

The Hodge-Arakelov Theory of Elliptic Curves, I

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§1. Main Results

(Comparison Isomorphisms and
Arithmetic Kodaira-Spencer
Morphism)

§2. Philosophy: In Search of an
Absolute Derivative

§1. Main Results

(A.) Simple Version of the Main Comparison Theorem

K : a field of characteristic 0

E : an elliptic curve/ K

E^\dagger : its universal extension

= { moduli of $(\mathcal{L}, \nabla_{\mathcal{L}})$:

$(\mathcal{L}, \nabla_{\mathcal{L}}) : \deg(\mathcal{L}) = 0$ }

$\stackrel{\text{char } 0}{=} H_{\text{DR}}^1(E, \mathcal{O}_E^\times)$

Over \mathbf{C} : $E^\dagger = H_{\text{DR}}^1(E, \mathcal{O}_E) / \Lambda$

= ' $E_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ ' (under. real an. man.)

where $\Lambda = H_{\text{sing}}^1(E, 2\pi i \mathbf{Z}) \cong \mathbf{Z}^2$

In general:

Tang. sp. to $E^\dagger = H_{\text{DR}}^1(E, \mathcal{O}_E)$

Char. 0: ${}_d E^\dagger \cong {}_d E \stackrel{\text{def}}{=} \ker [d] : E \rightarrow E$
(d : a positive integer)

(in mixed char., denominators arise)

$\eta \in E(K)$: torsion point of order m ,
s. t. d does not divide m

$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$

Theorem: The restriction morphism

$$\Gamma(E^\dagger, \mathcal{L})^{<d} \xrightarrow{\sim} \mathcal{L}|_{{}_d E^\dagger}$$

is an isomorphism.

Note:

- (1.) “ $< d$ ” denotes torsorial degree
(relative degree: E^\dagger / E) $< d$.
- (2.) Both sides are K -vector spaces
of dimension d^2 .
- (3.) Theorem false if d divides m .
(e.g., if $d = m = 1$, then
 $\Gamma(E, \mathcal{O}_E([0_E]) = \mathcal{L}) \rightarrow \mathcal{L}|_{0_E}$ is 0)
- (4.) Proof:
Mumford’s algebraic theta functions
+ Zhang’s theory of admiss. metrics
+ complicated degree computations

(B.) Integral Structures at “Arithmetic” Primes

In general:

$$0 \rightarrow \omega_E \rightarrow E^\dagger \rightarrow E \rightarrow 0$$

($\omega_E =$ invariant diffs. on E)

Near “point at infinity” ∞ :

$$E = \mathbf{G}_m / q^{\mathbf{Z}}$$

(“Tate curve”)

\implies Over power series in q (“hat”):

$$\widehat{E} = \widehat{\mathbf{G}}_m$$

$$\widehat{E}^\dagger = \widehat{\mathbf{G}}_m \times \widehat{\omega}_E = \widehat{\mathbf{G}}_m \times \left\langle \frac{dU}{U} \right\rangle$$

Integral structure at finite primes:

$$\mathcal{O}_{\widehat{E}}[T] = \bigoplus \mathcal{O}_{\widehat{E}} \cdot T^j \implies$$

$$\bigoplus \mathcal{O}_{\widehat{E}} \cdot \binom{T - \eta_\infty}{j}^{-\frac{1}{2}}$$

where T : coord. on ω_E , def'd by $\frac{dU}{U}$

...(p-adic analytically) extends

over all $\overline{\mathcal{M}}_{1,0}$, not just near ∞

Integral Structure Near ∞ :

$$\bigoplus \mathcal{O}_{\widehat{E}} \cdot \binom{T - \eta_\infty}{j}^{-\frac{1}{2}} \implies$$

$$\bigoplus \mathcal{O}_{\widehat{E}} \cdot q^{\approx -j^2/8d} \cdot \binom{T - \eta_\infty}{j}^{-\frac{1}{2}}$$

“Gaussian poles” (cf. e^{-x^2})

Important Theme:

Gaussian and its derivatives

(cf. Hermite polynomials)

...also, main obstruct. to Dio. applics.

