

ON TOPOLOGICAL AND COMBINATORIAL STRUCTURES OF POINTED STABLE CURVES OVER ALGEBRAICALLY CLOSED FIELDS OF POSITIVE CHARACTERISTIC

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Dedicated to the memory of my dear grandmother Jinlan Lin

ABSTRACT. In the present paper, we study anabelian geometry of curves over algebraically closed fields of positive characteristic. Let $X^\bullet = (X, D_X)$ be a pointed stable curve over an algebraically closed field of characteristic $p > 0$ and Π_{X^\bullet} the admissible fundamental group of X^\bullet . We prove that there exists a group-theoretical algorithm whose input datum is the admissible fundamental group Π_{X^\bullet} , and whose output data are the topological and the combinatorial structures associated to X^\bullet . This result can be regarded as a mono-anabelian version of the combinatorial Grothendieck conjecture in positive characteristic. Moreover, by applying this result, we construct clutching maps for moduli spaces of admissible fundamental groups.

Keywords: pointed stable curve, fundamental group, anabelian geometry, positive characteristic.

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INTRODUCTION

0.1. Anabelian geometry. Let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k , where X denotes the underlying curve which is a semi-stable curve over k , D_X denotes the set of marked points satisfying [K, Definition 1.1 (iv)], g_X denotes the genus of X , and n_X denotes the cardinality $\#D_X$ of D_X . Moreover, by choosing a suitable base point x of X^\bullet (i.e., a geometric point whose image is not contained in the singular locus of X), we have the admissible fundamental group (=geometric log étale fundamental group)

$$\pi_1^{\text{adm}}(X^\bullet, x)$$

of X^\bullet (see [Y4, Section 2] for the definitions of admissible coverings and admissible fundamental groups). In the present paper, since we only focus on the isomorphism classes of $\pi_1^{\text{adm}}(X^\bullet, x)$, we omit the base point x and write Π_{X^\bullet} for $\pi_1^{\text{adm}}(X^\bullet, x)$. The admissible fundamental group of a pointed stable curve is a natural generalization of tame fundamental group of a smooth pointed stable curve. In particular, if X^\bullet is smooth over k , then Π_{X^\bullet} is naturally isomorphic to the tame fundamental group $\pi_1^\dagger(X^\bullet, x)$. The main question of interest in the anabelian geometry of curves is, roughly speaking, the following:

How much geometric information about the isomorphism class of a pointed stable curve is contained in the knowledge of its fundamental group?

Suppose that the characteristic $\text{char}(k)$ of k is 0. The structure of Π_{X^\bullet} is well-known, which is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of type (g_X, n_X) ([V, Théorème 2.2 (c)]). In particular, Π_{X^\bullet} is a free profinite group with $2g_X + n_X - 1$ generators if $n_X > 0$. This means that the geometric information of X^\bullet cannot be deduced from the isomorphism class of Π_{X^\bullet} (i.e., no anabelian geometry exists in this situation).

0.2. Moduli spaces and anabelian geometry in positive characteristic. When $\text{char}(k) = p > 0$, the situation is quite different from that in characteristic 0, and the structure of Π_{X^\bullet} is no longer known. In the remainder of the introduction, we assume that $\text{char}(k) = p > 0$, and that $\overline{\mathbb{F}}_p$ is the algebraic closure of \mathbb{F}_p in k . The admissible fundamental group Π_{X^\bullet} is very mysterious. Since the late 1990s, some developments of M. Raynaud ([R]), F. Pop-M. Saïdi ([PS]), A. Tamagawa ([T1], [T2], [T3]), and the author of the present paper ([Y1], [Y2], [Y4]) showed evidence for very strong anabelian phenomena for curves over *algebraically closed fields of characteristic p* . In this situation, the Galois group of the base field is trivial, and the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kinds of anabelian phenomena go beyond Grothendieck's anabelian geometry ([G]), and show that the admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic p must encode “*moduli*” of the curve. Moreover, this is the reason that we do not have an explicit description of the admissible (or tame) fundamental group of any pointed stable curve in positive characteristic.

0.2.1. Let us explain some background about the theory of anabelian geometry of curves over algebraically closed fields of characteristic p from the point of view of moduli spaces that motivated the theory developed in the present paper.

Let X^\bullet be a pointed stable curve of type (g_X, n_X) over k and $o_X : D_X \xrightarrow{\sim} \{1, \dots, n_X\}$ a bijective map. We shall say (X^\bullet, o_X) an ordered pointed stable curve of type (g_X, n_X) over k (i.e., n -pointed stable curve defined in [K, Definition 1.1]). Let $\overline{\mathcal{M}}_{g,n}^{\text{ord}}$ be the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing ordered pointed stable curves of type (g, n) (in the sense of [K]) and $\mathcal{M}_{g,n}^{\text{ord}} \subseteq \overline{\mathcal{M}}_{g,n}^{\text{ord}}$ the open substack classifying smooth ordered pointed stable curves. We denote by $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}_{g,n}$) the quotient $[\overline{\mathcal{M}}_{g,n}^{\text{ord}}/S_n]$ (resp. $[\mathcal{M}_{g,n}^{\text{ord}}/S_n]$) by the natural action of the symmetric group S_n (in the sense of stacks). We write $\overline{M}_{g,n}^{\text{ord}}, M_{g,n}^{\text{ord}}, \overline{M}_{g,n}, M_{g,n}$ for the coarse moduli spaces of $\overline{\mathcal{M}}_{g,n}^{\text{ord}}, \mathcal{M}_{g,n}^{\text{ord}}, \overline{\mathcal{M}}_{g,n}, \mathcal{M}_{g,n}$, respectively.

Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q , and $k(q) \subseteq k'$ an algebraically closed field. Then the natural morphism $\text{Spec } k' \rightarrow \overline{M}_{g,n}$ determines a pointed stable curve X_q^\bullet of type (g, n) over k' . We denote by $\Pi_{X_q^\bullet}$ the admissible fundamental group of X_q^\bullet . Since the isomorphism class of $\Pi_{X_q^\bullet}$ does not depend on the choice of the base field k' , we may write Π_q for $\Pi_{X_q^\bullet}$. We denote by $\overline{\Pi}_{g,n}$ the set of isomorphism classes of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$. Then we have a surjective map

$$\pi_{g,n}^{\text{adm}} : \overline{\mathfrak{M}}_{g,n} \stackrel{\text{def}}{=} \overline{M}_{g,n} / \sim_{fe} \rightarrow \overline{\Pi}_{g,n}, [q] \mapsto [\Pi_q],$$

where \sim_{fe} denotes an equivalence relation determined by Frobenius actions which is called Frobenius equivalence (see [Y4, Definition 3.4]), $[q]$ denotes the image of q in $\overline{\mathfrak{M}}_{g,n}$, and $[\Pi_q]$ denotes the isomorphism class of Π_q in $\overline{\Pi}_{g,n}$.

0.2.2. One of main conjectures in the theory of anabelian geometry of curves in positive characteristic is the so-called *weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic p* (=the Weak Isom-version Conjecture) which was formulated by Tamagawa in the case of smooth pointed stable curves, and by the author in the general case. The Weak Isom-version Conjecture says that $\pi_{g,n}^{\text{adm}}$ is a *bijection*. This means that the moduli spaces of curves can be reconstructed group-theoretically as *sets* from the isomorphism classes of the admissible fundamental groups of curves.

0.2.3. Recently, the author observed that some further structures of moduli spaces of curves in positive characteristic can be deduced from fundamental groups. More precisely, by using two important group-theoretical formulas concerning generalized Hasse-Witt invariants obtained in [Y3], [Y5], in [Y6], the author introduced a topological space which is called *the moduli space of admissible fundamental groups of type (g, n)* , whose underlying set is $\overline{\Pi}_{g,n}$, and whose topology is determined by the sets of finite quotients of admissible fundamental groups of curves. We still use the notation $\overline{\Pi}_{g,n}$ to denote the moduli space of admissible fundamental groups of type (g, n) . Moreover, the author proved that $\pi_{g,n}^{\text{adm}}$ is a continuous map, and posed the so-called *Homeomorphism Conjecture* which says that $\pi_{g,n}^{\text{adm}}$ is a *homeomorphism*, where we regard $\overline{\mathfrak{M}}_{g,n}$ as a topological space whose topology is induced by the Zariski topology of $\overline{M}_{g,n}$. This means that the moduli spaces of curves can be reconstructed group-theoretically as *topological spaces* from the isomorphism classes of the admissible fundamental groups of curves. In [Y6], [Y7], the author proved that the Homeomorphism Conjecture holds when $\dim(\overline{M}_{g,n}) \leq 1$.

The Homeomorphism Conjecture supplies a point of view to see *what anabelian phenomena that we can reasonably expect* from pointed stable curves over algebraically closed fields of characteristic p based on the following philosophy:

The anabelian properties of pointed stable curves of type (g, n) over algebraically closed fields of characteristic p are equivalent to the topological properties of the topological space $\overline{\Pi}_{g,n}$.

0.3. Motivation and questions. Let $R \stackrel{\text{def}}{=} \{r_1, \dots, r_{n_1}\}$ and $S \stackrel{\text{def}}{=} \{s_1, \dots, s_{n_2}\}$ be distinct subsets of $\{1, \dots, n\}$ such that $r_1 < \dots < r_{n_1}$, that $s_1 < \dots < s_{n_2}$, and that $n_1 + n_2 = n$. Let $g_1, g_2, g \in \mathbb{Z}_{\geq 0}$ such that $g = g_1 + g_2$. In the case of moduli spaces of curves, we have the following important morphisms of moduli stacks (i.e., clutching morphisms defined in [K, Definition 3.8]):

$$\begin{aligned} \alpha'_{g_1, g_2, R, S} : \overline{\mathcal{M}}_{g_1, n_1+1}^{\text{ord}} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{g_2, n_2+1}^{\text{ord}} &\rightarrow \overline{\mathcal{M}}_{g, n}^{\text{ord}}, \\ \beta' : \overline{\mathcal{M}}_{g-1, n+2}^{\text{ord}} &\rightarrow \overline{\mathcal{M}}_{g, n}^{\text{ord}}. \end{aligned}$$

We see that $\alpha'_{g_1, g_2, R, S}$ and β' induce the following continuous maps of coarse moduli spaces:

$$\begin{aligned} \tilde{\alpha}_{g_1, g_2, R, S} : \overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}} &\rightarrow \overline{M}_{g, n}^{\text{ord}}, \\ \tilde{\beta} : \overline{M}_{g-1, n+2}^{\text{ord}} &\rightarrow \overline{M}_{g, n}^{\text{ord}}, \end{aligned}$$

where $\overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}}$ denotes the product as topological spaces. The clutching morphisms play important roles for studying the topological properties of moduli spaces of curves (e.g. for studying the p -rank stratification of moduli spaces of curves in positive characteristic (e.g. [AP1], [AP2], [FvdG])). Motivated by some questions in [Y6, Problem 3.9], we ask whether or not one can construct clutching morphisms for moduli spaces of admissible fundamental groups.

0.3.1. Write $\tilde{\pi}_{g,n}^{\text{adm}}$ for the composition of maps $\overline{M}_{g,n} \rightarrow \overline{\mathfrak{M}}_{g,n} \xrightarrow{\pi_{g,n}^{\text{adm}}} \overline{\Pi}_{g,n}$. More precisely, we have the following questions:

Question 0.1. Can we define a set $\overline{\Pi}_{g,n}^{\text{ord}}$ which can be reconstructed group-theoretically from $\overline{\Pi}_{g,n}$ such that the followings are satisfied:

(i) There exists a map $\pi_{g,n}^{\text{adm, ord}} : \overline{M}_{g,n}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}$ which fits into the following commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}^{\text{ord}} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm, ord}}} & \overline{\Pi}_{g,n}^{\text{ord}} \\ \downarrow & & \downarrow \\ \overline{M}_{g,n} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm}}} & \overline{\Pi}_{g,n}. \end{array}$$

(ii) There exist maps

$$\begin{aligned} \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} &\rightarrow \overline{\Pi}_{g, n}^{\text{ord}}, \\ \overline{\Pi}_{g-1, n+2}^{\text{ord}} &\rightarrow \overline{\Pi}_{g, n}^{\text{ord}} \end{aligned}$$

which are compatible with $\tilde{\pi}_{g,n}^{\text{adm, ord}}$, and which can be reconstructed group-theoretically from $\overline{\Pi}_{g,n}$.

Remark 0.1.1. For example, let $[\Pi_i] \in \overline{\Pi}_{g_i, n_i}$, $i \in \{1, 2\}$, and $[\Pi] \in \overline{\Pi}_{g,n}$. Moreover, suppose that Π is isomorphic to the admissible fundamental group of a pointed stable curve W^\bullet over an algebraically closed field of positive characteristic. If we want to define a clutching map $\overline{\Pi}_{g_1, n_1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2}^{\text{ord}} \rightarrow \overline{\Pi}_{g, n}^{\text{ord}}$, we should detect the following group-theoretically from Π : Whether or not Π_i , $i \in \{1, 2\}$, is isomorphic to the admissible fundamental group of a pointed stable curve associated to a sub-semi-graph (see 1.2 of the present paper) of the dual semi-graph Γ_{W^\bullet} of W^\bullet .

The above questions *cannot* be solved by using the classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [G], which focuses on a comparison between two geometric objects via their fundamental groups).

0.3.2. We maintain the notation introduced in 0.1. Moreover, write Γ_{X^\bullet} for the dual semi-graph of X^\bullet . The topological and combinatorial data associated to X^\bullet , roughly speaking, consist of the set of types of pointed stable sub-curves of X^\bullet , the set of admissible fundamental groups of pointed stable sub-curves of X^\bullet , and the set of sub-semi-graph of Γ_{X^\bullet} (e.g. (g_X, n_X) , the dual semi-graph Γ_{X^\bullet} of X^\bullet , the admissible fundamental groups of smooth pointed stable curves associated to irreducible components of X^\bullet , etc., see Definition 1.4 for precise definitions of topological and combinatorial data associated pointed stable curves). Then Question 0.1 is essentially equivalent to the following *mono-anabelian* problem:

Question 0.2. *Does there exist a group-theoretical algorithm whose input datum is an abstract topological group which is isomorphic to Π_{X^\bullet} , and whose output data are the topological and the combinatorial data associated to X^\bullet ?*

The philosophy of “mono-anabelian geometry” was introduced by S. Mochizuki ([M4]). The classical point of view of anabelian geometry focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term “group-theoretical”, in the classical point of view, means that “preserved by an arbitrary isomorphism between the fundamental groups under consideration”. The classical point of view is referred to as *bi-anabelian geometry*. On the other hand, mono-anabelian geometry focuses on the establishing a *group-theoretical algorithm* whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to the fundamental groups of given geometric objects of interest), and whose output datum is a geometry object which is isomorphic to the given geometric object (resp. a morphism of geometric objects which is isomorphic to the given geometric objects) of interest. In the point of view of mono-anabelian geometry, the term “group-theoretical algorithm” is used to mean that “the algorithm in a discussion is phrased in language that only depends on the topological group structure of the fundamental groups under consideration”. Mono-anabelian results are the strongest form in the theory of anabelian geometry, and we have “mono-anabelian results \Rightarrow bi-anabelian results”.

0.4. Main results.

0.4.1. The main theorem of the present paper is as follows (see Theorem 3.11 for a more precise statement):

Theorem 0.3. *We maintain the notation introduced above. Then there exists a group-theoretical algorithm whose input datum is an abstract topological group which is isomorphic to Π_{X^\bullet} , and the output data are the topological and the combinatorial data associated to X^\bullet (i.e., the topological and the combinatorial data associated to X^\bullet can be mono-anabelian reconstructed from Π_{X^\bullet}).*

As a consequence, we obtain the following corollary (see Corollary 3.12, and 1.2 of the present paper for the definitions of various data concerning Γ and $\Gamma \setminus L$):

Corollary 0.4. *We maintain the notation introduced in Theorem 3.11. Let W^\bullet be a pointed stable curve of type (g_W, n_W) over an algebraically closed field of positive characteristic and Π_{W^\bullet} the admissible fundamental group of W^\bullet . Then we can detect group-theoretically whether or not there exists a sub-semi-graph Γ of Γ_{X^\bullet} (resp. a semi-graph associated to a sub-semi-graph Γ of*

Γ_{X^\bullet} and a set of edges L of Γ) such that $(g_W, n_W) = (g_\Gamma, n_\Gamma)$ (resp. $(g_W, n_W) = (g_{\Gamma \setminus L}, n_{\Gamma \setminus L})$) and $\Pi_{W^\bullet} \xrightarrow{\sim} \Pi_{\widehat{\Gamma}}$ (resp. $\Pi_{W^\bullet} \xrightarrow{\sim} \Pi_{\widehat{\Gamma \setminus L}}$).

