any bugs and comments to Christoph Koutschan.

Literature

The theoretical background of the algorithms implemented in HolonomicFunctions and how to use the package, is described in

- C. Koutschan, **Advanced Applications of the Holonomic Systems Approach**, RISC, Johannes Kepler University, Linz. PhD Thesis. September 2009. [pdf]
- C. Koutschan, **A Fast Approach to Creative Telescoping**, Mathematics in Computer Science 4(2-3), pp. 259-266. 2010. [pdf]

The PhD thesis also contains a chapter about how to use the package. All the commands that are contained in HolonomicFunctions are in detail described in the documentation

- C. Koutschan, **HolonomicFunctions (User's Guide)**, Technical report no. 10-01 in RISC Report Series, University of Linz, Austria. January 2010. [pdf]

Some Applications

The package HolonomicFunctions has been applied in many different contexts, some of which are listed below.

- In Inverse inequality estimates with symbolic computation the HolonomicFunctions package was used to evaluate holonomic determinants that arose in numerical analysis.
- By means of the holonomic gradient method, the HolonomicFunctions package contributed to the analysis of wireless communication networks, as described in the papers MIMO zero-forcing performance evaluation using the holonomic gradient method and Exact ZF analysis and computer-algebra-aided evaluation in rank-1 LoS Rician fading.
- From version 1.5.1 on, HolonomicFunctions provides the closure property twisting q-holonomic sequences by complex roots of unity via the command DFiniteQSubstitute; more details, examples, and applications in quantum topology (Kashaev invariant of twist and pretzel knots) are presented in the corresponding paper (see the above link).
- In Advanced Computer Algebra for Determinants the package HolonomicFunctions was used to carry out Zeilberger's holonomic ansatz (and variations thereof) for determinant evaluations to solve three conjectures by George Andrews, Guoce Xin, and Christian Krattenthaler.
- The proofs of some evaluations of Pfaffians have been carried out with HolonomicFunctions.

Not all RISC combinatorics packages are bundled in RISCErgoSum:

but "HolonomicFunctions" is !

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University of Linz, Austria. January 2010. [pdf]

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- The proofs of some evaluations of Pfaffians have been carried out with HolonomicFunctions.
- The article Lattice Green's Functions of the Higher-Dimensional Face-Centered Cubic Lattices by Christoph Koutschan studies random walks in certain lattices; these studies involved heavy computer calculations.
- In physics, the HolonomicFunctions package has contributed to the evaluation of relativistic Coulomb integrals and to the study of fundamental laser modes in paraxial optics.
- In numerical analysis, finite element methods are used to construct approximate solutions to partial differential equations. In some instances, HolonomicFunctions was able to derive the differential-difference relations between the basis functions, that are necessary for an efficient implementation for Maxwell's equations; this work finally led to a registered patent.
- The Proof of George Andrews' and David Robbins' q-TSPP conjecture was a remarkable result by Christoph Koutschan, Manuel Kauers, and Doron Zeilberger, that settled a 25-years-old conjecture. The computations which established its computer proof were done by HolonomicFunctions. This work has been awarded the David P. Robbins Prize 2016 of the American Mathematical Society.
- The article "The integrals in Gradshteyn and Ryzhik. Part 18: Some automatic proofs" by Christoph Koutschan and Victor Moll uses HolonomicFunctions to deals with some integrals from the book by Gradshteyn and Ryzhik. The notebook GR18.nb (for Mathematica version 7) contains the computations for these examples.
- HolonomicFunctions was used in The 1958 Pekeris-Accad-WEIZAC Ground-Breaking Collaboration that computed Ground States of Two-Electron Atoms (and its 2010 Redux) by Christoph Koutschan and Doron Zeilberger.
- Ira Gessel's conjecture about the enumeration of certain random walks has been proven by the computer, using the package HolonomicFunctions. The corresponding article is Proof of Ira Gessel's Lattice Path Conjecture (Manuel Kauers, Christoph Koutschan, Doron Zeilberger), Proceedings of the National Academy of Sciences 106(28), pp. 11502-11505, July 2009.

The HolonomicFunctions package is registered in swMATH, where a more extensive list of papers using and citing the package can be found.

