A Brief Survey of the Geometry of Categories

§1. From Anabelian Geometry to the Geometry of Categories
§2. Categories of Arithmetic Log Schemes
§3. Categories of Metrized Line Bundles
§4. Categories of Squares, Rectangles, and Parallelograms
§5. Absolute Anabelian Geometry

---

§1. From Anabelian Geometry to the Geometry of Categories

'Anabelian geometry': $X \rightsquigarrow \pi_1(X) \overset{?}{\rightsquigarrow} X$.
(Grothendieck). (X: connected scheme) ($\pi_1(X)$: étale fund. gp.)

More generally: mathematical theater A $\rightsquigarrow$ math. theater B $\rightsquigarrow$ math. theater C
('Inter-universal Geometry'). $\rightsquigarrow \ldots \overset{?}{\rightsquigarrow}$ A

loop!

$\Downarrow$

one good approximation of 'a math. theater': **category**:
- simple enough to (re-)construct (just specify composition of arrows!)
- rich/complicated enough to encode most arith-geom. 'theaters' (cf. the rest of this talk!)

Motivation: ABC Conjecture of Diophantine geometry (cf. Grothendieck! — but quite different approach)

nevertheless, one common aspect: $\boxed{\text{pursuit of non-scheme-like geometries in arith. geom.! ('}\mathbb{F}_1\text{' not nec. }/\mathbb{Z}\text{!)}}$

— cf. '$\exists$ non-scheme-like aut. of G_K (abs. Gal. gp. of $K/\mathbb{Q}_p < \infty$), unlike G_F (abs. Gal. gp. of $F/\mathbb{Q} < \infty$)'

§2. Categories of arith. log schemes

X, Y noether. scheme.

$Sch(X) :=$ cat. of ~~fin. type~~ schemes$/X$, morphisms$/X$.

$\Downarrow$

Thm: $Sch(X) \xrightarrow{\sim} Sch(Y) \iff X \xrightarrow{\sim} Y$ (RIMS Preprint 1364, 1475)

Remarks: (i) if X fin. type$/\mathbb{Z}$, then considering arch. strs. (i.e., compact $H \subseteq X(\mathbb{C})$) $\Rightarrow$ arith. version.

(ii) $\exists$ log version via fs log schemes.

(iii) Thm $\Rightarrow$ still remain in classical geom. of schemes!

---

§3. Categories of metrized line bundles

For arch./nonarch. local flds. } category of:
gl. flds. } 'Spec (field) + metrized line bundle'

(omit details).

$\Downarrow$

'$Loc^{\times}$'$(\ldots)$: (i) can recover classical geometry of metrized line bundles (cf. Arak. theory)

(ii) in global case, 'product formula' is preserved by cat. equ's. (cf. Arak. theory)

(iii) New geometry! $\exists$ Frobenius: $Loc^{\times} \circlearrowleft$
(cf. pos. char.!) $(l.b.) \mapsto (l.b.)^{\otimes N}$

$\Downarrow$

can construct global version of '$0 \to \mathbb{Z}/N(1) \to E[N] \to \mathbb{Z}/N \to 0$' for Tate curve.

.... cf. Hodge theory!!

(iv) New geometry: can construct global no. fld.-type objects with inf. many primes omitted!!

§4. Categories of squares, rectangles, parallelograms

- X, Y: sober, loc. conn. top. sp $\rightsquigarrow$ $\mathrm{Open}(X)$: category of conn. open subsets. $U \subseteq X$, $U_1 \hookrightarrow U_2$ (over X).

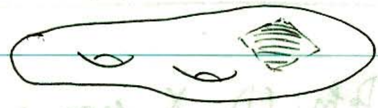
<u>Classical Thm:</u> $\mathrm{Open}(X) \xrightarrow{\sim} \mathrm{Open}(Y) \iff X \xrightarrow{\sim} Y$.

X: hyp. Riemann surface, equipped with (log) square differential φ_X

Y: $\ldots, \varphi_Y$.