Remark 0.4.1. We would like to mention that a special case of a *bi-abelian* version of Theorem 0.3 has been proved by the author (see [Y1, Theorem 1.2]). Roughly speaking, [Y1, Theorem 1.2] says that the following holds:

Let $i \in \{1, 2\}$, and let X_i^\bullet be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field of characteristic $p_i > 0$ and $\Pi_{X_i^\bullet}$ the admissible fundamental group of X_i^\bullet . Suppose that $\Pi_{X_1^\bullet} \xrightarrow{\sim} \Pi_{X_2^\bullet}$ is an isomorphism. Then the following data associated to X_i^\bullet are same (e.g. there exists an isomorphism of the dual semi-graphs $\Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$):

- $p_i, (g_{X_i}, n_{X_i}), \Gamma_{X_i^\bullet}$.
- the conjugacy class of the inertia group of every marked point of X_i^\bullet .
- the conjugacy class of the inertia group of every node of X_i^\bullet .
- the conjugacy class of the admissible fundamental group associated to an irreducible component of X_i^\bullet .

Let us explain the difference between [Y1, Theorem 1.2] and Theorem 0.3. [Y1, Theorem 1.2] and its proof tell us that the topological and the combinatorial data associated to X_1^\bullet and X_2^\bullet are same when their admissible fundamental groups are isomorphic. However, [Y1, Theorem 1.2] and its proof *cannot* tell us the relation between X_1^\bullet and X_2^\bullet when their admissible fundamental groups are not isomorphic, and *cannot* tell us how to produce the topological and the combinatorial data associated to X_i^\bullet by only using an abstract topological group which is isomorphic to $\Pi_{X_i^\bullet}$. Then we *cannot* deduce a similar result of Corollary 0.4 from [Y1, Theorem 1.2].

Remark 0.4.2. In this remark, we explain a similar result of Theorem 0.3 in characteristic 0 obtained by Y. Hoshi and Mochizuki. Let $k_i, i \in \{1, 2\}$, be an algebraically closed field of characteristic 0, X_i^\bullet a pointed stable curve over k_i , $\Pi_{X_i^\bullet}$ the admissible fundamental group of X_i^\bullet , $I_i \cong \widehat{\mathbb{Z}}$ a pro-cyclic group, and $\rho_i : I_i \rightarrow \text{Out}(\Pi_{X_i^\bullet}) \stackrel{\text{def}}{=} \text{Aut}(\Pi_{X_i^\bullet})/\text{Inn}(\Pi_{X_i^\bullet})$ an outer Galois representation. Hoshi and Mochizuki proved the following result (see [HM, Theorem A] for a more precise statement):

Suppose that $\rho_i : I_i \rightarrow \text{Out}(\Pi_{X_i^\bullet}), i \in \{1, 2\}$, is a certain outer Galois representation of NN-type ([HM, Definition 2.4]), that $\alpha : \Pi_{X_1^\bullet} \xrightarrow{\sim} \Pi_{X_2^\bullet}$ and $\beta : I_1 \xrightarrow{\sim} I_2$ are isomorphisms of profinite groups, and that the diagram

$$\begin{array}{ccc} I_1 & \xrightarrow{\rho_1} & \text{Out}(\Pi_{X_1^\bullet}) \\ \beta \downarrow & & \text{out}(\alpha) \downarrow \\ I_2 & \xrightarrow{\rho_2} & \text{Out}(\Pi_{X_2^\bullet}), \end{array}$$

is commutative. Then the data appeared in Remark 0.4.1 associated to X_1^\bullet and X_2^\bullet are same.

This result is called *the (bi-abelian) combinatorial Grothendieck conjecture in characteristic 0* which plays a central role in the theory of combinatorial anabelian geometry in characteristic 0. Then Theorem 0.3 can be regarded as a *mono-abelian version of combinatorial Grothendieck conjecture in positive characteristic*. The proof of Hoshi-Mochizuki requires the use of *the non-trivial outer Galois representations which is completely different from* the proof of Theorem 0.3.

The combinatorial Grothendieck conjecture and the theory of combinatorial anabelian geometry in characteristic 0 have many applications (e.g. anabelian geometry for higher dimensional varieties ([HMM]), Belyi-type results([M3], [HM]), mapping class groups ([HI]), Grothendieck-Teichmüller group ([HMM]), etc.). The author hopes that Theorem 0.3 plays a prominent role to establish a theory of combinatorial mono-anabelian geometry in positive characteristic.

0.4.2. By using Theorem 0.3, we solve Question 0.1 as follows (see Theorem 4.1, Theorem 4.3, and Theorem 4.5 for more precise statements):

Theorem 0.5. (i) *There exists a set $\overline{\Pi}_{g,n}^{\text{ord}}$ which can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$. Moreover, there are natural surjective maps*

$$\tilde{\pi}_{g,n}^{\text{adm,ord}} : \overline{M}_{g,n}^{\text{ord}} \twoheadrightarrow \overline{\Pi}_{g,n}^{\text{ord}}, \quad \overline{\Pi}_{g,n}^{\text{ord}} \twoheadrightarrow \overline{\Pi}_{g,n}$$

which fit into the following commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}^{\text{ord}} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm,ord}}} & \overline{\Pi}_{g,n}^{\text{ord}} \\ \downarrow & & \downarrow \\ \overline{M}_{g,n} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm}}} & \overline{\Pi}_{g,n}. \end{array}$$

(ii) Let $R \stackrel{\text{def}}{=} \{r_1, \dots, r_{n_1}\}$ and $S \stackrel{\text{def}}{=} \{s_1, \dots, s_{n_2}\}$ be distinct subsets of $\{1, \dots, n\}$ such that $r_1 < \dots < r_{n_1}$, that $s_1 < \dots < s_{n_2}$, and that $n_1 + n_2 = n$. Let $g_1, g_2, g \in \mathbb{Z}_{\geq 0}$ such that $g = g_1 + g_2$. There exists a map $\alpha_{g_1, g_2, R, S}^{\text{gp}} : \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}$ which fits into the following diagram

$$\begin{array}{ccc} \overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}} & \xrightarrow{\tilde{\alpha}_{g_1, g_2, R, S}} & \overline{M}_{g,n}^{\text{ord}} \\ \tilde{\pi}_{g_1, n_1+1}^{\text{adm,ord}} \times \tilde{\pi}_{g_2, n_2+1}^{\text{adm,ord}} \downarrow & & \tilde{\pi}_{g,n}^{\text{adm,ord}} \downarrow \\ \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} & \xrightarrow{\alpha_{g_1, g_2, R, S}^{\text{gp}}} & \overline{\Pi}_{g,n}^{\text{ord}}. \end{array}$$

Moreover, $\overline{\Pi}_{g_1, n_1+1}^{\text{ord}}$, $\overline{\Pi}_{g_2, n_2+1}^{\text{ord}}$, and $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$.

(iii) There exists a map $\beta^{\text{gp}} : \overline{\Pi}_{g-1, n+2}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}$ which fits into the following diagram

$$\begin{array}{ccc} \overline{M}_{g-1, n+2}^{\text{ord}} & \xrightarrow{\beta} & \overline{M}_{g,n}^{\text{ord}} \\ \tilde{\pi}_{g-1, n+2}^{\text{adm,ord}} \downarrow & & \tilde{\pi}_{g,n}^{\text{adm,ord}} \downarrow \\ \overline{\Pi}_{g-1, n+2}^{\text{ord}} & \xrightarrow{\beta^{\text{gp}}} & \overline{\Pi}_{g,n}^{\text{ord}}. \end{array}$$

Moreover, $\overline{\Pi}_{g-1, n+2}^{\text{ord}}$ and β^{gp} can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$.

Remark 0.5.1. In [Y8], we will prove that $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ and β^{gp} are continuous maps (in the sense of moduli spaces of admissible fundamental groups defined in [Y6]). Moreover, we will prove that the images of $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ and β^{gp} are closed subsets of $\overline{\Pi}_{g,n}$.

The author believes that $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ and β^{gp} will play important roles for studying the purity of the p -rank stratification of $\overline{\Pi}_{g,n}$ and the problems concerning the dimension of $\overline{\Pi}_{g,n}$ (see [Y6, Problem 3.9]).

0.5. Structure of the present paper. The present paper is organized as follows. In Section 1, we introduce some notation and recall some results which will be used in the present paper. In Section 2, we establish a correspondence between a subset of cohomology classes and the set of vertices (resp. the set of edges) of the dual semi-graph of a pointed stable curve. In Section 3, we prove Theorem 0.3. In Section 4, we prove Theorem 0.5.

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1. TOPOLOGICAL AND COMBINATORIAL DATA ASSOCIATED TO POINTED STABLE CURVES

1.1. Semi-graphs.

Definition 1.1. Let \mathbb{G} be a semi-graph ([M2, Section 1]).

(a) We shall denote by $v(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, and $e^{\text{cl}}(\mathbb{G})$ the set of vertices of \mathbb{G} , the set of open edges of \mathbb{G} , and the set of closed edges of \mathbb{G} , respectively. Let $e \in e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G})$ be an edge. We denote by $v(e) \subseteq v(\mathbb{G})$ the subset of vertices which are abutted by e . We shall say e a loop of \mathbb{G} if $e \in e^{\text{cl}}(\mathbb{G})$ and $\#(v(e)) = 1$, where $\#(-)$ denotes the cardinality of $(-)$. We denote by $e^{\text{lp}}(\mathbb{G}) \subseteq e^{\text{cl}}(\mathbb{G})$ the set of loops of \mathbb{G} .

(b) The semi-graph \mathbb{G} can be regarded as a topological space with natural topology induced by \mathbb{R}^2 . We define an *one-point compactification* \mathbb{G}^{cpt} of \mathbb{G} as follows: if $e^{\text{op}}(\mathbb{G}) = \emptyset$, we put $\mathbb{G}^{\text{cpt}} = \mathbb{G}$; otherwise, the set of vertices of \mathbb{G}^{cpt} is the disjoint union $v(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} v(\mathbb{G}) \sqcup \{v_\infty\}$, the set of closed edges of \mathbb{G}^{cpt} is $e^{\text{cl}}(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G})$, the set of open edges of \mathbb{G}^{cpt} is empty, and every edge $e \in e^{\text{op}}(\mathbb{G}) \subseteq e^{\text{cl}}(\mathbb{G}^{\text{cpt}})$ connects v_∞ with the vertex of \mathbb{G} that is abutted by e .

(c) Let $v \in v(\mathbb{G})$. We shall say that \mathbb{G} is *2-connected* at v if $\mathbb{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that \mathbb{G} is *2-connected* if \mathbb{G} is 2-connected at each $v \in v(\mathbb{G})$. Note that, if \mathbb{G} is connected, then \mathbb{G}^{cpt} is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\text{cpt}})$ if and only if \mathbb{G}^{cpt} is 2-connected.

(d) We put

$$b(v) \stackrel{\text{def}}{=} \sum_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v . We put

$$v(\mathbb{G})^{b \leq 1} \stackrel{\text{def}}{=} \{v \in v(\mathbb{G}) \mid b(v) \leq 1\},$$

and denote by $e^{\text{cl}}(\mathbb{G})^{b \leq 1}$ the set of closed edges of \mathbb{G} which meet a vertex of $v(\mathbb{G})^{b \leq 1}$. We put

$$b^{\text{cl}}(v) \stackrel{\text{def}}{=} \sum_{e \in e^{\text{cl}}(\mathbb{G})} b_e(v).$$

We shall say that a vertex v is *terminal* if the following conditions are satisfied: (i) \mathbb{G} is a connected semi-graph. (ii) \mathbb{G} is a tree (i.e., the Betti number of \mathbb{G} is 0). (iii) $b^{\text{cl}}(v) \leq 1$.

(e) Let \mathbb{G}' be a semi-graph. We shall say \mathbb{G}' a *sub-semi-graph* of \mathbb{G} if either $\mathbb{G}' = \{e\}$ for some $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$ or the following conditions hold: (i) \mathbb{G}' is connected and $v(\mathbb{G}') \neq \emptyset$. (ii) $v(\mathbb{G}') \subseteq v(\mathbb{G})$. (iii) $e^{\text{cl}}(\mathbb{G}') \subseteq e^{\text{cl}}(\mathbb{G})$ is the subset of closed edges such that $v(e) \subseteq v(\mathbb{G}')$. (iv) $e^{\text{op}}(\mathbb{G}') \subseteq (e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G})) \setminus e^{\text{cl}}(\mathbb{G}')$ is the subset of edges such that $\#(v(e) \cap v(\mathbb{G}')) = 1$.

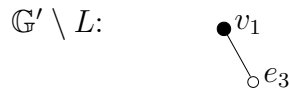
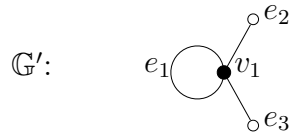
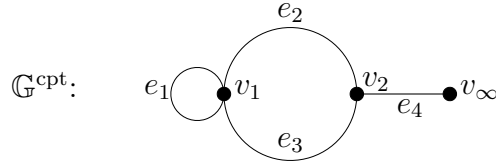
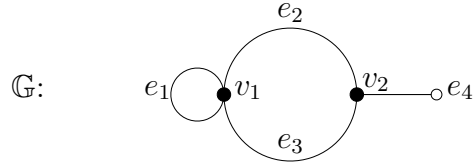
Note that (iii) and (iv) imply that, if e is a loop and $v(e) \subseteq v(\mathbb{G}')$, then $e \in e^{\text{cl}}(\mathbb{G}')$. If $\mathbb{G}' = \{e\}$ for some $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, we will use e to denote \mathbb{G}' .

(f) Let \mathbb{G}' be a sub-semi-graph of \mathbb{G} such that $v(\mathbb{G}') \neq \emptyset$ and $L \subseteq e^{\text{op}}(\mathbb{G}') \cup e^{\text{cl}}(\mathbb{G}')$ a subset of edges of \mathbb{G}' . We shall say $\mathbb{G}' \setminus L$ a *semi-graph associated to \mathbb{G}' and L* if $\mathbb{G}' \setminus L$ is connected (as a topological space), and the following conditions hold (i.e., removing L from \mathbb{G}'): (i) $v(\mathbb{G}' \setminus L) \stackrel{\text{def}}{=} v(\mathbb{G})$. (ii) $e^{\text{op}}(\mathbb{G}' \setminus L) \stackrel{\text{def}}{=} e^{\text{op}}(\mathbb{G}') \setminus L$. (iii) $e^{\text{cl}}(\mathbb{G}' \setminus L) \stackrel{\text{def}}{=} e^{\text{cl}}(\mathbb{G}') \setminus L$.

Remark 1.1.1. Suppose that \mathbb{G} is a connected semi-graph, and that \mathbb{G} is a tree. Then \mathbb{G}^{cpt} is 2-connected if and only if one of the following holds: (i) $\#(v(\mathbb{G})) = 1$; (ii) $\#(v(\mathbb{G})) = 2$ and $\#(e^{\text{op}}(\mathbb{G})) = 0$; (iii) $\#(v(\mathbb{G})) \geq 2$ and each terminal vertex of \mathbb{G} meets some open edge of \mathbb{G} .

Example 1.2. We give some examples of semi-graphs to explain Definition 1.1. We use the notation “ \bullet ” and “ \circ ” to denote a vertex and an open edge, respectively.

Let \mathbb{G} be a semi-graph, \mathbb{G}' a sub-semi-graph of \mathbb{G} such that $v(\mathbb{G}') = \{v_1\}$, and $L \stackrel{\text{def}}{=} \{e_1, e_2\}$ a subset of edges of \mathbb{G}' . Then we have the following:



1.2. Pointed stable curves and admissible fundamental groups.

1.2.1. Let p be a prime number, and let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $\text{char}(k) = p$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X , and

$n_X \stackrel{\text{def}}{=} \#D_X$. Write Γ_{X^\bullet} for the dual semi-graph of X^\bullet and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph Γ_{X^\bullet} .

1.2.2. We shall write Π_{X^\bullet} , $\Pi_{X^\bullet}^{\text{ét}}$, and $\Pi_{X^\bullet}^{\text{top}}$ for the admissible fundamental group of X^\bullet (see [Y4, Section 2] for the definitions of admissible coverings and admissible fundamental groups), the étale fundamental group of X , and the profinite completion of the topological fundamental group of Γ_{X^\bullet} , respectively. Then we have the following natural surjections

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X^\bullet}^{\text{top}}.$$

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. We write X_H^\bullet for the pointed semi-stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H , $\Gamma_{X_H^\bullet}$ for the dual semi-graph of X_H^\bullet , and r_{X_H} for the Betti number of $\Gamma_{X_H^\bullet}$. Then we obtain an admissible covering

$$f_H^\bullet : X_H^\bullet \rightarrow X^\bullet$$

over k induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and obtain a natural morphism of dual semi-graphs

$$f_H^{\text{sg}} : \Gamma_{X_H^\bullet} \rightarrow \Gamma_{X^\bullet}$$

induced by f_H^\bullet , where “sg” means “semi-graph”. We shall say that f_H^\bullet is étale if the underlying morphism $f_H : X_H \rightarrow X$ induced by f_H^\bullet is étale.