Ismail-Zhang formula: leads to the frontiers of what in the "holonomic universe" is computationally feasible today.

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Important classical expansion of the plane wave in terms of Gegenbauer polynomials $C_m^{(\nu)}(x)$:

$$e^{irx} = \left(\frac{2}{r}\right)^{\nu} \Gamma(\nu) \sum_{m=0}^{\infty} i^m (\nu + m) \, J_{\nu+m}(r) \, C_m^{(\nu)}(x).$$

q-analogue?

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Important classical expansion of the plane wave in terms of Gegenbauer polynomials $C_m^{(\nu)}(x)$:

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q-analogue? Mourad Ismail and Ruiming Zhang [1994] found:

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty\, \omega^{-\nu}}{(q^\nu; q)_\infty\, (-q\omega^2; q^2)_\infty}$$

$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m})\, q^{m^2/4}\, J_{\nu+m}^{(2)}(2\omega; q)\, C_m(x; q^\nu | q),$$

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty \, \omega^{-\nu}}{(q^\nu; q)_\infty \, (-q\omega^2; q^2)_\infty}$$

$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m}) \, q^{m^2/4} J_{\nu+m}^{(2)}(2\omega; q) \, C_m(x; q^\nu | q),$$

$$\mathcal{E}_q(x; i\omega) = \frac{(q; q)_\infty \, \omega^{-\nu}}{(q^\nu; q)_\infty \, (-q\omega^2; q^2)_\infty}$$

$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m}) \, q^{m^2/4} \, J^{(2)}_{\nu+m}(2\omega; q) \, C_m(x; q^\nu | q),$$

- $J^{(2)}_{\nu+m}(2\omega; q)$ is Jackson's (second) q-Bessel function:

$$J^{(2)}_\nu(z; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} q^{(\nu+n)n} \frac{(-1)^n (z/2)^{\nu+2n}}{(q; q)_n (q^{\nu+1}; q)_n}.$$

$$\mathcal{E}_q(x; i\omega) = \frac{(q; q)_\infty \, \omega^{-\nu}}{(q^\nu; q)_\infty \, (-q\omega^2; q^2)_\infty}$$

$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m}) \, q^{m^2/4} J_{\nu+m}^{(2)}(2\omega; q) \, C_m(x; q^\nu | q),$$

- $J_{\nu+m}^{(2)}(2\omega; q)$ is Jackson's (second) q-Bessel function:

$$J_\nu^{(2)}(z; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} q^{(\nu+n)n} \frac{(-1)^n (z/2)^{\nu+2n}}{(q; q)_n (q^{\nu+1}; q)_n}.$$

- $C_m(x; q^\nu | q)$, $x = \cos(\theta)$ are the continuous q-ultraspherical (q-Gegenbauer) polynomials :

$$C_m(\cos\theta; \beta | q) := \sum_{k=0}^{m} \frac{(\beta; q)_k \, (\beta; q)_{m-k}}{(q; q)_k \, (q; q)_{m-k}} e^{i(m-2k)\theta}.$$

$$\mathcal{E}_q(x; i\omega) = \frac{(q; q)_\infty \, \omega^{-\nu}}{(q^\nu; q)_\infty \, (-q\omega^2; q^2)_\infty}$$
$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m}) \, q^{m^2/4} J^{(2)}_{\nu+m}(2\omega; q) \, C_m(x; q^\nu | q),$$

_____

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty \, \omega^{-\nu}}{(q^\nu; q)_\infty \, (-q\omega^2; q^2)_\infty}$$

$$\times \sum_{m=0}^{\infty} i^m (1 - q^{\nu+m}) \, q^{m^2/4} J_{\nu+m}^{(2)}(2\omega; q) \, C_m(x; q^\nu | q),$$