Integral Structure at Arch. Primes:

To relate 'DR metric' to 'étale metric'

\implies approximate by comparison

to special functions — models:

Hermite polys. (slope = $\frac{1}{2}$)

Legendre polys. (slope = 1)

= lim (disc. Tchebycheff polys.)

Binomial polys. = $\binom{T}{r}$ (slope = 0)

slope = scaling factor as $d \rightarrow \infty$

(cf. Frobenius on cryst. coh.)

(C.) Arithmetic Kodaira-Spencer Morphism

Main Theorem is a sort of
function-theoretic comp. isom.:

linear fns. + completion of tors. pts.
 \implies get classical comp. isoms.:

Over \mathbf{C} :

$$\begin{array}{ccc} H_{\text{DR}}^1(E, \mathcal{O}_E) & \supseteq & H_{\text{sing}}^1(E, 2\pi i \cdot \mathbf{R}) \\ \downarrow & & \downarrow \\ E^\dagger & \supseteq & E_{\mathbf{R}} \end{array}$$

Over p -adics:

Hodge-Tate, DR comp. isoms:

$$'H_{\text{DR}}^1 \cong H_{\text{ét}}^1,'$$

also def'able by rest. to p^∞ tors. pts.

In general (global, \mathbf{C} , p -adics):

$$\left\{ \text{DR coh.} \right\} \xrightarrow{\sim} \left\{ \text{ét. coh.} \right\} \curvearrowright \underline{\text{Galois}}$$

\implies Galois acts on DR coh.!!

\implies Look at effect on Hodge filtr.

\implies Kodaira-Spencer morphism

motion in base

\mapsto induced motion of Hodge filtr.

Over \mathbf{C} :

“Galois” = $SL_2(\mathbf{R})$ on upp. half-plane
 \implies above ‘arith. KS’ = classical KS

Over p -adics:

$\text{Gal}(\mathbf{Z}_p[[T]]_{\mathbf{Q}_p})$

\approx Tang. bun. $(\mathbf{Z}_p[[T]]_{\mathbf{Q}_p})$

(Faltings’ theory of alm. et. extns.)

\implies above ‘arith. KS’ = classical KS

(cf. Serre-Tate theory)

Hodge-Arakelov (global) Case:

Gal(Base of Fam. of Ell. Curves $\otimes \mathbf{Q}$)

$\xrightarrow{\text{arith. KS}}$ {Arak.-theoretic flag bun.} !!

§2. Philosophy: In Search of an Absolute Derivative

(A.) From Differentiation to Comparison Isomorphisms

S : a scheme; $E \rightarrow S$ fam. of ell. curves

\implies classifying morphism $S \rightarrow \mathcal{M}_{1,0}$

\implies derivative (KS) $\Omega_{\mathcal{M}_{1,0}}|_S \rightarrow \Omega_S$

\Downarrow

Does \exists arithmetic/absolute analogue

$$'\Omega_{\mathcal{M}_{1,0}}|_S \rightarrow \Omega_{\mathbf{Z}/\mathbf{F}_1}'$$

(when $S = \text{Spec}(\mathbf{Z})$,

or $\text{Spec}(\mathcal{O}_F)$, $[F : \mathbf{Q}] < \infty$)?

Observe: $\Omega_{\mathcal{M}_{1,0}}|_S = \omega_E^{\otimes 2}$, and

$$\begin{aligned} \omega_E \hookrightarrow H_{\text{DR}}^1(E) &\xrightarrow{\nabla_{\text{GM}}} H_{\text{DR}}^1(E) \otimes \Omega_S \\ &\longrightarrow \tau_E \otimes \Omega_S \end{aligned}$$

$$\implies \Omega_{\mathcal{M}_{1,0}}|_S = \omega_E^{\otimes 2} \rightarrow \Omega_S \text{ (KS)}$$

(∇_{GM} : Gauss-Manin conn. on H_{DR}^1)