Moreover, if H is an open *normal* subgroup, then $\Gamma_{X_H^\bullet}$ admits an action of Π_{X^\bullet}/H induced by the natural action of Π_{X^\bullet}/H on X_H^\bullet . Note that the quotient of $\Gamma_{X_H^\bullet}$ by Π_{X^\bullet}/H coincides with Γ_{X^\bullet} , and that H is isomorphic to the admissible fundamental group $\Pi_{X_H^\bullet}$ of X_H^\bullet . We also use the notation $H^{\text{ét}}$ and H^{top} to denote $\Pi_{X_H^\bullet}^{\text{ét}}$ and $\Pi_{X_H^\bullet}^{\text{top}}$, respectively.

1.2.3. We define pointed stable curves associated to various semi-graphs introduced in Definition 1.1. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph (Definition 1.1 (e)). We write X_Γ for the semi-stable sub-curve of X (i.e., a closed subscheme of X which is a semi-stable curve) whose irreducible components are the irreducible components corresponding to the vertices of $v(\Gamma)$, and whose nodes are the nodes corresponding to the edges of $e^{\text{cl}}(\Gamma)$. Moreover, write D_{X_Γ} for the set of closed points $X_\Gamma \cap \{x_e\}_{e \in e^{\text{op}}(\Gamma) \subseteq e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})}$, where $x_e \in X$ denotes the closed point corresponding to $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We define a pointed stable curve

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma})$$

of type (g_Γ, n_Γ) over k . Note that the dual semi-graph of X_Γ^\bullet is naturally isomorphic to Γ . We shall say X_Γ^\bullet *the pointed stable curve of type (g_Γ, n_Γ) associated to Γ* , or the pointed stable curve associated to Γ for short. We denote by $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of X_Γ^\bullet .

1.2.4. Let $L \subseteq e^{\text{cl}}(\Gamma)$ such that $\Gamma \setminus L$ is a semi-graph associated to Γ and L (i.e., $\Gamma \setminus L$ is connected, see Definition 1.1 (f)), and N_L the set of nodes of X_Γ corresponding to L . We write

$$\text{nor}_L : X_{\Gamma \setminus L} \rightarrow X_\Gamma$$

for the normalization of X_Γ at N_L corresponding to L . Moreover, we put

$$D_{X_{\Gamma \setminus L}} \stackrel{\text{def}}{=} \text{nor}_L^{-1}(D_{X_\Gamma} \cup N_L).$$

We define a pointed stable curve of type $(g_{\Gamma \setminus L}, n_{\Gamma \setminus L})$ to be

$$X_{\Gamma \setminus L}^\bullet = (X_{\Gamma \setminus L}, D_{\Gamma \setminus L}).$$

Note that the dual semi-graph of $X_{\Gamma \setminus L}^\bullet$ is *not isomorphic* to $\Gamma \setminus L$. On the other hand, if write L^{op} for the set of open edges of the dual semi-graph of $X_{\Gamma \setminus L}^\bullet$ corresponding to $\text{nor}_L^{-1}(N_L)$, then there is a natural isomorphism $\Gamma_{X_{\Gamma \setminus L}^\bullet} \setminus L^{\text{op}} \xrightarrow{\sim} \Gamma \setminus L$.

We shall say $X_{\Gamma \setminus L}^\bullet$ *the pointed stable curve of type $(g_{\Gamma \setminus L}, n_{\Gamma \setminus L})$ associated to $\Gamma \setminus L$* , or the pointed stable curve associated to $\Gamma \setminus L$ for short. By the construction of $X_{\Gamma \setminus L}^\bullet$, we see that $r_{X_{\Gamma \setminus L}} = r_{X_\Gamma} - \#(L)$, $g_{\Gamma \setminus L} = g_\Gamma - \#(L)$, and $n_{\Gamma \setminus L} = n_\Gamma + 2\#(L)$. We denote by $\Pi_{X_{\Gamma \setminus L}^\bullet}$ the admissible fundamental group of $X_{\Gamma \setminus L}^\bullet$. Note that we have the following natural outer injections

$$\Pi_{X_{\Gamma \setminus L}^\bullet} \hookrightarrow \Pi_{X_\Gamma^\bullet} \hookrightarrow \Pi_{X^\bullet}.$$

1.2.5. Denote by $\Gamma_v \subseteq \Gamma_{X^\bullet}$ the sub-semi-graph such that $v(\Gamma_v) = \{v\}$. Write X_v for the irreducible component corresponding to v and $\text{nor}_v : \tilde{X}_v \rightarrow X_v$ for the normalization of X_v . We put

$$D_{\tilde{X}_v} \stackrel{\text{def}}{=} \text{nor}_v^{-1}((D_X \cap X_v) \cup (X_v \cap X^{\text{sing}})),$$

where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$. We see that $\tilde{X}_v = X_{\Gamma_v \setminus e^{\text{lp}}(\Gamma_v)}$ and $D_{\tilde{X}_v} = D_{X_{\Gamma_v \setminus e^{\text{lp}}(\Gamma_v)}}$. We shall say

$$\tilde{X}_v^\bullet \stackrel{\text{def}}{=} (\tilde{X}_v, D_{\tilde{X}_v}) = X_{\Gamma_v \setminus e^{\text{lp}}(\Gamma_v)}^\bullet$$

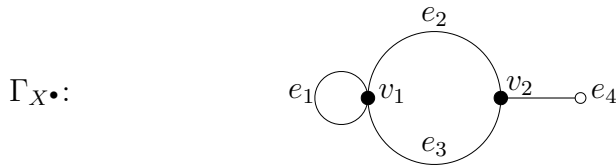
the smooth pointed stable curve of type (g_v, n_v) $\stackrel{\text{def}}{=} (g_{\Gamma_v \setminus e^{\text{lp}}(\Gamma_v)}, n_{\Gamma_v \setminus e^{\text{lp}}(\Gamma_v)})$ associated to v , or the smooth pointed stable curve associated to v for short. We denote by $\Pi_{\tilde{X}_v^\bullet}$ the admissible fundamental group of \tilde{X}_v^\bullet .

By the definition of sub-semi-graphs, we see that $X_{\Gamma_v}^\bullet = \tilde{X}_v^\bullet$ if and only if $\#(v(e)) = 2$ for all $e \in e^{\text{cl}}(\Gamma_v)$ (i.e., Γ_v does not contain loops).

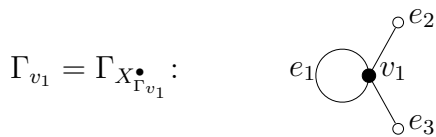
Suppose that $\Gamma_v \subseteq \Gamma$. Then we have the following natural outer injections

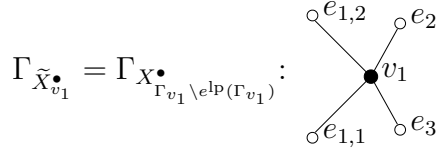
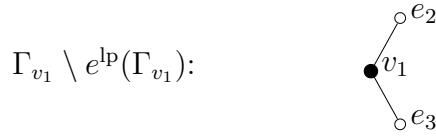
$$\Pi_{\tilde{X}_v^\bullet} \hookrightarrow \Pi_{X_{\Gamma_v}^\bullet} \hookrightarrow \Pi_{X_\Gamma^\bullet} \hookrightarrow \Pi_{X^\bullet}.$$

Example 1.3. Suppose that Γ_{X^\bullet} is a semi-graph as follows:



Then we have





1.3. Topological and combinatorial data.

1.3.1. We put

$$\widehat{X} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H, \quad D_{\widehat{X}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H}, \quad \Gamma_{\widehat{X}^\bullet} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X_H^\bullet}.$$

We shall say that

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

is the universal admissible covering associated to Π_{X^\bullet} , and that $\Gamma_{\widehat{X}^\bullet}$ is the dual semi-graph of \widehat{X}^\bullet . Note that we have that $\text{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of Π_{X^\bullet} . We denote by $\pi_X : \Gamma_{\widehat{X}^\bullet} \twoheadrightarrow \Gamma_{X^\bullet}$ the natural surjection.

1.3.2. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph, $L \subseteq e^{\text{cl}}(\Gamma)$ a subset of closed edges of Γ such that $\Gamma \setminus L$ is a semi-graph associated to Γ and L (i.e., $\Gamma \setminus L$ is connected), $\widehat{\Gamma} \subseteq \widehat{\Gamma}_{X^\bullet}$ a connected component of $\pi_X^{-1}(\Gamma)$, and $\widehat{\Gamma \setminus L}$ a connected component of $\pi_X^{-1}(\Gamma \setminus L)$. We denote by

$$\Pi_{\widehat{\Gamma}} \subseteq \Pi_{X^\bullet}, \quad \Pi_{\widehat{\Gamma \setminus L}} \subseteq \Pi_{X^\bullet}$$

the stabilizer subgroups of $\widehat{\Gamma}$ and $\widehat{\Gamma \setminus L}$, respectively.

Let $v \in v(\Gamma_{X^\bullet})$ and $\widehat{v} \in \pi_X^{-1}(v)$. We denote by $\Pi_{\widehat{v}} \subseteq \Pi_{X^\bullet}$ the stabilizer subgroup of \widehat{v} . We see that

$$\Pi_{\widehat{v}} = \Pi_{\widehat{\Gamma \setminus L}}$$

if $\Gamma = \Gamma_v$ and $L = e^{\text{lp}}(\Gamma_v)$.

1.3.3. By the theory of admissible fundamental groups, the following facts are well-known: $\Pi_{\widehat{\Gamma}}$ is isomorphic to $\Pi_{X_{\widehat{\Gamma}}^\bullet}$, $\Pi_{\widehat{\Gamma \setminus L}}$ is isomorphic to $\Pi_{X_{\Gamma \setminus L}^\bullet}$, and, in particular, $\Pi_{\widehat{v}}$ is (outer) isomorphic to $\Pi_{\widehat{X}_{v_1}^\bullet}$. Note that we have the following natural injections

$$\Pi_{\widehat{\Gamma \setminus L}} \hookrightarrow \Pi_{\widehat{\Gamma}} \hookrightarrow \Pi_{X^\bullet}$$

if $\widehat{\Gamma \setminus L} \subseteq \widehat{\Gamma}$. Moreover, if $\Gamma = \{e\}$ for some $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$, then $I_{\widehat{e}} \stackrel{\text{def}}{=} \Pi_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of X corresponding to e . Then we have that $I_{\widehat{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to- p quotient of $(-)$. Let

$e \in e^{\text{op}}(\Gamma_v) \cup e^{\text{cl}}(\Gamma_v)$ such that \widehat{e} abuts on \widehat{v} , and $\widehat{\Gamma}_v \subseteq \widehat{\Gamma}$. Then we have the following natural injections

$$I_{\widehat{e}} \hookrightarrow \Pi_{\widehat{v}} \hookrightarrow \Pi_{\widehat{\Gamma}_v} \hookrightarrow \Pi_{\widehat{\Gamma}} \hookrightarrow \Pi_{X^\bullet}.$$

1.3.4. We denote by $\text{Sub}(\Gamma_{X^\bullet})$ the set of sub-semi-graphs of Γ_{X^\bullet} and put

$$\text{CSub}(\Gamma_{X^\bullet}) \stackrel{\text{def}}{=} \{\Gamma \setminus L \mid \Gamma \setminus L \text{ is a semi-graph associated to } \Gamma \text{ and } L\}_{\Gamma \in \text{Sub}(\Gamma_{X^\bullet}), L \subseteq e^{\text{cl}}(\Gamma)}.$$

Furthermore, we put

$$\begin{aligned} \text{Sub}(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{\Pi_{\widehat{\Gamma}}\}_{\Gamma \in \text{Sub}(\Gamma_{X^\bullet})}, \\ \text{CSub}(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{\Pi_{\widehat{\Gamma \setminus L}}\}_{\Gamma \setminus L \in \text{CSub}(\Gamma_{X^\bullet})}. \end{aligned}$$

In particular, we denote by

$$\begin{aligned} \text{Ver}(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{\Pi_{\widehat{v}}\}_{\widehat{v} \in v(\Gamma_{X^\bullet})} \subseteq \text{CSub}(\Pi_{X^\bullet}), \\ \text{Edg}^{\text{op}}(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\text{op}}(\Gamma_{X^\bullet})} \subseteq \text{Sub}(\Pi_{X^\bullet}), \\ \text{Edg}^{\text{cl}}(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\text{cl}}(\Gamma_{X^\bullet})} \subseteq \text{Sub}(\Pi_{X^\bullet}). \end{aligned}$$

Note that $\text{Sub}(\Pi_{X^\bullet})$, $\text{CSub}(\Pi_{X^\bullet})$, $\text{Ver}(\Pi_{X^\bullet})$, $\text{Edg}^{\text{op}}(\Pi_{X^\bullet})$, and $\text{Edg}^{\text{cl}}(\Pi_{X^\bullet})$ admit natural actions of Π_{X^\bullet} (i.e., the conjugacy actions), and that we have the following natural bijections

$$\begin{aligned} \text{Sub}(\Pi_{X^\bullet})/\Pi_{X^\bullet} &\xrightarrow{\sim} \text{Sub}(\Gamma_{X^\bullet}), \\ \text{CSub}(\Pi_{X^\bullet})/\Pi_{X^\bullet} &\xrightarrow{\sim} \text{CSub}(\Gamma_{X^\bullet}), \\ \text{Ver}(\Pi_{X^\bullet})/\Pi_{X^\bullet} &\xrightarrow{\sim} v(\Gamma_{X^\bullet}), \\ \text{Edg}^{\text{op}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} &\xrightarrow{\sim} e^{\text{op}}(\Gamma_{X^\bullet}), \\ \text{Edg}^{\text{cl}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} &\xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet}). \end{aligned}$$

Definition 1.4. We shall say that $\{(g_\Gamma, n_\Gamma)\}_{\Gamma \in \text{Sub}(\Gamma_{X^\bullet})}$ and $\{(g_{\Gamma \setminus L}, n_{\Gamma \setminus L})\}_{\Gamma \setminus L \in \text{CSub}(\Gamma_{X^\bullet})}$ are the *topological data* associated to X^\bullet , and that $\text{Sub}(\Gamma_{X^\bullet})$, $\text{CSub}(\Gamma_{X^\bullet})$, $\text{Sub}(\Pi_{X^\bullet})$, and $\text{CSub}(\Pi_{X^\bullet})$ are the *combinatorial data* associated to X^\bullet .

1.4. **The limit of p -averages.** Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{p^t-1} the kernel of the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}$, where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$. The following important group-theoretical invariant was introduced by Tamagawa ([T2]). We put

$$\text{Avr}_p(\Pi_{X^\bullet}) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \frac{\dim_{\mathbb{F}_p}(K_{p^t-1}^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})},$$

and shall say that $\text{Avr}_p(\Pi_{X^\bullet})$ is the *limit of p -averages of Π_{X^\bullet}* . The following formula concerning $\text{Avr}_p(\Pi_{X^\bullet})$ plays a fundamental role in the theory of (tame or admissible) anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Theorem 1.5. *We maintain the notation introduced above. Suppose that $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected (Definition 1.1 (c)). Then we have*

$$\text{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet})^{b \leq 1} + \#e^{\text{cl}}(\Gamma_{X^\bullet})^{b \leq 1}.$$

Proof. We maintain the notation introduced in [Y3, Theorem 5.2]. Note that $\#v(\Gamma_{X^\bullet})^{b \leq 1} = \#V_{X^\bullet}^{\text{tre}}$ and $\#e^{\text{cl}}(\Gamma_{X^\bullet})^{b \leq 1} = \#E_{X^\bullet}^{\text{tre}}$. Then the theorem follows from [Y3, Theorem 5.2]. \square

Remark 1.5.1. Suppose that $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected. Note that $\#v(\Gamma_{X^\bullet})^{b \leq 1} \neq 0$ if one of the following conditions holds: (i) X^\bullet is smooth and $e^{\text{op}}(\Gamma_{X^\bullet}) \leq 1$; (ii) $D_X = \emptyset$ (i.e., $\#e^{\text{op}}(\Gamma_{X^\bullet}) = 0$), $\#e^{\text{cl}}(\Gamma_{X^\bullet}) = 1$, and $\#v(\Gamma_{X^\bullet}) = 2$. In particular, if $\#v(\Gamma_{X^\bullet})^{b \leq 1} \neq 0$, we have $\text{Avr}_p(\Pi_{X^\bullet}) = g_X - 1$.