- The basic exponential function $\mathcal{E}_q(x; i\omega)$ [Ismail & Zhang, 1994];
  For $x = \cos(\theta)$ and $|\omega| < 1$, in terms of basic sine and cosine,

$$\mathcal{E}_q(x; i\omega) := C_q(x; \omega) + i \, S_q(x; \omega),$$

where

$$C_q(x; \omega) := \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \sum_{j=0}^{\infty} \frac{(-qe^{2i\theta}; q^2)_j \, (-qe^{-2i\theta}; q^2)_j}{(q; q^2)_j \, (q^2; q^2)_j} (-\omega^2)^j,$$

and

$$S_q(x; \omega) := \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \frac{2q^{1/4}\omega}{1 - q} \cos(\theta) \sum_{j=0}^{\infty} \frac{(-qe^{2i\theta}; q^2)_j \, (-qe^{-2i\theta}; q^2)_j}{(q^3; q^2)_j \, (q^2; q^2)_j} (-\omega^2)^j.$$

$$\mathcal{E}_q(x; i\omega) := C_q(x; \omega) + i\, S_q(x; \omega),$$

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and

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NOTE. It is not difficult to check that

$$\lim_{q \to 1-} C_q\big(x; \omega(1-q)/2\big) = \cos(\omega x),$$

$$\lim_{q \to 1-} S_q\big(x; \omega(1-q)/2\big) = \sin(\omega x).$$

With the help of

```
In[1]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)

written by Christoph Koutschan

Copyright Research Institute for Symbolic Computation (RISC),

Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

it is possible to provide a computer-assisted proof of the Ismail-Zhang formula. To this end, for each side of the identity one computes generators of an ideal of annihilating operators.

Details: C. Koutschan and P., "Holonomic Tools for Basic Hypergeometric Functions", 2016.

Not all RISC combinatorics packages are bundled in RISCErgoSum:

Not all RISC combinatorics packages are bundled in RISCErgoSum:

Browser address bar: https://risc.jku.at/sw/RaduRK/

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UNIVERSITÄT LINZ

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RaduRK

RaduRK: Ramanujan-Kolberg Program

Computer Algebra for Combinatorics

Authors

Nicolas Smoot

RaduRK is a Mathematica implementation of an algorithm developed by Cristian-Silviu Radu. The algorithm takes as input an arithmetic sequence a(n) generated from a large class of q-Pochhammer quotients, together with a given arithmetic progression mn+j, and the level of a given congruence subgroup. The algorithm produces expressions for the generating function of a(mn+j) in terms of Q-linear combinations of Dedekind eta quotients which are modular over the subgroup. Identities of this form include famous results by Ramanujan which demonstrate the divisibility properties of p(5n+4) and p(7n+5). The algorithm relies on certain powerful finiteness conditions imposed by the study of modular functions, and illustrates the utility of the subject to computational number theory. The package has been developed by Nicolas Allen Smoot.

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The package

For using the package, download the file RaduRK.m, which can be found at
- RaduRK.m

Place this file into a directory where Mathematica will find it. For a demonstration of how to use the package see
- RKSupplement1.nb.

For more ambitious examples, see:
- RKSupplement2.nb.

The package requires 4ti2 and math4ti2.m, see installation instructions.

Literature

Instructions for the proper installation for these packages and RaduRK can be found in the following paper:
