Tempered Anabelian Geometry

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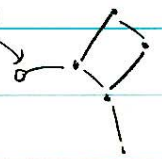
§1. Tempered Fundamental Groups of Semi-graphs of Anabelioids

allow half-open edges



Γ

Consider a semi-graph of (connected) anabelioids (= (multi-) Galois category, considered geometrically — e.g., allow 'fin. et coverings')

$$\mathcal{G} = (\Gamma, \{\mathcal{G}_v\}, \{\mathcal{G}_e\}, \{b : \mathcal{G}_e \to \mathcal{G}_v\})$$

vertices   edges   branches of edges

(cf. 'graph of groups')

$\Downarrow$

allow ctbly many connected components

$\mathcal{B}^{cov}(\mathcal{G})$: category of $\{S_v, \varphi_e\}$: $S_v \in \mathcal{G}_v^{\top}$, $\varphi_e : b_1^* S_{v_1} \xrightarrow{\sim} b_2^* S_{v_2}$ in $\mathcal{G}_e^{\top}$

$e = (b_1, b_2)$, $b_1 \leadsto v_1$, $b_2 \leadsto v_2$

Semi-graph of anab'ds $\mathcal{H} \to \mathcal{G}$.

$$\mathcal{B}^{cov}(\mathcal{G}) \supseteq \mathcal{B}^{temp}(\mathcal{G}) \supseteq \mathcal{B}(\mathcal{G})$$

$\mathcal{B}^{temp}(\mathcal{G})$ = {tempered coverings} — dominated by a 'combinatorial' (= graph-theoretic) covering of a finite covering (cf. work of André!)

$\mathcal{B}(\mathcal{G})$ = {finite coverings} ← Galois category $\leadsto \pi_1^{profinite}(\mathcal{G}) = \hat{\pi}_1(\mathcal{G})$

$\boxed{\pi_1^{temp}(\mathcal{G})}$ ← 'temperoid'

$\mathcal{B}^{temp}(\Pi)$: $\Pi = \varprojlim$ (ctbl. discrete groups)

↳ category of ctbl. discrete sets with continuous Π-action.

has similar properties to Galois categories!

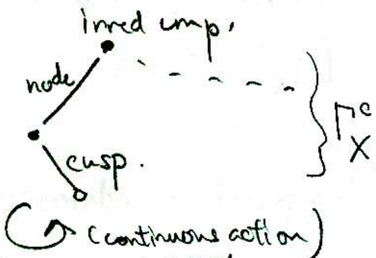
e.g. $\mathrm{Hom}^{out}(\Pi_1, \Pi_2) \xrightarrow{\sim} \mathrm{Mor}(\mathcal{B}^{temp}(\Pi_1), \mathcal{B}^{temp}(\Pi_2))$

↳ 'pull-back functor that preserves fin. limits, ctbl. colimits (considered up to isoms. of functors)

<u>Example</u>: $K/\mathbb{Q}_p < \infty$, X: hyperbolic curve $/K$, $\exists$ stable model $\mathfrak{X}$

$\Rightarrow$ $\mathfrak{X}_k \rightsquigarrow$ <u>semi-graph of anab'ds</u> $\mathcal{G}^c_X$

(k = res. fld. of K) ($\exists$ pro-Σ version)



$G_K = \mathrm{Gal}(\overline{K}/K)$

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§2. <u>Maximal Compact S'gps</u>:

$\mathcal{G}$ '<u>nice</u>' semi-gr. of anab'ds.
↳ (e.g., <u>Example</u>).

<u>Thm</u>: · <u>maximal compact s/gps.</u> of $\pi_1^{temp}(\mathcal{G}) \longleftrightarrow$ <u>vertical s/gps.</u> of $\pi_1^{temp}(\mathcal{G})$ ('$\hat{\pi}_1(\mathcal{G}_v)$')

· <u>nontrivial intersections</u> of two distinct max. compact s/gps. of $\pi_1^{temp}(\mathcal{G})$ $\longleftrightarrow$ <u>edge-like s/gps</u> of $\pi_1^{temp}(\mathcal{G})$ ('$\hat{\pi}_1(\mathcal{G}_e)$')

<u>Pf</u>: Apply result of Serre, <u>Trees</u>: finite group $\curvearrowright$ tree.