Remark 1.5.2. Let Δ be an arbitrary profinite group and $m, N \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup $D_N(\Delta)$ of Δ to be the topological closure of $[\Delta, \Delta]\Delta^N$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ . Moreover, we define the closed normal subgroup $D_N^{(m)}(\Delta)$ of Δ inductively by $D_N^{(1)}(\Delta) \stackrel{\text{def}}{=} D_N(\Delta)$ and $D_N^{(i+1)}(\Delta) \stackrel{\text{def}}{=} D_N(D_N^{(i)}(\Delta))$, $i \in \{1, \dots, m-1\}$. Let $\ell \neq p$ be a prime number. We put

$$H \stackrel{\text{def}}{=} D_\ell^{(3)}(\Pi_{X^\bullet}).$$

Then we see that the following conditions hold (e.g. [Y6, Lemma 5.4]): (i) $(\#(\Pi_{X^\bullet}/H), p) = 1$; (ii) the genus of $\tilde{X}_{H,v}$ is positive for each $v \in v(\Gamma_{X_H^\bullet})$; (iii) $\Gamma_{X_H^\bullet}$ is 2-connected and $\#(v(\Gamma_{X_H^\bullet})^{b \leq 1}) = 0$.

1.4.1. Let $f^\bullet : Y^\bullet \rightarrow X^\bullet$ be an admissible covering over k , $f : Y \rightarrow X$ the underlying morphism induced by f^\bullet , and $\deg(f)$ the degree of f . For any $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\text{op}}(\Gamma_{X^\bullet})$), write x_e for the node (resp. marked point) of X^\bullet corresponding to e . We put

$$\begin{aligned} e_f^{\text{cl,ra}} &\stackrel{\text{def}}{=} \{e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e) = 1\}, \\ e_f^{\text{cl,\acute{e}t}} &\stackrel{\text{def}}{=} \{e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e) = \deg(f)\}, \\ e_f^{\text{op,ra}} &\stackrel{\text{def}}{=} \{e \in e^{\text{op}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e) = 1\}, \\ v_f^{\text{ra}} &\stackrel{\text{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#\text{Irr}(f^{-1}(X_v)) = 1\}, \\ v_f^{\text{sp}} &\stackrel{\text{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#\text{Irr}(f^{-1}(X_v)) = \deg(f)\}, \end{aligned}$$

where $\text{Irr}(-)$ denotes the set of irreducible components of $(-)$. If the Galois closure of f^\bullet is a Galois admissible covering whose Galois group is a p -group, then the definition of admissible coverings implies that $\#e_f^{\text{cl,ra}} = \#e_f^{\text{op,ra}} = 0$.

1.4.2. We have the following lemma.

Lemma 1.6. *Let $f^\bullet : Y^\bullet \rightarrow X^\bullet$ be a Galois admissible covering over k and Γ_{Y^\bullet} the dual semi-graph of Y^\bullet . Suppose that $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected, and that $\Gamma_{Y^\bullet}^{\text{cpt}}$ is not 2-connected. Then there exists a unique vertex $v \in v(\Gamma_{X^\bullet})$ such that $f^{-1}(X_v)$ is irreducible.*

Proof. Let $f^{\text{sg}} : \Gamma_{Y^\bullet} \rightarrow \Gamma_{X^\bullet}$ be the map of dual semi-graphs induced by f^\bullet and $v \in v(\Gamma_{X^\bullet})$ an arbitrary vertex of Γ_{X^\bullet} . Since $\Gamma_{Y^\bullet}^{\text{cpt}}$ is not 2-connected, we have $\#v(\Gamma_{X^\bullet}) \geq 2$.

Suppose that $D_X \neq \emptyset$. Then we have $v(\Gamma_{X^\bullet}^{\text{cpt}}) = v(\Gamma_{X^\bullet}) \cup \{v_{X,\infty}\}$ and $v(\Gamma_{Y^\bullet}^{\text{cpt}}) = v(\Gamma_{Y^\bullet}) \cup \{v_{Y,\infty}\}$. Moreover, f^{sg} can be extended to a map

$$f^{\text{sg,cpt}} : \Gamma_{Y^\bullet}^{\text{cpt}} \rightarrow \Gamma_{X^\bullet}^{\text{cpt}}$$

such that $f^{\text{sg,cpt}}(v_{Y,\infty}) = v_{X,\infty}$. Since $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected, we obtain that $\Gamma_{X^\bullet}^{\text{cpt}} \setminus \{v\}$ is connected, and that $(f^{\text{sg,cpt}})^{-1}(\Gamma_{X^\bullet}^{\text{cpt}} \setminus \{v\})$ is connected. Then, for each $w \in (f^{\text{sg,cpt}})^{-1}(v)$, there exists a closed edge of $\Gamma_{Y^\bullet}^{\text{cpt}}$ which meets w and $(f^{\text{sg,cpt}})^{-1}(\Gamma_{X^\bullet}^{\text{cpt}} \setminus \{v\})$. We obtain that $\Gamma_{Y^\bullet}^{\text{cpt}}$ is 2-connected. This contradicts our assumptions. Then we may assume that $D_X = \emptyset$.

Since we assume that $D_X = \emptyset$, we have $\Gamma_{X^\bullet}^{\text{cpt}} = \Gamma_{X^\bullet}$ and $\Gamma_{Y^\bullet}^{\text{cpt}} = \Gamma_{Y^\bullet}$. Let v_1 and v_2 be vertices of $v(\Gamma_{X^\bullet})$ distinct from each other. Suppose that $(f^{\text{sg}})^{-1}(v_1)$ and $(f^{\text{sg}})^{-1}(v_2)$ are connected. Since Γ_{X^\bullet} is 2-connected, $\Gamma_{X^\bullet} \setminus \{v\}$ is connected. If $v \notin \{v_1, v_2\}$, we see that $(f^{\text{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v\})$

is connected (since v_1 and v_2 are contained in $\Gamma_{X^\bullet} \setminus \{v\}$), and that, for each $w \in (f^{\text{sg}})^{-1}(v)$, there exists a closed edge of Γ_{Y^\bullet} which meets w and $(f^{\text{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v\})$. This means that Γ_{Y^\bullet} is 2-connected at w for each $w \in (f^{\text{sg}})^{-1}(v)$. Suppose that $v = v_1$. Then we have that $\Gamma_{X^\bullet} \setminus \{v_1\}$ is connected, and that $(f^{\text{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v_1\})$ is connected (since v_2 is contained in $\Gamma_{X^\bullet} \setminus \{v_1\}$). Thus Γ_{Y^\bullet} is 2-connected at $(f^{\text{sg}})^{-1}(v_1)$. Similar arguments to the arguments given in the proof above imply that Γ_{Y^\bullet} is 2-connected at $(f^{\text{sg}})^{-1}(v_2)$. Then Γ_{Y^\bullet} is 2-connected. This contradicts our assumptions.

Suppose that $(f^{\text{sg}})^{-1}(v')$ is not connected for each $v' \in v(\Gamma_{X^\bullet})$. Then we have the following:

Claim: Let $w \in (f^{\text{sg}})^{-1}(v)$. Then $\Gamma_{Y^\bullet} \setminus \{w\}$ is connected.

Let us prove the claim. Since we focus only on v and its inverse images $(f^{\text{sg}})^{-1}(v)$ and Γ_{X^\bullet} is 2-connected (i.e., $\Gamma_{X^\bullet} \setminus \{v\}$ is connected), to verify the claim, it's sufficient to prove the claim when $v(\Gamma_{X^\bullet}) = \{v, v^*\}$ (e.g. by replacing X^\bullet by the deformation of X^\bullet along the set of nodes corresponding to the set of closed edges $e^{\text{cl}}(\Gamma_{X^\bullet}) \setminus e(v)$, where $e(v)$ denotes the set of edges of Γ_{X^\bullet} which abuts to v).

Let $e_{v,v^*} \in e^{\text{cl}}(\Gamma_{X^\bullet})$. Then e_{v,v^*} meets v and v^* . Since $(f^{\text{sg}})^{-1}(v')$ is not connected for each $v' \in v(\Gamma_{X^\bullet})$, we see that $(f^{\text{sg}})^{-1}(e_{v,v^*})$ is a *loop* in Γ_{Y^\bullet} (i.e., the element of the topological fundamental group $\pi_1^{\text{top}}(\Gamma_{Y^\bullet})$ induced by $(f^{\text{sg}})^{-1}(e_{v,v^*})$ is not *trivial*), and that $(f^{\text{sg}})^{-1}(v)$ and $(f^{\text{sg}})^{-1}(v^*)$ are contained in $(f^{\text{sg}})^{-1}(e_{v,v^*})$. Then $\Gamma_{Y^\bullet} \setminus \{w\}$ is connected for all $w \in (f^{\text{sg}})^{-1}(v)$.

On the other hand, the above claim contradicts our assumptions that Γ_{Y^\bullet} is not 2-connected (i.e., there exists $v'' \in v(\Gamma_{X^\bullet})$ such that $\Gamma_{Y^\bullet} \setminus \{w''\}$ is not connected for some $w'' \in (f^{\text{sg}})^{-1}(v'')$). We complete the proof of the lemma. \square

2. COHOMOLOGY CLASSES, SETS OF VERTICES, AND SETS OF EDGES

2.0.1. Settings. Let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, Π_{X^\bullet} the admissible fundamental group of X^\bullet , Γ_{X^\bullet} the dual semi-graph of X^\bullet , and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph Γ_{X^\bullet} .

2.1. Sets of vertices. Some results of this subsection are also contained in [Y1, Section 3].

2.1.1. Let ℓ be a prime number. We put

$$v(\Gamma_{X^\bullet})^{>0,\ell} \stackrel{\text{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \dim_{\mathbb{F}_\ell}(H_{\text{ét}}^1(\tilde{X}_v, \mathbb{F}_\ell)) > 0\} \subseteq v(\Gamma_{X^\bullet}),$$

where \tilde{X}_v denotes the normalization of X_v (1.2.5). Write $M_{X^\bullet}^{\text{ét}}$ and $M_{X^\bullet}^{\text{top}}$ for $\text{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{F}_\ell)$ and $\text{Hom}(\Pi_{X^\bullet}^{\text{top}}, \mathbb{F}_\ell)$, respectively (1.2.2). Note that there is a natural injection $M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{ét}}$ induced by the natural surjection $\Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X^\bullet}^{\text{top}}$. Moreover, we put

$$M_{X^\bullet}^{\text{nt}} \stackrel{\text{def}}{=} \text{coker}(M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{ét}}),$$

where “nt” means that “non-top”.

2.1.2. The elements of $M_{X^\bullet}^{\text{ét}}$ correspond to étale, Galois abelian coverings of X^\bullet of degree ℓ . We denote by $V_{X^\bullet, \ell}^* \subseteq M_{X^\bullet}^{\text{ét}}$ the subset of elements whose image in $M_{X^\bullet}^{\text{nt}}$ is not 0. Let $\alpha \in V_{X^\bullet, \ell}^*$. We denote by

$$X_\alpha^\bullet = (X_\alpha, D_{X_\alpha}) \rightarrow X^\bullet$$

the étale covering (i.e., the morphism of underlying curves is étale) corresponding to the element α and denote by $\Gamma_{X_\alpha^\bullet}$ the dual semi-graph of X_α^\bullet . Then we have a map

$$\iota : V_{X,\ell}^* \rightarrow \mathbb{Z}, \alpha \mapsto \#v(\Gamma_{X_\alpha^\bullet}).$$

Furthermore, we put

$$\begin{aligned} V_{X,\ell}^* &\stackrel{\text{def}}{=} \{\alpha \in V_{X,\ell}^* \mid \iota \text{ attains its maximum}\} \\ &= \{\alpha \in V_{X,\ell}^* \mid \iota(\alpha) = \ell \#v(\Gamma_{X^\bullet}) - \ell + 1\} \\ &= \{\alpha \in V_{X,\ell}^* \mid \#v_{f_\alpha}^{\text{ra}} = 1\} \text{(1.4.1)}. \end{aligned}$$

For each $\alpha \in V_{X,\ell}^*$, $\iota(\alpha) = \ell \#v(\Gamma_{X^\bullet}) - \ell + 1$ implies that there exists a unique irreducible component $Z \subseteq X_\alpha$ whose decomposition group under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is not trivial. Let $v_\alpha \in v(\Gamma_{X^\bullet})$ such that $X_{v_\alpha} = f_\alpha(Z)$. Then we have $v_\alpha \in v(\Gamma_{X^\bullet})^{>0,\ell}$. This means that $V_{X,\ell}^* = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$.

2.1.3. Let S, S' be sets. We shall call $f : S \rightarrow S'$ a *quasi-map* if f is a map from some subset $S_1 \subseteq S$ to S' . Moreover, suppose that S^{\max} is the maximal subset of S such that f is a map from S^{\max} to S' . Let $S^* \stackrel{\text{def}}{=} S \setminus S^{\max}$. Then we shall write $f(s) = \emptyset$ for all $s \in S^*$.

Let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f_H^{\text{sg}} : \Gamma_{X_H^\bullet} \rightarrow \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f_H^\bullet : X_H^\bullet \rightarrow X^\bullet$ over k corresponding to H . We define a quasi-map (i.e., we allow that an element maps to empty set)

$$f_H^{\text{ver},\ell} : v(\Gamma_{X_H^\bullet})^{>0,\ell} \rightarrow v(\Gamma_{X^\bullet})^{>0,\ell}$$

as follows: Let $v_H \in v(\Gamma_{X_H^\bullet})^{>0,\ell}$ and $v \stackrel{\text{def}}{=} f_H^{\text{sg}}(v_H) \in v(\Gamma_{X^\bullet})$. Then we have that $f_H^{\text{ver},\ell}(v_H) = v$ if $\dim_{\mathbb{F}_\ell}(\text{Hom}(\Pi_{X_v}^{\text{ét}}, \mathbb{F}_\ell)) \neq 0$; otherwise, $f_H^{\text{ver},\ell}(v_H) = \emptyset$. Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $v(\Gamma_{X_H^\bullet})^{>0,\ell}$ admits a natural action of Π_{X^\bullet}/H .

Proposition 2.1. *We define a pre-equivalence relation \sim on $V_{X,\ell}^*$ as follows:*

Let $\alpha, \beta \in V_{X,\ell}^$. We have that $\alpha \sim \beta$ if $\lambda\alpha + \mu\beta \in V_{X,\ell}^*$ for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in V_{X,\ell}^*$.*

Then \sim is an equivalence relation on $V_{X,\ell}^$. Moreover, we have a natural bijection*

$$\kappa_{X,\ell} : V_{X,\ell} \stackrel{\text{def}}{=} V_{X,\ell}^* / \sim \xrightarrow{\sim} v(\Gamma_{X^\bullet})^{>0,\ell}, [\alpha] \mapsto v_\alpha,$$

where $[\alpha]$ denotes the image of α in $V_{X,\ell}$.

Proof. Since $V_{X,\ell} = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$, we may suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \neq \emptyset$. Let $\alpha, \beta \in V_{X,\ell}^*$.

If $v_\alpha = v_\beta$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $v_{\lambda\alpha + \mu\beta} = v_\alpha = v_\beta$. Thus, $\alpha \sim \beta$.

On the other hand, if $\alpha \sim \beta$, we have $v_\alpha = v_\beta$; otherwise, there exist two irreducible components of $X_{\alpha+\beta}^\bullet$ whose decomposition groups under the actions of $\mathbb{Z}/\ell\mathbb{Z}$ are not trivial (i.e., $\alpha + \beta \notin V_{X,\ell}^*$). Thus, $\alpha \sim \beta$ if and only if $v_\alpha = v_\beta$. This means that \sim is an equivalence relation on $V_{X,\ell}^*$.