- N. Smoot, "On the Computation of Identities Relating Partition Numbers in Arithmetic Progressions with Eta Quotients: An Implementation of Radu's Algorithm".

For details concerning the design of the algorithm, consult the following:
- S. Radu, "An Algorithmic Approach to Ramanujan's Congruences," Ramanujan Journal, 20, pp. 215-251 (2009).
- S. Radu, "An Algorithmic Approach to Ramanujan-Kolberg Identities," Journal of Symbolic Computation, 68, pp. 225-253 (2015).

Bugs

Please report any bugs or other suggestions to Nicolas Smoot.

Phone: +43 732 2468 9921
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Visitor Address
Research Institute for Symbolic Computation (RISC)
Schloss Hagenberg (Castle of Hagenberg)

Radu's Ramanujan-Kolberg Algorithm
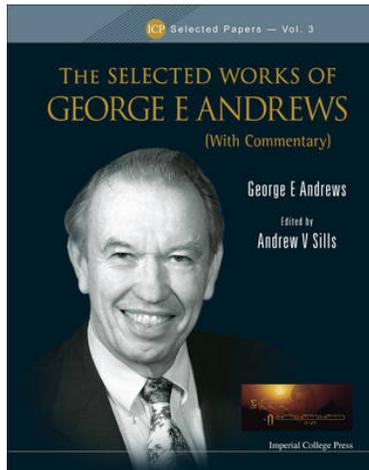
G.E. Andrews and R. Roy, "Ramanujan's Method in q-Series Congruences", 1997:

"Ramanujan is the discoverer of the surprising fact that the partition function, $p(n)$, satisfies numerous congruences. Among the infinite family of such congruences, the two simplest examples are

$$p(5n + 4) \equiv 0 \pmod 5 \tag{1.1}$$

and

$$p(7n + 5) \equiv 0 \pmod 7 . \text{"} \tag{1.2}$$

$p(4) = 5: \ 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$

Ramanujan's Congruences

$$
\begin{aligned}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}
\end{aligned}
$$

Ramanujan's Congruences

$$
\begin{aligned}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(5^2 n + 24) &\equiv 0 \pmod{5^2}, \\
p(5^3 n + 99) &\equiv 0 \pmod{5^3},
\end{aligned}
$$

etc.

PROOF (mod 5 congruence).

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

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"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities . . . ; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting . . ." [G.H. Hardy]

`http://www-history.mcs.st-andrews.ac.uk/BigPictures/Hardy_5.jpeg`

PROOF (mod 7 congruence).

$$\sum_{n=0}^{\infty} p(7n+5)q^n$$

$$= 7 \prod_{j=1}^{\infty} \frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^7}{(1-q^j)^8}.$$

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NOTE 1. Radu's Ramanujan-Kolberg algorithm computes both of Ramanujan's [1919] witness identities.

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NOTE 1. Radu's Ramanujan-Kolberg algorithm computes both of Ramanujan's [1919] witness identities.

NOTE 2. Nicolas Smoot produced a beautiful implementation of Radu's algorithm.

QUESTION. What about the 11 congruence?

Computer Algebra Session 3

Pick up and open Smoot's Mathematica notebook "RKSupplement1.nb":

RKSupplement1.nb - Wolfram Mathematica 11.3

Edit Insert Format Cell Graphics Evaluation Palettes Window Help
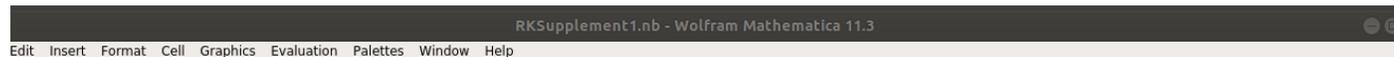
Supplement (I) to the article
"An Implementation of Radu's Ramanujan--Kolberg Algorithm"
by Nicolas Allen Smoot,
Research Institute for Symbolic Computation (RISC),
Johannes Kepler University Linz

Begin with the program RaduRK

In[1]:= << RaduRK`

math4ti2: Mathematica interface to 4ti2 (http://www.4ti2.de/)
Copyright (C) 2017, Ralf Hemmecke <ralf@hemmecke.org>
Copyright (C) 2017, Silviu Radu <sradu@risc.jku.at>

RaduRK: Ramanujan--Kolberg Program Version 2.8
Copyright (C) 2020, Nicolas Allen Smoot <nicolas.smoot@risc.jku.at>
Research Institute for Symbolic Computation
Johannes Kepler Universität, Linz

