INTER-UNIVERSAL TEICHMÜLLER THEORY: A PROGRESS REPORT

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October 2010

The analogy between *number fields* and *function fields of curves* (e.g., hyperbolic curves) *over finite fields* is quite classical. In the present talk, we survey work in progress concerning a theory developed by the lecturer during the last decade — in the spirit of this analogy — whose goal is to construct an analogue for *number fields "equipped with an elliptic curve"* of the ***p*-adic Teichmüller theory** developed by the lecturer during the early 1990's for *hyperbolic curves over a finite field "equipped with a nilpotent ordinary indigenous bundle"*. From an even more classical point of view, one may think of this theory as a sort of analogue for number fields of classical complex Teichmüller theory, in which canonical deformations of the holomorphic structure of a hyperbolic Riemann surface of finite type are constructed by dilating one of the two underlying real dimensions of the Riemann surface, while leaving the other dimension fixed (i.e., "undeformed").

In the case of number fields equipped with an elliptic curve, one thinks of the ring structure of the number field as a sort of "arithmetic holomorphic structure". One then constructs *canonical deformations of this arithmetic holomorphic structure* — i.e., analogues of the canonical liftings of p-adic Teichmüller theory — by applying the general theory of Frobenioids, as well as the theory of the *Frobenioid-theoretic theta function* (developed in earlier papers by the lecturer). At a more concrete level, if one thinks of the ring structure (i.e., "arithmetic holomorphic structure") of the given number field as consisting of *"two underlying combinatorial dimensions"* corresponding to **addition** and **multiplication**, then working with Frobenioids corresponds, roughly speaking, to performing operations with the *multiplicative monoids* involved (i.e., multiplicative portions of the rings involved) — in a fashion motivated by the theory of log structures; in particular, such operations are *not necessarily compatible* with the additive portions of the ring structures involved. Alternatively, if one thinks of the ring structure (i.e., "arithmetic holomorphic structure") of the various local fields that arise as localizations of the given number field as consisting of *"two underlying combinatorial dimensions"* corresponding to the group of units and the value group, then one may think of these canonical deformations of the arithmetic holomorphic structure as deformations in which the *value groups* are **(canonically!) dilated** — by means of the **theta function** — while the *units* are left undeformed.

Since such "arithmetic Teichmüller dilations" are *manifestly incompatible with the ring structure* of the given number field, it follows that they are not compatible, in general, with various classical scheme-theoretic constructions performed over the number field which depend on this ring structure. In particular, these arithmetic Teichmüller dilations fail to be compatible with the various **basepoints** *of arithmetic fundamental groups* involved (e.g., Galois groups) which are defined by considering scheme-theoretic geometric points. The resulting incompatibility of (conventional scheme-theoretic) basepoints on either side of the "arithmetic Teichmüller dilation" gives rise to numerous indeterminacies; these indeterminacies lead naturally to the introduction of tools from anabelian geometry. It is this fundamental aspect of the theory that is referred to by the term **"inter-universal"**.

The (expected) **main theorem** *of inter-universal Teichmüller theory* consists of a fairly explicit computation, up to certain relatively mild indetermacies, of the "arithmetic Teichmüller deformations of a number field equipped with an elliptic curve" discussed above by applying various results obtained in previous papers by the lecturer concerning local and global absolute anabelian geometry, tempered anabelian geometry, and the étale theta function. This passage from the **Frobenioid-theoretic definition** of the arithmetic deformations involved to a more explicit **Galois-theoretic description** may be thought of as a sort of global arithmetic analogue of the classical computation of the Gaussian integral (i.e., $\int_{-\infty}^{\infty} e^{-x^2} dx$) by means of the passage from *cartesian* to *polar* coordinates. Inequalities of interest in diophantine geometry may then be obtained as (expected) corollaries of this (expected) main theorem.