$\mathrm{Loc}^{\mathfrak{P}}(X, \varphi_X)$: category of coverings of X + parallelograms (w.r.t. $\int \varphi_X^{\frac{1}{2}}$.)

(+morphisms$/X$)



(omit details).

also $\mathrm{Loc}^{\mathfrak{S}}$, $\mathrm{Loc}^{\mathfrak{R}}$: squares, rectangles

<u>Thm:</u> $\square \in \{\mathfrak{S}, \mathfrak{R}, \mathfrak{P}\}$.

$$\mathrm{Loc}^{\square}(X, \varphi_X) \xrightarrow{\sim} \mathrm{Loc}^{\square}(Y, \varphi_Y) \iff \begin{cases} \cdot\ X \xrightarrow{\sim} Y \text{ biholom./complex conj.} \ (\square = \mathfrak{S}, \mathfrak{R}) \\ \cdot\ X \xrightarrow{\sim} Y \text{ Teich. map/complex conj.} \ (\square = \mathfrak{P}). \end{cases}$$

(RIMS Preprint 1483)

---

§5. Absolute Anabelian Geometry

X: hyp. curve $/K$, $[K:\mathbb{Q}_p] < \infty$ $\rightsquigarrow$ $1 \to \Delta_X \to \Pi_X \to G_K \to 1$

Y: $\text{---} /L$ --- $\quad 1 \to \Delta_Y \to \Pi_Y \to G_L \to 1$

<u>Question:</u> Despite "$\mathrm{Aut}(K) \subsetneq \mathrm{Out}(G_K)$", can one recover X from the profinite gp. Π_X?

Still unknown!

Partial results:

Thm. (p-adic Teich. theory).

$\exists\, \Pi_X \xrightarrow{\sim} \Pi_Y \Rightarrow$ (i) $K/\mathbb{Q}_p$ unram. X: canonical curve $\Big\} \Longleftrightarrow \Big\{$ L: ... Y: ...

(cf. Serre-Tate for ab. vars.)

(RIMS Preprint 1379)

(ii) if (i) holds, then $\exists\, X \xrightarrow{\sim} Y$. weak!

Defn: X abs.: $\exists\, \Pi_X \xrightarrow{\sim} \Pi_Y \Rightarrow \exists\, X \xrightarrow{\sim} Y$

$\boxed{\text{Cor: For } p \geq 5, \text{ abs. curves Zar. dense in } \mathcal{M}_{g,r}(\overline{\mathbb{Q}}_p),}$

Defn: (i) X geom. quasi-Belyi type: $\exists\, X \xrightarrow{\text{dom.}} \mathbb{P}^1 \setminus 3\,\text{pts}$.

(ii) X quasi-Belyi type: $(\downarrow)$ + def'd/ number fld.

Thm: X, Y quasi Belyi $\Rightarrow$

$\boxed{\Pi_X \xrightarrow{\sim} \Pi_Y \iff X \xrightarrow{\sim} Y}$

(RIMS Preprint 1490)

Rmk: recent unwritten expected Thm. of Tamagawa: OK, for only one of X, Y geom. quasi-Belyi

all rely essentially on $\ulcorner \mathbb{P}^1 \setminus 3\text{pts.}$ + Belyi$\lrcorner$

$\mathbb{Z} \ni n \geq \begin{matrix} 2 & \text{if } X \text{ affine} \\ 3 & \text{if } X \text{ proper} \end{matrix}$ $\Big\|$ $C_X := \underbrace{X \times \dots \times X}_{n} \setminus \text{diagonals}$. 'config. space'

(unwritten) Expected Thm: X, Y genus $\geq 2 \Rightarrow$

$\boxed{\Pi_{C_X} \xrightarrow{\sim} \Pi_{C_Y} \iff C_X \xrightarrow{\sim} C_Y}$

cf. work of H. Nakamura, N. Takao.