Next, establish the two key variables q and t ("generic" variables):

In[2]:= `{SetVar1[q], SetVar2[t]}`

Out[2]= `{q, t}`

- For each case we work over a modular curve $X_0(N)$, with N a positive integer.

- Take an arithmetic function a(n) defined by a generating function which can be constructed with an integer M and an integer-valued vector r indexed over the divisors of M:

$$\sum_{n=0}^{\infty} a(n)\, q^n := \prod_{\delta \mid M} \left(q^\delta; q^\delta\right)_{\infty}^{r_\delta}$$

Ex. M = 1 and $r_\delta = -1$,

$$\sum_{n=0}^{\infty} p(n)\, q^n := \prod_{\delta \mid 1} \left(q^\delta; q^\delta\right)_{\infty}^{-1} = \prod_{k=1}^{n} \frac{1}{1 - q^k}$$

- Finally, take the linear progression mn+j, as n goes over the nonnegative integers.

$$\sum_{n=0}^{\infty} a(mn + j)\, q^n \qquad\qquad \text{Ex. } m = 5 \text{ and } j = 4, \quad \sum_{n=0}^{\infty} p(5n + 4)\, q^n$$

Classical Identities

Section 3.1 (refers to Smoot's article "An Implementation of Radu's...")

Start with the ordinary partition function, with the progression 5n+4 - as in the example above. If we work over N=5, we retrieve Ramanujan's famous identity.

Ex. N = 5, and M = 1, r_δ = -1 , and m = 5 and j = 4,

$$\sum_{n=0}^{\infty} p\,(5\,n+4)\,q^n \text{ with } \sum_{n=0}^{\infty} p\,(n)\,q^n := \prod_{\delta\,|\,1} \left(q^\delta\,q^\delta\right)_\infty^{-1}$$

Input: RK[N, M,r, m,j] = RK[5, 1,{-1}, 5,4] with r=(r_δ).

In[7]:= RK[5, 1, {-1}, 5, 4]

In[7]:= **RK[5, 1, {-1}, 5, 4]**

$$\prod_{\delta\mid M} (q^\delta;q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n)\,q^n$$

$$f_1(q) \cdot \prod_{j'\in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j')\,q^n = \sum_{g\in AB} g\cdot p_g(t)$$

Modular Curve: $X_0(N)$

N:	5
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{1, \{-1\}\}$
m:	5
$P_{m,r}(j)$:	$\{4\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^6}{\left(\left(q^5;q^5\right)_\infty\right)^5}$
t:	$\dfrac{((q;q)_\infty)^6}{q\left(\left(q^5;q^5\right)_\infty\right)^6}$
AB:	$\{1\}$
$\{p_g(t): g\in AB\}$	$\{5\}$
Common Factor:	5

Out[7]=

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

after substituting the output data, turns into

$$\frac{((q;q)_\infty)^6}{((q^5;q^5)_\infty)^5} \cdot \sum_{n=0}^{\infty} p(5n+4) q^n = \sum_{g \in \{1\}} g \cdot p_g(t) = 1 \cdot 5$$

In standard notation,

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \cdot \prod_{k=1}^{\infty} \frac{\left(1 - q^{5k}\right)^5}{\left(1 - q^k\right)^6} \qquad [\boldsymbol{Ramanujan}]$$

Why is the output given in the format

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j') q^n = \sum_{g \in AB} g \cdot p_g(t) \quad ?$$

Why is the output given in the format

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$?

ANSWER: <u>Radu's algorithm</u> computes an integer set $P_{m,r}(j)$ with $j \in P_{m,r}(j)$, and an eta quotient $f_1(q)$ such that

$$f := f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j') q^n$$

is a MODULAR FUNCTION f for $X_0(N)$ where f has poles (if any) only at ∞; i.e., $f \in M(X_0(N))$.

$q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}$

If f can be presented as a linear combination

linear combs. of

$$f = \sum_{g \in AB} g \cdot p_g(t)$$

of eta quotients $g \in AB$, with $g \in M(X_0(N))$ and with coefficients being polynomials $p_g(x) \in \mathbb{C}[x]$, the algorithm finds it.