\Downarrow

Since $\exists H_{\text{DR}}^1$, Hodge filtr. ($\omega_E \hookrightarrow H_{\text{DR}}^1$)
in arith. case, need analogue of ∇_{GM}

\implies Recall de Rham isomorphism
(= comparison isomorphism/ \mathbf{C}):

S : Riemann surface \implies

$$H_{\text{DR}}^1(E/S) \cong H_{\text{sing}}^1(E/S, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{O}_S$$

\implies sections of $H_{\text{sing}}^1(E/S, \mathbf{Z})$ are
horizontal for ∇_{GM}

$\implies \nabla_{\text{GM}}$ is the unique conn. for which
sects. of $H_{\text{sing}}^1(E/S, \mathbf{Z})$ are horiz.

\Downarrow

Knowledge of comp. isom. \implies

Knowledge of ∇_{GM}

Conclusion: To construct arith. KS,
suffices to construct arith. comp. isom.

(B.) Function-Theoretic Comparison Isomorphisms

So what form should a (global)
arith. comp. isom. (ACI) take?

(e.g., over \mathbf{C} : $\otimes \mathbf{C}$;
over p -adics: $\otimes B_{\text{DR}}, B_{\text{crys}}$, etc.)

In geometric case/ \mathbf{C} : one implicit sign
of exist. of ∇_{GM} is a sort of 'stability':

$$0 \rightarrow \omega_E \rightarrow H_{\text{DR}}^1(E/S) \rightarrow \tau_E \rightarrow 0$$

If this sequ. split — i.e.,

H_{DR}^1 is 'unstable' — then
 $\exists \nabla$ on ω_E (= ample l.b.): ABSURD!

\implies Even if can't translate ' ∇ ' into
arith. case, can translate stability
— i.e., of Arakelov bundles
= usual v.b. + metric

\implies 'Stability' (e.g., over \mathbf{Z})
= 'equidistrib. of matter in lattice'

Note: Arakelov degree large (small)
 \iff matter dense (sparse)

\Downarrow

Expected Form I of ACI:

$\left\{ \text{Matter Distrib. in DR coh.} \right\}$
 $\cong \left\{ \text{Matter Distrib. in étale coh.} \right\}$

Note: RHS is ‘equidist.’ by ‘Galois’
 \implies By ACI, LHS also ‘equidist.’

In no. theory, ‘matter distributions’
typically measured by ‘test fns.’ \implies

Expected Form II of ACI:

$$\left\{ \text{test fns. on DR coh.} \right\} \\ \cong \left\{ \text{test fns. on étale coh.} \right\}$$

where ‘ \cong ’ is isometry at all primes
of a number field (cf. Arak. theory)

... = the content of the main theorem!!

‘Hodge-Arakelov Comp. Isom.’

‘Split’ distrib. of matter:

‘Equidist.’ distrib. of matter:

(C.) Discretization and the Meaning of Nonlinearity

Note: To measure distrib. in this case, need nonlinear test fns. — cf. linearity of Hodge theory/ \mathbf{C} , p -adics, additive approach to motive theory.

Reasons for Nonlinearity:

- (1.) In Arakelov theory, things tend to become nonlinear (e.g., $H^0(\mathcal{L})$).
- (2.) Nonlinear symmetries of noncomm. torus \approx theta gp. \approx Heisenberg alg. (cf. Gaussians!)

Also, related to discreteness:

Hodge-Arakelov Comp. Isom. =
'discretization of loc. Hodge theories'

— e.g.,

Hodge theory/ \mathbf{C} \approx 'calculus on $E_{\mathbf{R}}$ '

HACI \approx 'discrete calc. on tors. pts.'

\implies periods analogous to

$$2\pi i = \lim_{d \rightarrow \infty} d \cdot (e^{2\pi i/d} - 1)$$

$$= \lim_{d \rightarrow \infty} \text{('theta fns.' on } \mathbf{G}_m \text{ evaluated on tors. pts. of } \mathbf{G}_m)$$