Next, we prove that the map

$$\kappa_{X,\ell} : V_{X,\ell} \rightarrow v(\Gamma_{X^\bullet})^{>0,\ell}, [\alpha] \mapsto v_\alpha$$

is a bijection. It is easy to see that $\kappa_{X,\ell}$ is an injection. On the other hand, for any irreducible component $X_v \in v(\Gamma_{X^\bullet})^{>0,\ell}$, we see that there is a Galois étale covering $f^\bullet : Y^\bullet \rightarrow X^\bullet$ (i.e., the underlying morphism f is étale) whose Galois group is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ such that X_v is the unique irreducible component of X^\bullet whose inverse image $f^{-1}(X_v)$ is connected. Then the

cardinality of the set of irreducible components of Y^\bullet is equal to $\ell(\#v(\Gamma_{X^\bullet}) - 1) + 1$. Thus, Y^\bullet induces an element of $V_{X,\ell}$. This implies that $\kappa_{X,\ell}$ is a surjection. We complete the proof of the proposition. \square

Remark 2.1.1. Let ℓ and ℓ' be prime numbers distinct from each other. Write

$$V_{X,\ell}, V_{X,\ell'}$$

for the sets associated to ℓ and ℓ' defined above, respectively. Suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \subseteq v(\Gamma_{X^\bullet})^{>0,\ell'}$ (note that $v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet})^{>0,\ell'}$ if ℓ and ℓ' are not equal to p). Then we may define a natural injection

$$V_{X,\ell} \hookrightarrow V_{X,\ell'}$$

which fits into the following commutative diagram

$$\begin{array}{ccc} V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell} \\ \downarrow & & \downarrow \\ V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^\bullet})^{>0,\ell'} \end{array}$$

as follows: For each $\alpha \in V_{X,\ell}$ and each $\alpha' \in V_{X,\ell'}$, we write $X_\alpha^\bullet \rightarrow X^\bullet$ and $X_{\alpha'}^\bullet \rightarrow X^\bullet$ for the Galois admissible coverings corresponding to α and α' , respectively. We consider the following connected Galois admissible covering

$$X_\alpha^\bullet \times_{X^\bullet} X_{\alpha'}^\bullet \rightarrow X^\bullet$$

over k whose Galois group is isomorphic to $\mathbb{Z}/\ell\ell'\mathbb{Z}$, where $X_\alpha^\bullet \times_{X^\bullet} X_{\alpha'}^\bullet$ denotes the fiber product in the category of pointed stable curves. Then it is easy to see that $v_\alpha = v_{\alpha'}$ if and only if the cardinality of the set of irreducible components of $X_\alpha^\bullet \times_{X^\bullet} X_{\alpha'}^\bullet$ is equal to

$$\ell\ell'(\#v(\Gamma_{X^\bullet}) - 1) + 1.$$

Then we obtain a natural injection

$$V_{X,\ell} \hookrightarrow V_{X,\ell'}, [\alpha] \mapsto [\alpha'],$$

where the cardinality of the set of irreducible components of $X_\alpha^\bullet \times_{X^\bullet} X_{\alpha'}^\bullet$ is equal to $\ell\ell'(\#v(\Gamma_{X^\bullet}) - 1) + 1$. In particular, if ℓ and ℓ' are not equal to p , then the injection $V_{X,\ell} \hookrightarrow V_{X,\ell'}$ constructed above is a bijection.

Remark 2.1.2. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup,

$$f_H^\bullet : X_H^\bullet = (X_H, D_{X_H}) \rightarrow X^\bullet$$

the admissible covering over k with degree $\deg(f_H)$ corresponding to H , $\Gamma_{X_H^\bullet}$ the dual semi-graph of X_H^\bullet , and ℓ a prime number such that $(\ell, \deg(f_H)) = 1$.

Write $V_{X_H,\ell}$ and $V_{X,\ell}$ for the sets defined above. Then we claim that the natural injection $H \hookrightarrow \Pi_{X^\bullet}$ induces a quasi-map

$$\gamma_H^{\text{ver},\ell} : V_{X_H,\ell} \rightarrow V_{X,\ell}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X_H^\bullet})^{>0,\ell} \\ \gamma_H^{\text{ver},\ell} \downarrow & & f_H^{\text{ver},\ell} \downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell}. \end{array}$$

Moreover, suppose that $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup. Then $V_{X_H,\ell}$ admits an action of Π_{X^\bullet}/H such that $\kappa_{X_H,\ell}$ is compatible with Π_{X^\bullet}/H -actions (i.e., $\kappa_{X_H,\ell}$ is Π_{X^\bullet}/H -equivariant).

We prove the claim. Let $[\alpha_X] \in V_{X,\ell}$. Then α_X induces an element $\beta_{X_H} \in \text{Hom}(H, \mathbb{F}_\ell)$ via the natural homomorphism $\text{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow \text{Hom}(H, \mathbb{Z}/\ell\mathbb{Z})$ such that β_{X_H} can be written as

$$\sum_{\beta \in L_{\alpha_X}} c_\beta \beta, \quad c_\beta \in \mathbb{F}_\ell^\times,$$

where L_{α_X} is a subset of $V_{X_H,\ell}^*$ such that, if $\beta_1, \beta_2 \in L_{\alpha_X}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$.

Let $[\alpha_{X_H}] \in V_{X_H,\ell}$. Then we define $\gamma_H^{\text{ver},\ell}([\alpha_{X_H}]) = [\alpha_X]$ if there exists $[\alpha_X] \in V_{X,\ell}$ such that $[\beta] = [\alpha_{X_H}]$ (i.e., $\beta \sim \alpha_{X_H}$) for some $\beta \in L_{\alpha_X}$. Otherwise, we put $\gamma_H^{\text{ver},\ell}([\alpha_{X_H}]) = \emptyset$. It is easy to check that $\gamma_H^{\text{ver},\ell}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X_H^\bullet})^{>0,\ell} \\ \gamma_H^{\text{ver},\ell} \downarrow & & \downarrow f_H^{\text{ver},\ell} \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell} \end{array}$$

is commutative.

Moreover, suppose that H is an open normal subgroup of Π_{X^\bullet} . The natural exact sequence

$$1 \rightarrow H \rightarrow \Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}/H \rightarrow 1$$

induces an outer representation

$$\Pi_{X^\bullet}/H \rightarrow \text{Out}(H) \stackrel{\text{def}}{=} \frac{\text{Aut}(H)}{\text{Inn}(H)}.$$

Then we obtain an action of Π_{X^\bullet}/H on $V_{X_H,\ell}^* \subseteq \text{Hom}(H^{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) = \text{Hom}(H^{\text{ét,ab}}, \mathbb{Z}/\ell\mathbb{Z})$ induced by the outer representation. Let $\sigma \in \Pi_{X^\bullet}/H$ and $\alpha_{X_H}, \alpha'_{X_H} \in V_{X_H,\ell}^*$. Then we have that $\alpha_{X_H} \sim \alpha'_{X_H}$ if and only if $\sigma(\alpha_{X_H}) \sim \sigma(\alpha'_{X_H})$. Thus, we obtain an action of Π_{X^\bullet}/H on $V_{X_H,\ell}$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is easy to check that the above commutative diagram is compatible with the Π_{X^\bullet}/H -actions.

Remark 2.1.3. We maintain the notation introduced in Remark 2.1.2. In this remark, we explain that $\gamma_H^{\text{ver},\ell} : V_{X_H,\ell} \rightarrow V_{X,\ell}$ defined above can be described in another way which will be used in the remainder of the present paper.

Write Q_{α_X} for the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \xrightarrow{\alpha_X} \mathbb{F}_\ell$. Let $\beta \in V_{X_H,\ell}^*$. Write Q_β for the kernel $H \twoheadrightarrow H^{\text{ét}} \xrightarrow{\beta} \mathbb{F}_\ell$. Note that $X_{Q_{\alpha_X} \cap Q_\beta}^\bullet$ is isomorphic to a connected component of $X_{Q_\beta}^\bullet \times_{X^\bullet} X_{Q_{\alpha_X}}^\bullet$, and that $X_{Q_{\alpha_X} \cap H}^\bullet$ is isomorphic to $X_H^\bullet \times_{X^\bullet} X_{Q_{\alpha_X}}^\bullet$. Then we see that $\beta \in L_{\alpha_X}$ (see Remark 2.1.2 for L_{α_X}) if and only if one of the following statements holds: (1) $Q_\beta = Q_{\alpha_X} \cap H$; (2)

$$\#v(\Gamma_{X_{Q_{\alpha_X} \cap Q_\beta}^\bullet}) = \ell \#v(\Gamma_{X_{Q_{\alpha_X} \cap H}^\bullet}).$$

Note that (1) (resp. (2)) happens if X_H^\bullet is (resp. is not) irreducible. Namely, we have the following:

Let $[\alpha_{X_H}] \in V_{X_H,\ell}$. Then $\gamma_H^{\text{ver},\ell}([\alpha_{X_H}]) = [\alpha_X]$ if and only if one of the following holds: (1) there exists $\beta \in V_{X_H,\ell}^*$ such that $\beta \sim \alpha_{X_H}$ and $Q_\beta = Q_{\alpha_X} \cap H$; (2) there exists $\beta \in V_{X_H,\ell}^*$ such that $\beta \sim \alpha_{X_H}$, and that

$$\#v(\Gamma_{X_{Q_{\alpha_X} \cap Q_\beta}^\bullet}) = \ell \#v(\Gamma_{X_{Q_{\alpha_X} \cap H}^\bullet}).$$

2.2. Sets of edges.

2.2.1. Assumptions. We maintain the notation introduced in 2.0.1. Moreover, in this subsection, we suppose that the genus g_v of \tilde{X}_v^\bullet (1.2.5) is *positive* for each $v \in v(\Gamma_{X^\bullet})$, and that $\Gamma_{X^\bullet}^{\text{cpt}}$ is *2-connected*.

2.2.2. We shall say that

$$\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \rightarrow X^\bullet)$$

is an *edge-triple* associated to X^\bullet if the following conditions are satisfied:

- (i) ℓ and d are prime numbers distinct from each other and from p ;
- (ii) $\ell \equiv 1 \pmod{d}$; this means that all d th roots of unity are contained in \mathbb{F}_ℓ ; moreover, we write $\mu_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of d th roots of unity;
- (iii) $f_X^\bullet : Y^\bullet \stackrel{\text{def}}{=} (Y, D_Y) \rightarrow X^\bullet$ is a Galois *étale* covering (i.e., the underlying morphism $f_X : Y \rightarrow X$ is étale) whose Galois group is isomorphic to μ_d such that $\#v_{f_X}^{\text{sp}} = 0$ (1.4.1) holds (note that since $g_v, v \in v(\Gamma_{X^\bullet})$, is positive, f_X^\bullet exists).

2.2.3. In the remainder of this subsection, we fix an edge-triple $\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \rightarrow X^\bullet)$ associated to X^\bullet . Let $\Pi_{Y^\bullet} \subseteq \Pi_{X^\bullet}$ be the admissible fundamental group of Y^\bullet . Write $M_{Y^\bullet}^{\text{ét}}$ and M_{Y^\bullet} for $\text{Hom}(\Pi_{Y^\bullet}^{\text{ét}}, \mathbb{F}_\ell)$ and $\text{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell)$, respectively. We obtain a natural injection $M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}$ induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi_{Y^\bullet}^{\text{ét}}$. Then we have an exact sequence

$$0 \rightarrow M_{Y^\bullet}^{\text{ét}} \rightarrow M_{Y^\bullet} \rightarrow M_{Y^\bullet}^{\text{ra}} \stackrel{\text{def}}{=} \text{coker}(M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}) \rightarrow 0$$

with a natural action of μ_d .

Let $M_{Y^\bullet, \mu_d}^{\text{ra}} \subseteq M_{Y^\bullet}^{\text{ra}}$ be the subset of elements on which μ_d acts via the character $\mu_d \subseteq \mathbb{F}_\ell^\times$ and $E_{\mathfrak{T}_{X^\bullet}}^* \subseteq M_{Y^\bullet}$ the subset of elements that map to nonzero elements of $M_{Y^\bullet, \mu_d}^{\text{ra}}$. Let $\alpha \in E_{\mathfrak{T}_{X^\bullet}}^*$. Write

$$g_\alpha^\bullet : Y_\alpha^\bullet \rightarrow Y^\bullet$$

for the admissible covering corresponding to the element α and $\Gamma_{Y_\alpha^\bullet}$ for the dual semi-graph of Y_α^\bullet . Then we obtain a map

$$\epsilon : E_{\mathfrak{T}_{X^\bullet}}^* \rightarrow \mathbb{Z}, \quad \alpha \mapsto \#(e^{\text{op}}(\Gamma_{Y_\alpha^\bullet}) \cup e^{\text{cl}}(\Gamma_{Y_\alpha^\bullet})).$$

We define two subsets of $E_{\mathfrak{T}_{X^\bullet}}^*$ as follows:

$$E_{\mathfrak{T}_{X^\bullet}}^{\text{cl},*} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}}^* \mid \#e_{g_\alpha}^{\text{cl,ra}} = d, \#e_{g_\alpha}^{\text{op,ra}} = 0\} \quad (1.4.1),$$

$$E_{\mathfrak{T}_{X^\bullet}}^{\text{op},*} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}}^* \mid \#e_{g_\alpha}^{\text{cl,ra}} = 0, \#e_{g_\alpha}^{\text{op,ra}} = d\},$$

where “cl” means “closed edge”, and “op” means “open edge”. Note that $E_{\mathfrak{T}_{X^\bullet}}^{\text{cl},*}$ and $E_{\mathfrak{T}_{X^\bullet}}^{\text{op},*}$ are not empty. For each $\alpha \in E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}$ (resp. $\alpha \in E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op},*}$), since the image of α is contained in $M_{Y^\bullet, \mu_d}^{\text{ra}}$, we obtain that the action of μ_d on the set

$$\{y_e\}_{e \in e_{g_\alpha}^{\text{cl,ra}}} \subseteq \text{Nod}(Y^\bullet) \quad (\text{resp. } \{y_e\}_{e \in e_{g_\alpha}^{\text{op,ra}}} \subseteq D_Y)$$

is transitive, where $\text{Nod}(-)$ denotes the set of nodes of $(-)$, and y_e denotes the node (resp. the marked point) of Y^\bullet corresponding to e . Then there exists a unique node (resp. marked point) x_α of X^\bullet such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e \in e_{g_\alpha}^{\text{cl,ra}}}$ (resp. $y_e \in \{y_e\}_{e \in e_{g_\alpha}^{\text{op,ra}}}$). We denote by $e_\alpha \in e^{\text{cl}}(\Gamma_{X^\bullet})$ the closed edge (resp. $e_\alpha \in e^{\text{op}}(\Gamma_{X^\bullet})$ the open edge) corresponding to x_α .

Proposition 2.2. *We maintain the notation introduced above. We define a pre-equivalence relation \sim on $E_{\mathfrak{T}_{X^\bullet}}^{\text{cl},*}$ (resp. $E_{\mathfrak{T}_{X^\bullet}}^{\text{op},*}$) as follows:*

Let $\alpha, \beta \in E_{\mathfrak{T}_{X^\bullet}}^{\text{cl},}$ (resp. $\alpha, \beta \in E_{\mathfrak{T}_{X^\bullet}}^{\text{op},*}$). Then $\alpha \sim \beta$ if $\lambda\alpha + \mu\beta \in E_{\mathfrak{T}_{X^\bullet}}^{\text{cl},*}$ (resp. $E_{\mathfrak{T}_{X^\bullet}}^{\text{op},*}$) for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in E_{\mathfrak{T}_{X^\bullet}}^*$.*

Then the pre-equivalence relation \sim on $E_{\mathfrak{T}_X^\bullet}^{\text{cl},*}$ (resp. $E_{\mathfrak{T}_X^\bullet}^{\text{op},*}$) defined above is an equivalence relation. Moreover, we have a natural bijection

$$\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}} : E_{\mathfrak{T}_X^\bullet}^{\text{cl}} \stackrel{\text{def}}{=} E_{\mathfrak{T}_X^\bullet}^{\text{cl},*} / \sim \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet}), [\alpha] \mapsto e_\alpha$$

$$\text{(resp. } \vartheta_{\mathfrak{T}_X^\bullet}^{\text{op}} : E_{\mathfrak{T}_X^\bullet}^{\text{op}} \stackrel{\text{def}}{=} E_{\mathfrak{T}_X^\bullet}^{\text{op},*} / \sim \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X^\bullet}), [\alpha] \mapsto e_\alpha),$$

where $[\alpha]$ denotes the image of α in $E_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ (resp. $E_{\mathfrak{T}_X^\bullet}^{\text{op}}$).

Proof. Let $\alpha, \beta \in E_{\mathfrak{T}_X^\bullet}^{\text{cl},*}$. If $e_{g_\alpha}^{\text{cl,ra}} = e_{g_\beta}^{\text{cl,ra}}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $e_{g_{\lambda\alpha + \mu\beta}}^{\text{cl,ra}} = e_{g_\alpha}^{\text{cl,ra}} = e_{g_\beta}^{\text{cl,ra}}$. Thus, $\alpha \sim \beta$.

On the other hand, if $\alpha \sim \beta$, then we have $e_{g_\alpha}^{\text{cl,ra}} = e_{g_\beta}^{\text{cl,ra}}$; otherwise, we obtain $\#e_{g_{\alpha+\beta}}^{\text{cl,ra}} = 2d$. Thus, $\alpha \sim \beta$ if and only if $e_{g_\alpha}^{\text{cl,ra}} = e_{g_\beta}^{\text{cl,ra}}$. This means that \sim is an equivalence relation on $E_{\mathfrak{T}_X^\bullet}^{\text{cl},*}$.