```
In[9]:= RK[7, 1, {-1}, 7, 5]
```

In[9]:= **RK[7, 1, {-1}, 7, 5]**

$$\prod_{\delta \mid M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^\infty a(n) q^n$$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^\infty a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

Modular Curve: $X_0(N)$

Out[9]=

N:	7
$\{M, (r_\delta)_{\delta \mid M}\}$:	$\{1, \{-1\}\}$
m:	7
$P_{m,r}(j)$:	$\{5\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^8}{q\left(\left(q^7;q^7\right)_\infty\right)^7}$
t:	$\dfrac{((q;q)_\infty)^4}{q\left(\left(q^7;q^7\right)_\infty\right)^4}$
AB:	$\{1\}$
$\{p_g(t): g \in AB\}$	$\{49 + 7\,t\}$
Common Factor:	7

$$f_1 (q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a (mn + j') q^n = \sum_{g \in AB} p_g (t) \cdot g$$

after substituting the output data, turns into

$$\frac{((q; q)_{\infty})^8}{q((q^7; q^7)_{\infty})^7} \cdot \sum_{n=0}^{\infty} p (7n + 5) q^n = \sum_{g \in \{1\}} g \cdot p_g (t) = \left(49 + 7 \frac{((q; q)_{\infty})^4}{q((q^7; q^7)_{\infty})^4}\right) \cdot 1$$

In standard notation,

$$\sum_{n=0}^{\infty} p (7n + 4) q^n = 49 \cdot q \prod_{k=1}^{\infty} \frac{(1 - q^{7k})^7}{(1 - q^k)^8} + 7 \cdot \prod_{k=1}^{\infty} \frac{(1 - q^{7k})^3}{(1 - q^k)^4} \qquad [\textbf{\textit{Ramanujan}}]$$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} p_g(t) \cdot g$$

after substituting the output data, turns into

$$\frac{((q;q)_\infty)^8}{q((q^7;q^7)_\infty)^7} \cdot \sum_{n=0}^{\infty} p(7n+5) q^n = \sum_{g \in \{1\}} g \cdot p_g(t) = \left(49 + 7 \frac{((q;q)_\infty)^4}{q((q^7;q^7)_\infty)^4}\right) \cdot 1$$

In standard notation,

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NOTE. From modular function point of view, the cases $N = 5$ and $N = 7$ are simple:

$$M(X_0(N)) = \mathbb{C}[t] \, (= (\mathbb{C}[t] \cdot 1)$$

E.g., if $N = 7$,

$$t = \frac{1}{q} \left(\frac{(q;q)_\infty}{(q^7;q^7)_\infty}\right)^4 .$$

Section 3.3. [What happens for p(11n+6)?]

We examine next the ordinary partition function, with the progression 11n+6.
● Start with N=11.

In[10]:= **RK[11, 1, {-1}, 11, 6]**

$$\prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^\infty a(n) q^n$$

$$\boxed{f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^\infty a(mn+j') q^n = \sum_{g \in AB} g \cdot p_g(t)}$$