$\Rightarrow \exists$ fixed vertex or edge

+ standard arguments.

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<u>Corollary</u>: Can reconstruct underlying <u>graph</u> (= semi-graph \ open edges) of $\mathcal{G}$ from <u>top. gp.</u> $\pi_1^{temp}(\mathcal{G})$ ('<u>anabelian</u>')

$\exists$ 'arithmetic version': $\overset{\text{profinite gp.}}{A} \underset{\text{cont. action}}{\curvearrowright} \mathcal{Y} \rightsquigarrow \boxed{1 \to \pi_1^{temp}(\mathcal{Y}) \to \pi_1^{temp}(\overline{\mathcal{Y}}) \to A \to 1}$

$\overline{\mathcal{Y}} = (\mathcal{Y}, A \curvearrowright \mathcal{Y})$

$\Downarrow$

'maximal' $\rightsquigarrow$ 'arith. maximal' (only consider compact s/gps. of $\pi_1^{temp}(\overline{\mathcal{Y}})$ that

'nontrivial' $\rightsquigarrow$ 'arith. nontrivial' surject onto $\exists$ open s/gp. of A).

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§3. Applications

Thm A: X, Y hyperbolic curves / nonarch. local flds, $\exists$ stable model
$\rightsquigarrow \Delta_X^{temp}, \Delta_Y^{temp}$ (geometric tempered fund. gp. — cf. André)

$$\forall\ \Delta_X^{temp} \xrightarrow{\sim} \Delta_Y^{temp} \rightsquigarrow \mathcal{Y}_X^c \xrightarrow{\sim} \mathcal{Y}_Y^c$$

Pf: (Cor. of §2) + (wild ram'd. covering $\rightsquigarrow \exists$ irred comp. / cusp $\rightsquigarrow$ open edges) //

Rmk: can combine with results of Tamagawa for π_1^{tame}(hyp. curves / $\overline{\mathbb{F}}_p$).

$X/_K$, $Y/_L$: hyp. curves ; K, L nonarch. loc. flds.

Thm B: $\begin{pmatrix} \text{dominant mors.} \\ X \to Y \end{pmatrix} \xrightarrow{\sim}$ (outer homs. of DOF-type $\Pi_X^{temp} \to \Pi_Y^{temp}$ over $G_K \to G_L$ that arises from $L \hookrightarrow K$)

image is dense in an open s/gp. of fin. index

Thm C: $\left(\Pi_X^{temp} \overset{outer}{\xrightarrow{\sim}} \Pi_Y^{temp}\right) \xleftrightarrow{\sim} \left(\Pi_X \overset{outer}{\xrightarrow{\sim}} \Pi_Y\right)$

(absolute p-adic anab. geom.)

profinite fund. gps.

Thm D: Suppose that X 'of strictly Belyi type': def'd / no. flds, isogenous to genus 0 curve ($\exists$ common fin. ét. covering)

Let $\Pi_X^{temp} \xrightarrow{\alpha} \Pi_Y^{temp}$ be an isom. Then:

(i) α preserves decomp. gps. of closed pts.

(ii) Y also of strictly Belyi type.

Pf: uses (arith.) max. compact s/gp, geometry of fin. gp. $\curvearrowright$ tree.

(– cf. 'real section conjecture', geometry of 'straight line spaces')

Rmk: $\exists$ profinite version of Thm D: Thm $\hat{D}$.

Thm E: Let X, Y, α be as in Thm D. Then α arises geometrically.

('absolute p-adic Groth. Conj.')

Pf: (Thm D + ...) or (Thm $\hat{D}$ + ... + Thm C)

Rmk: (i) first strong abs. pGC result!

(ii) only for ctbly many curves (false for arb. X, Y??)

(iii) $\exists$ wk abs pGC for canonical curves of p-adic Teich. theory – also only ctbly many