Next, let us prove that $\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ is a bijection. It is easy to see that $\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ is an injection. On the other hand, for each $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, the structure of the maximal pro- ℓ admissible fundamental groups implies that there is a Galois admissible covering of $h^\bullet : Z^\bullet \rightarrow Y^\bullet$ such that the element corresponding to h^\bullet is contained in $E_{\mathfrak{T}_X^\bullet}^{\text{cl},*}$. Then $\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ is a surjection.

Similar arguments to the arguments given in the proof above imply that the ‘‘resp’’ part holds. This completes the proof of the proposition. \square

Remark 2.2.1. In this remark, we prove that the sets

$$E_{\mathfrak{T}_X^\bullet}^{\text{cl}}, E_{\mathfrak{T}_X^\bullet}^{\text{op}}$$

do not depend on the choices of \mathfrak{T}_X^\bullet in the following sense. We only treat the case of closed edges. Let

$$\mathfrak{T}_X^* \stackrel{\text{def}}{=} (\ell^*, d^*, f_X^{\bullet,*} : Y^{\bullet,*} \rightarrow X^\bullet)$$

be an arbitrary edge-triple associated to X^\bullet . Hence we obtain $E_{\mathfrak{T}_X^*}^{\text{cl}}$ and a natural bijection

$$\vartheta_{\mathfrak{T}_X^*}^{\text{cl}} : E_{\mathfrak{T}_X^*}^{\text{cl}} \rightarrow e^{\text{cl}}(\Gamma_{X^\bullet}).$$

We will show that there exists a bijection $E_{\mathfrak{T}_X^*}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ which fits into the following commutative diagram

$$\begin{array}{ccc} E_{\mathfrak{T}_X^*}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{T}_X^*}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \parallel \\ E_{\mathfrak{T}_X^\bullet}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}). \end{array}$$

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then we may define a bijection

$$E_{\mathfrak{T}_X^*}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_X^\bullet}^{\text{cl}}$$

which is compatible with the bijections $\vartheta_{\mathfrak{T}_X^*}^{\text{cl}}$ and $\vartheta_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ as follows: Let $\alpha \in E_{\mathfrak{T}_X^\bullet}^{\text{cl}}$ and $\alpha^* \in E_{\mathfrak{T}_X^*}^{\text{cl}}$. Write $Y_\alpha^\bullet \rightarrow Y^\bullet$ and $Y_{\alpha^*}^\bullet \rightarrow Y^{\bullet,*}$ for the Galois admissible coverings corresponding to α and α^* , respectively. We consider the following connected Galois admissible covering

$$Y_\alpha^\bullet \times_{X^\bullet} Y_{\alpha^*}^\bullet \rightarrow X^\bullet$$

over k with Galois group $\mathbb{Z}/dd^*\ell^*\mathbb{Z}$. Then we see that $e_\alpha = e_{\alpha^*}$ if and only if the cardinality of the set of nodes of $Y_\alpha^\bullet \times_{X^\bullet} Y_{\alpha^*}^\bullet$ is equal to

$$dd^*(\ell^*(\#e^{\text{cl}}(\Gamma_{X^\bullet}) - 1) + 1).$$

Then we define a map

$$E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}}, [\alpha^*] \mapsto [\alpha]$$

by choosing α such that the cardinality of the set of nodes of $Y_{\alpha^*}^\bullet \times_{X^\bullet} Y_{\alpha^*}^\bullet$ is equal to $dd^*(\ell\ell^*(\#e^{\text{cl}}(\Gamma_{X^\bullet}) - 1) + 1)$.

Next, let us prove the general case. Let

$$\mathfrak{T}_{X^\bullet}^{**} \stackrel{\text{def}}{=} (\ell^{**}, d^{**}, f^{\bullet, **} : Y^{\bullet, **} \rightarrow X^\bullet)$$

be an edge-triple associated to X^\bullet such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain $E_{\mathfrak{T}_{X^\bullet}^{**}}^{\text{cl}}$ and a bijection $\vartheta_{\mathfrak{T}_{X^\bullet}^{**}}^{\text{cl}} : E_{\mathfrak{T}_{X^\bullet}^{**}}^{\text{cl}} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet})$. Then the proof above implies that there are two bijections

$$E_{\mathfrak{T}_{X^\bullet}^{**}}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}} \text{ and } E_{\mathfrak{T}_{X^\bullet}^{**}}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_{X^\bullet}^*}^{\text{cl}}.$$

Thus, we obtain the desired map $E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}}$.

Remark 2.2.2. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup, $f_H^\bullet : X_H^\bullet \rightarrow X^\bullet$ the Galois admissible covering over k with degree $\deg(f_H)$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and $\Gamma_{X_H^\bullet}$ the dual semi-graph of X_H^\bullet . Moreover, we have two natural maps

$$\begin{aligned} f_H^{\text{cl}} &: e^{\text{cl}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{cl}}(\Gamma_{X^\bullet}), \\ f_H^{\text{op}} &: e^{\text{op}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{op}}(\Gamma_{X^\bullet}) \end{aligned}$$

induced by f_H^\bullet .

Let $\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \rightarrow X^\bullet)$ be an edge-triple associated to X^\bullet such that $(\ell, \deg(f_H)) = (d, \deg(f_H)) = 1$. Then we obtain an edge-triple

$$\mathfrak{T}_{X_H^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_{X_H}^\bullet : Z^\bullet \stackrel{\text{def}}{=} Y^\bullet \times_{X^\bullet} X_H^\bullet \rightarrow X_H^\bullet)$$

associated to X_H^\bullet induced by the edge-triple \mathfrak{T}_{X^\bullet} . Moreover, we obtain two natural maps

$$\begin{aligned} f_H^{\text{cl}} &: e^{\text{cl}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{cl}}(\Gamma_{X^\bullet}), \\ f_H^{\text{op}} &: e^{\text{op}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{op}}(\Gamma_{X^\bullet}) \end{aligned}$$

induced by f_H^\bullet . Then we claim that the natural injection $H \hookrightarrow \Pi_{X^\bullet}$ induces surjective maps

$$\begin{aligned} \gamma_{\mathfrak{T}_{X^\bullet}, H}^{\text{cl}} &: E_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}} \rightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}}, \\ \gamma_{\mathfrak{T}_{X^\bullet}, H}^{\text{op}} &: E_{\mathfrak{T}_{X_H^\bullet}}^{\text{op}} \rightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{op}} \end{aligned}$$

which fit into the following commutative diagrams:

$$\begin{array}{ccc} E_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X_H^\bullet}) \\ \gamma_{\mathfrak{T}_{X^\bullet}, H}^{\text{cl}} \downarrow & & f_H^{\text{cl}} \downarrow \\ E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{T}_{X^\bullet}}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}), \\ E_{\mathfrak{T}_{X_H^\bullet}}^{\text{op}} & \xrightarrow{\vartheta_{\mathfrak{T}_{X_H^\bullet}}^{\text{op}}} & e^{\text{op}}(\Gamma_{X_H^\bullet}) \\ \gamma_{\mathfrak{T}_{X^\bullet}, H}^{\text{op}} \downarrow & & f_H^{\text{op}} \downarrow \\ E_{\mathfrak{T}_{X^\bullet}}^{\text{op}} & \xrightarrow{\vartheta_{\mathfrak{T}_{X^\bullet}}^{\text{op}}} & e^{\text{op}}(\Gamma_{X^\bullet}), \end{array}$$

respectively. Moreover, suppose that $H \subseteq \Pi_{X^\bullet}$ is an open *normal* subgroup. Then $E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$ and $E_{\mathfrak{S}_{X_H}^\bullet}^{\text{op}}$ admit actions of Π_{X^\bullet}/H , respectively, such that $\vartheta_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$ and $\vartheta_{\mathfrak{S}_{X_H}^\bullet}^{\text{op}}$ are compatible with Π_{X^\bullet}/H -actions (i.e., $\vartheta_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$ and $\vartheta_{\mathfrak{S}_{X_H}^\bullet}^{\text{op}}$ are Π_{X^\bullet}/H -equivariant), respectively.

We prove the claim. We only treat the case of closed edges. Let $\alpha_X \in E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$. Then α_X induces an element $\beta_{X_H} \in \text{Hom}(\Pi_{Z^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$ via the natural homomorphism $\text{Hom}(\Pi_{Y^\bullet}, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow \text{Hom}(\Pi_{Z^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$ such that β_{X_H} can be written as

$$\sum_{\beta \in J_{\alpha_X}} c_\beta \beta, \quad c_\beta \in \mathbb{F}_\ell^\times,$$

where $\Pi_{Z^\bullet} \stackrel{\text{def}}{=} \Pi_{Y^\bullet} \cap H$, and J_{α_X} is a subset of $E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}, \star}$ such that, if $\beta_1, \beta_2 \in J_{\alpha_X}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$.

Let $[\alpha_{X_H}] \in E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$. We define

$$\gamma_{\mathfrak{S}_{X^\bullet}, H}^{\text{cl}}([\alpha_{X_H}]) = [\alpha_X]$$

if $[\beta] = [\alpha_{X_H}]$ for some $\beta \in J_{\alpha_X}$. It is easy to check that $\gamma_{\mathfrak{S}_{X^\bullet}, H}^{\text{cl}}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X_H^\bullet}) \\ \gamma_{\mathfrak{S}_{X^\bullet}, H}^{\text{cl}} \downarrow & & f_H^{\text{cl}} \downarrow \\ E_{\mathfrak{S}_{X^\bullet}^\bullet}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{S}_{X^\bullet}^\bullet}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}) \end{array}$$

is commutative.

Moreover, suppose that H is an open normal subgroup of Π_{X^\bullet} . Since Π_{Z^\bullet} is an open normal subgroup of Π_{X^\bullet} , we have

$$\Pi_{X^\bullet}/\Pi_{Z^\bullet} \cong \Pi_{X^\bullet}/H \times \mathbb{Z}/d\mathbb{Z}.$$

Then the natural exact sequence

$$1 \rightarrow \Pi_{Z^\bullet} \rightarrow \Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}/\Pi_{Z^\bullet} \rightarrow 1$$

induces an outer representation $\Pi_{X^\bullet}/H \hookrightarrow \Pi_{X^\bullet}/\Pi_{Z^\bullet} \rightarrow \text{Out}(\Pi_{Z^\bullet})$. Thus, we obtain an action of Π_{X^\bullet}/H on $E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}, \star} \subseteq \text{Hom}(\Pi_{Z^\bullet}, \mathbb{Z}/\ell\mathbb{Z}) = \text{Hom}(\Pi_{Z^\bullet}^{\text{ab}}, \mathbb{Z}/\ell\mathbb{Z})$ induced by the outer representation.

Let $\sigma \in \Pi_{X^\bullet}/H$ and $\alpha_{X_H}, \alpha'_{X_H} \in E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}, \star}$. We observe that $\alpha_{X_H} \sim \alpha'_{X_H}$ if and only if $\sigma(\alpha_{X_H}) \sim \sigma(\alpha'_{X_H})$. Thus, we obtain an action of Π_{X^\bullet}/H on $E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}}$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is easy to check that the above commutative diagram is compatible with the Π_{X^\bullet}/H -actions.

Remark 2.2.3. We maintain the notation introduced in Remark 2.2.2. In this remark, we explain that $\gamma_{\mathfrak{S}_{X^\bullet}, H}^{\text{cl}} : E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}} \rightarrow E_{\mathfrak{S}_{X^\bullet}^\bullet}^{\text{cl}}$ defined above can be described in another way which will be used in the remainder of the present paper.

Write P_{α_X} for the kernel of α_X . Let $\beta \in E_{\mathfrak{S}_{X_H}^\bullet}^{\text{cl}, \star}$. Write P_β for the kernel β . Note that $X_{P_{\alpha_X} \cap P_\beta}^\bullet$ is isomorphic to a connected component of $X_{P_\beta}^\bullet \times_{X^\bullet} X_{P_{\alpha_X}}^\bullet$, and that $X_{P_{\alpha_X} \cap H}^\bullet$ is isomorphic to $X_H^\bullet \times_{X^\bullet} X_{P_{\alpha_X}}^\bullet$. Then we see that $\beta \in J_{\alpha_X}$ (see Remark 2.2.2 for J_{α_X}) if and only if one of the following statements holds: (1) $P_\beta = P_{\alpha_X} \cap H$; (2)

$$\#e^{\text{cl}}(\Gamma_{X_{P_{\alpha_X} \cap P_\beta}^\bullet}) = \ell \#e^{\text{cl}}(\Gamma_{X_{P_{\alpha_X} \cap H}^\bullet}).$$

Note that (1) (resp. (2)) happens when $\#e^{\text{cl}}(\Gamma_{X_H^\bullet}) = 1$ (resp. $\#e^{\text{cl}}(\Gamma_{X_H^\bullet}) > 1$). Namely, we have the following:

Let $[\alpha_{X_H}] \in E_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}}$. Then $\gamma_{\mathfrak{T}_{X^\bullet}, H}^{\text{cl}}([\alpha_{X_H}]) = [\alpha_X]$ if and only if one of the following statements holds: (1) there exists $\beta \in E_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}, \star}$ such that $\beta \sim \alpha_{X_H}$ and $P_\beta = P_{\alpha_X} \cap H$; (2) there exists $\beta \in E_{\mathfrak{T}_{X_H^\bullet}}^{\text{cl}, \star}$ such that $\beta \sim \alpha_{X_H}$, and that

$$\#e^{\text{cl}}(\Gamma_{X_{P_{\alpha_X} \cap P_\beta}^\bullet}) = \ell \#e^{\text{cl}}(\Gamma_{X_{P_{\alpha_X} \cap H}^\bullet}).$$

2.2.4. Let $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\text{op}}(\Gamma_{X^\bullet})$). We put

$$E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star} \mid e_\alpha = e\}$$

(resp. $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star} \mid e_\alpha = e\}$).

Then, for each $e, e' \in e^{\text{cl}}(\Gamma_{X^\bullet})$ (resp. $e, e' \in e^{\text{op}}(\Gamma_{X^\bullet})$) distinct from each other, we have

$$E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star} \cap E_{\mathfrak{T}_{X^\bullet}, e'}^{\text{cl}, \star} = \emptyset \quad (\text{resp. } E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star} \cap E_{\mathfrak{T}_{X^\bullet}, e'}^{\text{op}, \star} = \emptyset).$$

Thus, we have

$$E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star} = \bigsqcup_{e \in e^{\text{cl}}(\Gamma_{X^\bullet})} E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star} \quad (\text{resp. } E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star} = \bigsqcup_{e \in e^{\text{op}}(\Gamma_{X^\bullet})} E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star}).$$

For each $m \in \mathbb{Z}_{\geq 0}$, we put

$$E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star, m} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star} \mid \#v_{g_\alpha}^{\text{sp}} = m\}$$

(resp. $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star, m} \stackrel{\text{def}}{=} \{\alpha \in E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star} \mid \#v_{g_\alpha}^{\text{sp}} = m\}$).

If e is a closed edge corresponding to a node which is contained in two irreducible components of Y^\bullet distinct from each other, then $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star, m} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet}) - 1$. If e is a closed edge corresponding to a node which is contained in a unique irreducible component of Y^\bullet , then $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star, m} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet})$. If e is an open edge, then $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star, m} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet})$.

Lemma 2.3. *For each $m \in \mathbb{Z}_{\geq 0}$, if $E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star, m} \neq \emptyset$ (resp. $E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star, m} \neq \emptyset$), then the composition of maps*

$$E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star, m} \hookrightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star} \twoheadrightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet}),$$

(resp. $E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star, m} \hookrightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star} \twoheadrightarrow E_{\mathfrak{T}_{X^\bullet}}^{\text{op}} \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X^\bullet}))$)

is a surjection. In particular, we have that $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{cl}, \star, m} \neq \emptyset$ if $E_{\mathfrak{T}_{X^\bullet}}^{\text{cl}, \star, m} \neq \emptyset$ (resp. $E_{\mathfrak{T}_{X^\bullet}, e}^{\text{op}, \star, m} \neq \emptyset$ if $E_{\mathfrak{T}_{X^\bullet}}^{\text{op}, \star, m} \neq \emptyset$).