```
Modular Curve: X₀(N)
```

Out[10]=

N:	11
$\{M, (r_\delta)_{\delta|M}\}$:	$\{1, \{-1\}\}$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

Modular Curve: $X_0(N)$

Out[10]=

N:	11
$\{M, (r_\delta)_{\delta \mid M}\}$:	$\{1, \{-1\}\}$
m:	11
$P_{m,r}(j)$:	$\{6\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^{12}}{q^4 ((q^{11};q^{11})_\infty)^{11}}$
t:	$\dfrac{((q;q)_\infty)^{12}}{q^5 ((q^{11};q^{11})_\infty)^{12}}$
AB:	$\{1\}$
$\{p_g(t): g \in AB\}$	No Membership

$\{p_g(t): g{\in}AB\}$	No Membership
Common Factor:	None

We find no identity over $X_0(11)$. This is in part why no simple witness identity for p(11n+6) exists, in contrast to the cases of p(5n+4), p(7n+5).

⇒ Let's next try N=22.

In[12]:= `Timing[RK[22, 1, {-1}, 11, 6]]`

$$\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty}a(n)q^n$$

$$\boxed{f_1(q) \cdot \prod_{j'{\in}P_{m,r}(j)} \sum_{n=0}^{\infty}a(mn+j')q^n = \sum_{g{\in}AB}g\cdot p_g(t)}$$

Modular Curve: $X_0(N)$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

Modular Curve: $X_0(N)$

Out[12]= $\{34.3996,$

N:	22
$\{M, (r_\delta)_{\delta \mid M}\}$:	$\{1, \{-1\}\}$
m:	11
$P_{m,r}(j)$:	$\{6\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^{10} \left((q^2;q^2)_\infty\right)^2 \left((q^{11};q^{11})_\infty\right)^{11}}{q^{14} \left((q^{22};q^{22})_\infty\right)^{22}}$
t:	$-\dfrac{(q^2;q^2)_\infty \left((q^{11};q^{11})_\infty\right)^{11}}{8\,q^5\,(q;q)_\infty \left((q^{22};q^{22})_\infty\right)^{11}} + \dfrac{\left((q^2;q^2)_\infty\right)^8 \left((q^{11};q^{11})_\infty\right)^4}{11\,q^5\,((q;q)_\infty)^4 \left((q^{22};q^{22})_\infty\right)^8} + \dfrac{3\,((q;q)_\infty)^7 \left((q^{11};q^{11})_\infty\right)^3}{88\,q^5 \left((q^2;q^2)_\infty\right)^3 \left((q^{22};q^{22})_\infty\right)^7}$ $\}$
AB:	$\left\{1, -\dfrac{(q^2;q^2)_\infty \left((q^{11};q^{11})_\infty\right)^{11}}{8\,q^5\,(q;q)_\infty \left((q^{22};q^{22})_\infty\right)^{11}} + \dfrac{2\left((q^2;q^2)_\infty\right)^8 \left((q^{11};q^{11})_\infty\right)^4}{11\,q^5\,((q;q)_\infty)^4 \left((q^{22};q^{22})_\infty\right)^8} - \dfrac{5\,((q;q)_\infty)^7 \left((q^{11};q^{11})_\infty\right)^3}{88\,q^5 \left((q^2;q^2)_\infty\right)^3 \left((q^{22};q^{22})_\infty\right)^7}, \right.$ $\left. \dfrac{5\,(q^2;q^2)_\infty \left((q^{11};q^{11})_\infty\right)^{11}}{4\,q^5\,(q;q)_\infty \left((q^{22};q^{22})_\infty\right)^{11}} - \dfrac{3\left((q^2;q^2)_\infty\right)^8 \left((q^{11};q^{11})_\infty\right)^4}{11\,q^5\,((q;q)_\infty)^4 \left((q^{22};q^{22})_\infty\right)^8} + \dfrac{((q;q)_\infty)^7 \left((q^{11};q^{11})_\infty\right)^3}{44\,q^5 \left((q^2;q^2)_\infty\right)^3 \left((q^{22};q^{22})_\infty\right)^7}\right\}$
$\{p_g(t): g \in AB\}$	$\{6776 + 9427\,t + 15477\,t^2 + 13332\,t^3 + 1078\,t^4,$ $-9581 + 594\,t + 5390\,t^2 + 187\,t^3, -6754 + 5368\,t + 2761\,t^2 + 11\,t^3\}$
Common Factor:	11

Section 3.2

- What about, for instance,

$$\sum_{n=0}^{\infty} p\,(5\,n + 1)\,q^n\ ?$$

[...] Kolberg originally computed the following identities.

In[15]:= **RK[5, 1, {-1}, 5, 1]**

$$\prod_{\delta\,|\,M}(q^{\delta};q^{\delta})_{\infty}^{r_\delta}\ =\ \sum_{n=0}^{\infty}a\,(n)\,q^n$$

$$f_1\,(q)\ \cdot\ \prod_{j'\in P_{m,r}\,(j)}\ \sum_{n=0}^{\infty}a\,(mn + j')\,q^n\ =\ \sum_{g\in AB}g\cdot p_g\,(t)$$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

Modular Curve: $X_0(N)$

Out[15]=

N:	5
$\{M, (r_\delta)_{\delta \mid M}\}$:	$\{1, \{-1\}\}$
m:	5
$P_{m,r}(j)$:	$\{1, 2\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^{12}}{q\left(\left(q^5;q^5\right)_\infty\right)^{10}}$
t:	$\dfrac{((q;q)_\infty)^{6}}{q\left(\left(q^5;q^5\right)_\infty\right)^{6}}$
AB:	$\{1\}$
$\{p_g(t): g \in AB\}$	$\{25 + 2\,t\}$
Common Factor:	None

There is no "Common Factor" (i.e., no divisibility), but the program delivered an identity which was first established by Kolberg [1957]:

Observe that we obtained,

$$j = 1 \in P_{m,r}(j) = \{1, 2\}.$$

Together with the other output data this gives,

$$\frac{1}{q} \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^{10}} \cdot \sum_{n=0}^\infty p(5n+1)q^n \cdot \sum_{n=0}^\infty p(5n+2)q^n = 25 + 2t$$

with

$$t = \frac{1}{q} \left(\frac{(q;q)_\infty}{(q^5;q^5)_\infty} \right)^6.$$

NOTE. As already remarked,

$$M(X_0(5)) = \mathbb{C}[t], \quad \text{where } t \text{ is as above.}$$

Conclusion

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♠ Modular Functions and Forms:
  Ramanujan type congruences, Radu's algorithm, etc.