Proof. The lemma follows immediately from the structures of maximal pro-prime-to- p admissible fundamental groups. \square

2.2.5. We note that the edge-triple

$$\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \rightarrow X^\bullet)$$

associated to X^\bullet is equivalent to a triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, y),$$

where $y \in \text{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{F}_d)$ induced by the Galois admissible covering f_X^\bullet . We shall say that $\mathfrak{T}_{\Pi_{X^\bullet}}$ an edge-triple associated to Π_{X^\bullet} . Then we also use the notation

$$E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}, \star}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op}, \star}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op}}, E_{\mathfrak{T}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star}, E_{\mathfrak{T}_{\Pi_{X^\bullet}, e}}^{\text{op}, \star}, E_{\mathfrak{T}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star, m}, E_{\mathfrak{T}_{\Pi_{X^\bullet}, e}}^{\text{op}, \star, m}, \vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}}, \vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op}}$$

to denote $E_{\mathfrak{S}_{X^\bullet}}^*$, $E_{\mathfrak{S}_{X^\bullet}}^{\text{cl},*}$, $E_{\mathfrak{S}_{X^\bullet}}^{\text{op},*}$, $E_{\mathfrak{S}_{X^\bullet}}^{\text{cl}}$, $E_{\mathfrak{S}_{X^\bullet}}^{\text{op}}$, $E_{\mathfrak{S}_{X^\bullet},e}^{\text{cl},*}$, $E_{\mathfrak{S}_{X^\bullet},e}^{\text{op},*}$, $E_{\mathfrak{S}_{X^\bullet},e}^{\text{cl},*m}$, $E_{\mathfrak{S}_{X^\bullet},e}^{\text{op},*m}$, $\vartheta_{\mathfrak{S}_{X^\bullet}}^{\text{cl}}$, $\vartheta_{\mathfrak{S}_{X^\bullet}}^{\text{op}}$, respectively.

3. MONO-ABELIAN COMBINATORIAL GROTHENDIECK CONJECTURE IN POSITIVE CHARACTERISTIC

We maintain the notation introduced in previous sections.

3.1. Mono-abelian reconstructions. First, let us define the term “mono-abelian reconstruction”.

Definition 3.1. Let $i \in \{1, 2\}$, and let \mathcal{F}_i be a geometric object and $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object \mathcal{F}_i .

Let $\text{Inv}_{\mathcal{F}_i}$ be an invariant depending on the isomorphism class of \mathcal{F}_i (in a certain category), we shall say that $\text{Inv}_{\mathcal{F}_i}$ can be *mono-abelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\text{Inv}_{\mathcal{F}_i}$.

Let $\text{Add}_{\mathcal{F}_i}$ be an additional structure (e.g. a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on \mathcal{F}_i . We shall say that $\text{Add}_{\mathcal{F}_i}$ can be *mono-abelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\text{Add}_{\mathcal{F}_i}$.

We shall say that a map (or a morphism) $\text{Add}_{\mathcal{F}_1} \rightarrow \text{Add}_{\mathcal{F}_2}$ can be *mono-abelian reconstructed* from $\Pi_{\mathcal{F}_1} \rightarrow \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_1} \rightarrow \Pi_{\mathcal{F}_2}$, and whose output datum is $\text{Add}_{\mathcal{F}_1} \rightarrow \text{Add}_{\mathcal{F}_2}$.

3.1.1. One of the main difficulties of establishing a group-theoretical algorithm for reconstructing the topological and the combinatorial structures associated to X^\bullet is that, for each open subgroup $H \subseteq \Pi_{X^\bullet}$, we need to prove that the profinite completion of the topological fundamental group of $\Gamma_{X_H^\bullet}$ and the étale fundamental group of the underlying curve of X_H^\bullet (or the weight-monodromy filtration of the first ℓ -adic étale cohomology group of X_H , where $\ell \neq p$) can be mono-abelian reconstructed from H . When the base field is an arithmetic field, the weight-monodromy filtration can be mono-abelian reconstructed by applying the theory of “weight”. In our situation (i.e., the base field is an algebraically closed field), we have the following key observation:

The formula for $\text{Avr}_p(H)$ of H plays a role of (outer) Galois representations in the theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

3.1.2. We maintain the notation introduced in 2.0.1. In order to simplify the formula of $\text{Avr}_p(\Pi_{X^\bullet})$, we introduce the following condition for X^\bullet .

Condition A . We shall say that X^\bullet *satisfies Condition A* if the following conditions hold: (i) g_v is positive for each $v \in v(\Gamma_{X^\bullet})$; (ii) $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected (Definition 1.1 (c)); (iii) $\#v(\Gamma_{X^\bullet})^{b \leq 1} = 0$ and $\#e^{\text{cl}}(\Gamma_{X^\bullet})^{b \leq 1} = 0$ (Definition 1.1 (d)).

3.2. Reconstructions of various additional structures.

3.2.1. Settings. We maintain the notation introduced in 2.0.1. Moreover, in the remainder of this section, we suppose that X^\bullet *satisfies Condition A unless indicated otherwise*. Note that Theorem 1.5 and Condition A imply that

$$\text{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X.$$

3.2.2. Firstly, we have the following lemmas.

Lemma 3.2. (i) The data $p \stackrel{\text{def}}{=} \text{char}(k)$, g_X , $n_X = \#e^{\text{op}}(\Gamma_{X^\bullet})$, r_X , and $\Pi_{X^\bullet}^{\text{top},p}$ can be mono-anabelian reconstructed from Π_{X^\bullet} , where $\Pi_{X^\bullet}^{\text{top},p}$ denotes the maximal pro- p quotient of $\Pi_{X^\bullet}^{\text{top}}$ (1.2.2).

(ii) The set $v(\Gamma_{X^\bullet})^{>0,p}$ (2.1.1) can be mono-anabelian reconstructed from Π_{X^\bullet} .

(iii) Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup. Suppose that $\Gamma_{X_H^\bullet}^{\text{cpt}}$ is 2-connected. Then the natural map

$$v(\Gamma_{X_H^\bullet})^{>0,p} \rightarrow v(\Gamma_{X^\bullet})^{>0,p}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(iv) The cardinality $\#v(\Gamma_{X^\bullet})$ of $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} .

Proof. (i) A similar result has been proved in [Y1, Proposition 6.1 and Lemma 6.4], for readers' convenience, we put the proof here. If $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_\ell) = \dim_{\mathbb{F}_{\ell'}}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_{\ell'})$ holds for every two prime numbers ℓ and ℓ' , then $g_X = 2g_X + n_X - 1$ if $n_X > 0$, and $g_X = 2g_X$ if $n_X = 0$. Thus, either $(g_X, n_X) = (0, 1)$ or $(g_X, n_X) = (0, 0)$ holds. Since Π_{X^\bullet} is the admissible fundamental group of a pointed stable curve, this is a contradiction. Thus, p is the unique prime number such that $\dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_p) \neq \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_\ell)$ holds for each prime number $\ell \neq p$.

Let H be any open normal subgroup of Π_{X^\bullet} . We note that, if Π_{X^\bullet}/H is a p -group, then the decomposition group in Π_{X^\bullet}/H of every irreducible component of X_H^\bullet is trivial if and only if

$$g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X).$$

Thus, Theorem 1.5 implies that we may detect whether the equality

$$g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X)$$

holds or not, group-theoretically from Π_{X^\bullet} and H if $\Gamma_{X_H^\bullet}^{\text{cpt}}$ is 2-connected. We put

$$\begin{aligned} \text{Top}_p(\Pi_{X^\bullet}) &\stackrel{\text{def}}{=} \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \Pi_{X^\bullet}/H \text{ is a } p\text{-group} \\ &\text{and, for any characteristic subgroup } Q \subseteq \Pi_{X^\bullet}, \\ &g_{X_{H \cap Q}} - r_{X_{H \cap Q}} = \#(\Pi_{X^\bullet}/(H \cap Q))(g_{X_Q} - r_{X_Q})\}. \end{aligned}$$

Note that Lemma 1.6 implies that $\Gamma_{X_{H \cap Q}^\bullet}^{\text{cpt}}$ is 2-connected. Then by applying Theorem 1.5, we have that $\text{Top}_p(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Thus, we obtain that

$$\Pi_{X^\bullet}^{\text{top},p} = \Pi_{X^\bullet} / \left(\bigcap_{H \in \text{Top}_p(\Pi_{X^\bullet})} H \right)$$

can be mono-anabelian reconstructed from Π_{X^\bullet} . Moreover, we have that

$$r_X = \dim_{\mathbb{Q}}(\Pi_{X^\bullet}^{\text{top},p,\text{ab}} \otimes \mathbb{Q})$$

can be reconstructed group-theoretically from Π_{X^\bullet} . By Theorem 1.5 again, the genus

$$g_X = \text{Avr}_p(\Pi_{X^\bullet}) + r_X$$

can be mono-anabelian reconstructed from Π_{X^\bullet} .

Let $\ell \neq p$ be a prime number. If $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_\ell) \neq 2g_X$, then we have

$$n_X = \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_\ell) - 2g_X + 1.$$

Suppose that $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_\ell) = 2g_X$. Then $n_X = 0$ if, for any open normal subgroup $H \subseteq \Pi_{X^\bullet}$, $\dim_{\mathbb{F}_\ell}(H^{\text{ab}} \otimes \mathbb{F}_\ell) = 2g_{X_H}$. Otherwise, we have $n_X = 1$. We complete the proof of (i).

(ii) Since each Galois admissible covering of degree p is étale, by applying (i), we obtain that $V_{X,p}^*$ (2.1.2) can be mono-anabelian reconstructed from Π_{X^\bullet} . Then to verify (ii), Proposition

2.1 implies that it is sufficient to prove that $V_{X,p}^*$ (2.1.2) can be mono-anabelian reconstructed from Π_{X^\bullet} . Let $\alpha \in V_{X,p}^*$, X_α^\bullet the Galois admissible covering corresponding to α , $\Gamma_{X_\alpha^\bullet}$ the dual semi-graph of X_α^\bullet , and r_{X_α} the Betti number of $\Gamma_{X_\alpha^\bullet}$. Moreover, let $0 \neq \gamma \in \text{Hom}(\Pi_{X^\bullet}^{\text{top},p}, \mathbb{F}_p)$ if $\Pi_{X^\bullet}^{\text{top},p}$ is not trivial, X_γ^\bullet the Galois admissible covering corresponding to γ , $X_{\alpha,\gamma}^\bullet$ the pointed stable curve $X_\alpha^\bullet \times_{X^\bullet} X_\gamma^\bullet$, $\Gamma_{X_{\alpha,\gamma}^\bullet}$ the dual semi-graph of $X_{\alpha,\gamma}^\bullet$, $\Gamma_{X_\gamma^\bullet}$ the dual semi-graph of X_γ^\bullet , $r_{X_{\alpha,\gamma}}$ the Betti number of $\Gamma_{X_{\alpha,\gamma}^\bullet}$, and r_{X_γ} the Betti number of $\Gamma_{X_\gamma^\bullet}$. Then we have the following claim:

Claim:

$$\#v(\Gamma_{X_\alpha^\bullet}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if

$$r_{X_\alpha} = pr_X.$$

Moreover, suppose that $r_X \neq 0$. Then

$$\#v(\Gamma_{X_\alpha^\bullet}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if

$$r_{X_{\alpha,\gamma}} = pr_{X_\gamma} + p^2 - 2p + 1.$$

Let us prove the claim. Since $r_{X_\alpha} = \#e^{\text{cl}}(\Gamma_{X_\alpha^\bullet}) - \#v(\Gamma_{X_\alpha^\bullet}) + 1$ and $r_X = \#e^{\text{cl}}(\Gamma_{X^\bullet}) - \#v(\Gamma_{X^\bullet}) + 1$, we have that $r_{X_\alpha} = pr_X$ holds if and only if

$$\#e^{\text{cl}}(\Gamma_{X_\alpha^\bullet}) - \#v(\Gamma_{X_\alpha^\bullet}) = p\#e^{\text{cl}}(\Gamma_{X^\bullet}) - p(\#v(\Gamma_{X^\bullet}) - 1) - 1.$$

Since $\#e^{\text{cl}}(\Gamma_{X_\alpha^\bullet}) = p\#e^{\text{cl}}(\Gamma_{X^\bullet})$, we have

$$\#v(\Gamma_{X_\alpha^\bullet}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if $r_{X_\alpha} = pr_X$.

Suppose that $r_X \neq 0$. Since $0 \neq \gamma \in \text{Hom}(\Pi_{X^\bullet}^{\text{top},p}, \mathbb{F}_p)$, we have

$$r_{X_{\alpha,\gamma}} = p\#e^{\text{cl}}(\Gamma_{X_\alpha^\bullet}) - p\#v(\Gamma_{X_\alpha^\bullet}) + 1.$$

Then

$$r_{X_{\alpha,\gamma}} = pr_{X_\gamma} + p^2 - 2p + 1 = p(p\#e^{\text{cl}}(\Gamma_{X^\bullet}) - p\#v(\Gamma_{X^\bullet}) + 1) + p^2 - 2p + 1$$

if and only if

$$\#e^{\text{cl}}(\Gamma_{X_\alpha^\bullet}) - \#v(\Gamma_{X_\alpha^\bullet}) = p\#e^{\text{cl}}(\Gamma_{X^\bullet}) - p(\#v(\Gamma_{X^\bullet}) - 1) - 1$$

if and only if

$$\#v(\Gamma_{X_\alpha^\bullet}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1.$$

This completes the proof of the claim.

If $r_X = 0$ (i.e., Γ_{X^\bullet} is a tree), then by applying Remark 1.1.1 and Remark 1.5.1, Condition A implies that either each terminal vertex (Definition 1.1 (d)) of Γ_{X^\bullet} meets some open edge of Γ_{X^\bullet} or $\#v(\Gamma_{X^\bullet}) = 1$. Then we observe that the one-point compactification of the dual semi-graph of each connected Galois admissible covering of X^\bullet is 2-connected. Then by the first part of the claim above and (i), we obtain that $V_{X,p}^*$ can be mono-anabelian reconstructed from Π_{X^\bullet} . If $r_X \neq 0$, then $\Gamma_{X_\alpha^\bullet}^{\text{cpt}}$ is *not* 2-connected in general. Moreover, by the choice of γ , we see that the natural map of dual semi-graphs $f_\gamma^{\text{sg}} : \Gamma_{X_\gamma^\bullet} \rightarrow \Gamma_{X^\bullet}$ induced by the admissible covering $X_\gamma^\bullet \rightarrow X^\bullet$ is a topological covering. In particular, $\#((f_\gamma^{\text{sg}})^{-1}(v)) > 1$ and $\#((f_{\alpha,\gamma}^{\text{sg}})^{-1}(v)) > 1$ for all $v \in v(\Gamma_{X^\bullet})$, where $f_{\alpha,\gamma}^{\text{sg}} : \Gamma_{X_{\alpha,\gamma}^\bullet} \rightarrow \Gamma_{X^\bullet}$ denotes the natural map of dual semi-graphs induced by the admissible covering $X_{\alpha,\gamma}^\bullet \rightarrow X^\bullet$. Then Lemma 1.6 implies that $\Gamma_{X_\gamma^\bullet}^{\text{cpt}}$ and $\Gamma_{X_{\alpha,\gamma}^\bullet}^{\text{cpt}}$ are 2-connected. Then the ‘‘moreover’’ part of the claim above and (i) imply that $V_{X,p}^*$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Thus, by Proposition 2.1, the set $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . This completes the proof of (ii).

(iii) Since $\Gamma_{X_H^\bullet}^{\text{cpt}}$ is 2-connected, we obtain that X_H^\bullet satisfies Condition A. Moreover, by replacing X^\bullet by X_H^\bullet , (ii) implies that $v(\Gamma_{X_H^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from H . Then (iii) follows from Remark 2.1.2 and (ii).

(iv) Since $V_{X_{Q,p}^\bullet}^* \subseteq \text{Hom}(Q^{\text{ab}}, \mathbb{F}_p)$ for each open normal subgroup $Q \subseteq \Pi_{X^\bullet}$, $V_{X_{Q,p}^\bullet}^*$ admits a natural action of Π_{X^\bullet}/Q via the natural outer representation

$$\Pi_{X^\bullet}/Q \rightarrow \text{Out}(Q) \rightarrow \text{Aut}(Q^{\text{ab}})$$

induced by the natural exact sequence

$$1 \rightarrow Q \rightarrow \Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}/Q \rightarrow 1.$$

We have the following:

Claim: There is an open normal subgroup $Q \subseteq \Pi_{X^\bullet}$ such that the p -rank of $\tilde{X}_{Q,v}^\bullet$ is positive for each $v \in v(\Gamma_{X^\bullet})$.

Let us prove the claim. Since we assume that X^\bullet satisfies Condition A, $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected. Then [Y3, Corollary 3.5] implies that the natural homomorphism $\Pi_{\tilde{X}_v^\bullet}^{\text{ab}} \otimes \mathbb{Z}/m\mathbb{Z} \rightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/m\mathbb{Z}$, $v \in v(\Gamma_{X^\bullet})$, is injective for all $m \in \mathbb{Z}$ prime to p . By applying Theorem 1.5 to $\Pi_{\tilde{X}_v^\bullet}^{\text{ab}}$, there exists $m_v \in \mathbb{Z}_{>0}$ prime to p such that the p -rank of smooth pointed stable curve corresponding to the kernel of the natural surjection $\Pi_{\tilde{X}_v^\bullet} \rightarrow \Pi_{\tilde{X}_v^\bullet}^{\text{ab}} \otimes \mathbb{Z}/m_v\mathbb{Z} (\hookrightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/m_v\mathbb{Z})$ is positive. We put $d \stackrel{\text{def}}{=} \max\{m_v\}_{v \in v(\Gamma_{X^\bullet})}$ and

$$Q \stackrel{\text{def}}{=} \ker(\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/d\mathbb{Z}).$$

Then we see immediately that Q satisfies the conditions of the claim. This completes the proof of the claim.

Let Q' be an open normal subgroup $Q' \subseteq \Pi_{X^\bullet}$ satisfying the conditions of the above claim. Moreover, we may assume that $X_{Q'}^\bullet$ satisfies Condition A. Then we obtain $V_{X_{Q',p}^\bullet} \xrightarrow{\sim} v(\Gamma_{X_{Q'}^\bullet})^{>0,p} = v(\Gamma_{X_{Q'}^\bullet})$. Thus, we have

$$\#v(\Gamma_{X^\bullet}) = \max\{\#(V_{X_{Q,p}^\bullet}/(\Pi_{X^\bullet}/Q)) \mid Q \subseteq \Pi_{X^\bullet} \text{ open normal}\}.$$

This completes the proof of the lemma. \square

Lemma 3.3. *The data $\#e^{\text{cl}}(\Gamma_{X^\bullet})$, $\Pi_{X^\bullet}^{\text{top}}$, and $\Pi_{X^\bullet}^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} .*

Proof. By Lemma 3.2 (i) (iv), we obtain that r_X and $\#v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Then

$$\#e^{\text{cl}}(\Gamma_{X^\bullet}) = r_X + \#v(\Gamma_{X^\bullet}) - 1$$

and

$$\#e^{\text{op}}(\Gamma_{X^\bullet}) = n_X$$

can be also mono-anabelian reconstructed from Π_{X^\bullet} . We put

$$\text{Et}(\Pi_{X^\bullet}) \stackrel{\text{def}}{=} \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \text{for each proper characteristic open normal subgroup}$$

$$Q, \text{ we have } \#e^{\text{cl}}(\Gamma_{X_{H \cap Q}^\bullet}) = \#(\Pi_{X^\bullet}/(H \cap Q))\#e^{\text{cl}}(\Gamma_{X^\bullet})$$

$$\text{and } \#e^{\text{op}}(\Gamma_{X_{H \cap Q}^\bullet}) = \#(\Pi_{X^\bullet}/(H \cap Q))\#e^{\text{op}}(\Gamma_{X^\bullet})\}.$$

Note that, for each proper characteristic open normal subgroup Q , since $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected, Lemma 1.6 implies that $\Gamma_{X_{H \cap Q}^\bullet}^{\text{cpt}}$ is 2-connected. Moreover, $X_{H \cap Q}^\bullet$ satisfies Condition A. Then

$\#e^{\text{cl}}(\Gamma_{X_{H \cap Q}^\bullet})$ and $\#e^{\text{op}}(\Gamma_{X_{H \cap Q}^\bullet})$ can be mono-anabelian reconstructed from $H \cap Q$. Thus $\text{Et}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} . This implies that

$$\Pi_{X^\bullet}^{\text{ét}} = \Pi_{X^\bullet} / \bigcap_{H \in \text{Et}(\Pi_{X^\bullet})} H$$

can be mono-anabelian reconstructed from Π_{X^\bullet} . On the other hand, we put

$\text{Top}(\Pi_{X^\bullet}) \stackrel{\text{def}}{=} \{H \subseteq \Pi_{X^\bullet}^{\text{ét}} \text{ open normal} \mid \text{for each proper characteristic open normal subgroup } Q,$

$$g_{X_{H \cap Q}} - r_{X_{H \cap Q}} = \#(\Pi_{X^\bullet} / (H \cap Q))(g_X - r_X)\}.$$

Note that since $X_{H \cap Q}^\bullet$ satisfies Condition A, Lemma 3.2 (i) implies that $\text{Top}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Thus we have that

$$\Pi_{X^\bullet}^{\text{top}} = \Pi_{X^\bullet}^{\text{ét}} / \bigcap_{H \in \text{Top}(\Pi_{X^\bullet})} H$$

can be mono-anabelian reconstructed from Π_{X^\bullet} . This completes the proof of the lemma. \square

Lemma 3.4. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup. Then the data g_{X_H} , n_{X_H} , r_{X_H} , $\#e^{\text{cl}}(\Gamma_{X_H^\bullet})$, and $\#v(\Gamma_{X_H^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . Furthermore, we have that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H .*

Proof. Suppose that $r_X = 0$. Then by applying Remark 1.1.1, Condition A implies that either each terminal vertex of Γ_{X^\bullet} meets some open edge of Γ_{X^\bullet} or $\#v(\Gamma_{X^\bullet}) = 1$ holds. Then we observe that the one-point compactification of the dual semi-graph of each connected Galois admissible covering of X^\bullet is 2-connected. Then X_H^\bullet satisfies Condition A. Thus, the lemma follows from Lemma 3.2 and Lemma 3.3.

Suppose that $r_X \neq 0$. Then $\Gamma_{X_H^\bullet}^{\text{cpt}}$ is *not* 2-connected in general. Since p can be mono-anabelian reconstructed from Π_{X^\bullet} , we may choose a prime number $\ell \neq p$ such that $(\ell, \#(\Pi_{X^\bullet}/H)) = 1$. Let $0 \neq \gamma \in \text{Hom}(\Pi_{X^\bullet}^{\text{top}}, \mathbb{F}_\ell)$, H_γ the kernel of $\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{top}} \xrightarrow{\gamma} \mathbb{F}_\ell$, $X_{H_\gamma}^\bullet \rightarrow X^\bullet$ the admissible covering corresponding to H_γ , $X_{H \cap H_\gamma}^\bullet$ the pointed stable curve $X_H^\bullet \times_{X^\bullet} X_{H_\gamma}^\bullet$, $\Gamma_{X_{H \cap H_\gamma}^\bullet}$ the dual semi-graph of $X_{H \cap H_\gamma}^\bullet$, and $r_{X_{H \cap H_\gamma}}$ the Betti number of $\Gamma_{X_{H \cap H_\gamma}^\bullet}$. By Lemma 1.6, $\Gamma_{X_{H \cap H_\gamma}^\bullet}^{\text{cpt}}$ and $\Gamma_{X_{H_\gamma}^\bullet}^{\text{cpt}}$ are 2-connected. Moreover, $X_{H \cap H_\gamma}^\bullet$ satisfies Condition A. Then $g_{X_{H \cap H_\gamma}}$, $n_{X_{H \cap H_\gamma}}$, $r_{X_{H \cap H_\gamma}}$, $\#e^{\text{cl}}(\Gamma_{X_{H \cap H_\gamma}^\bullet})$, and $\#v(\Gamma_{X_{H \cap H_\gamma}^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} and $H \cap H_\gamma$.

We note that

$$n_{X_{H \cap H_\gamma}} = \ell n_{X_H}, \quad \#e^{\text{cl}}(\Gamma_{X_{H \cap H_\gamma}^\bullet}) = \ell \#e^{\text{cl}}(\Gamma_{X_H^\bullet}), \quad \#v(\Gamma_{X_{H \cap H_\gamma}^\bullet}) = \ell \#v(\Gamma_{X_H^\bullet}),$$

$$r_{X_{H \cap H_\gamma}} = \ell r_{X_H} - \ell + 1, \quad \text{and} \quad g_{X_{H \cap H_\gamma}} = \ell(g_{X_H} - 1) + 1.$$

Then g_{X_H} , n_{X_H} , r_{X_H} , $\#e^{\text{cl}}(\Gamma_{X_H^\bullet})$, and $\#v(\Gamma_{X_H^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . Moreover, similar arguments to the arguments given in the proof of Lemma 3.3 imply that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . \square

Proposition 3.5. (i) *Let ℓ be an arbitrary prime number. Then the sets $V_{X, \ell}^*$ defined in 2.1.2 and the $V_{X, \ell}$ defined in Proposition 2.1 can be mono-anabelian reconstructed from Π_{X^\bullet} .*

(ii) *Let ℓ', ℓ'' be prime numbers distinct from each other such that $\ell'' \neq p$. Then there is a natural injection*

$$V_{X, \ell'} \hookrightarrow V_{X, \ell''}$$

which fits into the following commutative diagram

$$\begin{array}{ccc} V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^\bullet})^{>0,\ell'} \\ \downarrow & & \downarrow \\ V_{X,\ell''} & \xrightarrow{\kappa_{X,\ell''}} & v(\Gamma_{X^\bullet})^{>0,\ell''}. \end{array}$$

Moreover, the injection can be mono-anabelian reconstructed from Π_{X^\bullet} .

(iii) The set of vertices $v(\Gamma_{X^\bullet})$ of Γ_{X^\bullet} can be mono-anabelian reconstructed from Π_{X^\bullet} .

Proof. (i) If $V_{X,\ell}^*$ can be mono-anabelian reconstructed from Π_{X^\bullet} , then Proposition 2.1 implies that $V_{X,\ell}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Thus, we only need to treat the case of $V_{X,\ell}^*$.

By Lemma 3.3, we obtain that $\Pi_{X^\bullet}^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . By replacing Π_{X^\bullet} and p by $\Pi_{X^\bullet}^{\text{ét}}$ and ℓ , respectively, then similar arguments to the arguments given in the proof of Lemma 3.2 (i) imply $\Pi_{X^\bullet}^{\text{top},\ell}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Moreover, by replacing $\Pi_{X^\bullet}^{\text{top},p}$ and p by $\Pi_{X^\bullet}^{\text{top},\ell}$ and ℓ , respectively, then similar arguments to the arguments given in the proof of Lemma 3.2 (ii) imply (i) holds.

(ii) Let $\alpha' \in V_{X,\ell'}^*$ and $\alpha'' \in V_{X,\ell''}^*$. Write $X_{\alpha'}^\bullet$ and $X_{\alpha''}^\bullet$ for the pointed stable curves corresponding to α' and α'' , $H_{\alpha'}$ and $H_{\alpha''}$ for the open subgroups of Π_{X^\bullet} corresponding to $X_{\alpha'}^\bullet$ and $X_{\alpha''}^\bullet$ (i.e., the kernels of $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \mathbb{F}_{\ell'}$ and $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \mathbb{F}_{\ell''}$), respectively. Then we obtain that

$$X_{\alpha'}^\bullet \times_{X^\bullet} X_{\alpha''}^\bullet$$

is a connected pointed stable curve corresponding to the open normal subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Lemma 3.4 implies that the cardinality of the set of irreducible components of $X_{\alpha'}^\bullet \times_{X^\bullet} X_{\alpha''}^\bullet$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''}$ and Π_{X^\bullet} . Then (ii) follows from Remark 2.1.1.

(iii) Lemma 3.2 (i) implies that p can be mono-anabelian reconstructed from Π_{X^\bullet} . Then we may choose a prime number ℓ distinct from p . Moreover, since X^\bullet satisfies Condition A, we have

$$v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet}).$$

Then (iii) follows from (i), (ii), and Proposition 2.1. \square

Proposition 3.6. Let $\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, y)$ be an arbitrary edge-triple associated to Π_{X^\bullet} (2.2.5), H_y the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \mathbb{F}_d$, and $f^\bullet : Y^\bullet \rightarrow X^\bullet$ the Galois admissible covering corresponding to H_y . Then the sets

$$E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op},*}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{op}}$$

defined in 2.2.3 and Proposition 2.2, respectively, can be mono-anabelian reconstructed from Π_{X^\bullet} and H_y .

Proof. We only treat $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}$ and $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}}$. Moreover, by Proposition 2.2, it's sufficient to treat the case of $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}$. Note that the construction of Y^\bullet implies that $\#((f^{\text{sg}})^{-1}(v)) = 1$ for all $v \in v(\Gamma_{X^\bullet})$, where $f^{\text{sg}} : \Gamma_{Y^\bullet} \rightarrow \Gamma_{X^\bullet}$ denotes the natural map of dual semi-graphs induced by f^\bullet . Since $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected, Lemma 1.6 implies that $\Gamma_{Y^\bullet}^{\text{cpt}}$ is 2-connected. Moreover, since X^\bullet satisfies Condition A, we have that Y^\bullet satisfies Condition A. By the definition of $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*$, Lemma 3.3 implies that the set $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*$ (2.2.3) can be mono-anabelian reconstructed from Π_{X^\bullet} and H_y . Hence, to verify the proposition, it is sufficient to prove that the set $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*} \subseteq E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*$ (2.2.3)

can be mono-anabelian reconstructed from Π_{X^\bullet} and H_y . Let $\alpha \in E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*$, $H_\alpha \subseteq H_y$ the kernel of α , $Y_\alpha^\bullet \rightarrow Y^\bullet$ the admissible covering corresponding to H_α , and $\Gamma_{Y_\alpha^\bullet}$ the dual semi-graph of Y_α^\bullet . We observe that

$$\alpha \in E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}$$

if and only if

$$\#e^{\text{cl}}(\Gamma_{Y_\alpha^\bullet}) = \ell(\#e^{\text{cl}}(\Gamma_{Y^\bullet}) - d) + d, \quad \#e^{\text{op}}(\Gamma_{Y_\alpha^\bullet}) = \ell\#e^{\text{op}}(\Gamma_{Y^\bullet}).$$

Since $H_\alpha \subseteq H_y$ (resp. $H_y \subseteq \Pi_{X^\bullet}$) is an open normal subgroup, by Lemma 3.4, we have that $\#e^{\text{cl}}(\Gamma_{Y_\alpha^\bullet})$ and $\#e^{\text{op}}(\Gamma_{Y_\alpha^\bullet})$ (resp. $\#e^{\text{cl}}(\Gamma_{Y^\bullet})$ and $\#e^{\text{op}}(\Gamma_{Y^\bullet})$) can be mono-anabelian reconstructed from H_α and H_y (resp. H_y and Π_{X^\bullet}). Then we obtain that $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl},*}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H_y . This completes the proof of the proposition. \square

Next, we generalize Lemma 3.4 to the case where H is an arbitrary open subgroup of Π_{X^\bullet} .

Proposition 3.7. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. Then the data g_{X_H} , n_{X_H} , r_{X_H} , $\#e^{\text{cl}}(\Gamma_{X_H^\bullet})$, and $\#v(\Gamma_{X_H^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . Furthermore, we have that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H .*

Proof. Let $N \subseteq H$ be a proper open characteristic subgroup of Π_{X^\bullet} . Then X_N^\bullet satisfies Condition A. Since N is a normal open subgroup of Π_{X^\bullet} , Lemma 3.2 and Lemma 3.4 imply that the data g_{X_N} , n_{X_N} , r_{X_N} , $\#e^{\text{cl}}(\Gamma_{X_N^\bullet})$, $\#v(\Gamma_{X_N^\bullet})$, N^{top} , and $N^{\text{ét}}$ can be mono-anabelian reconstructed from N . Moreover, by Proposition 3.5, we obtain that $v(\Gamma_{X_N^\bullet})$ can be mono-anabelian reconstructed from N , and that $v(\Gamma_{X_N^\bullet})$ admits a natural action of H/N . Then we obtain that

$$\#v(\Gamma_{X_H}) = \#(v(\Gamma_{X_N})/(H/N)).$$

Let $\mathfrak{T}_N \stackrel{\text{def}}{=} (\ell, d, y)$ be an arbitrary edge-triple associated to N , N_y the kernel of $N \rightarrow N^{\text{ét}} \xrightarrow{y} \mathbb{F}_d$, and $f^\bullet : Y_N^\bullet \rightarrow X_N^\bullet$ the Galois admissible covering corresponding to N_y . Then Proposition 3.6 implies that

$$E_{\mathfrak{T}_N}^{\text{cl}}, E_{\mathfrak{T}_N}^{\text{op}}$$

can be mono-anabelian reconstructed from Π_{X^\bullet} and N_y . Moreover, $E_{\mathfrak{T}_N}^{\text{cl}}$ and $E_{\mathfrak{T}_N}^{\text{op}}$ admit natural actions of H/N . Then we obtain that

$$\#e^{\text{cl}}(\Gamma_{X_H^\bullet}) = \#(E_{\mathfrak{T}_N}^{\text{cl}}/(H/N)), \quad n_{X_H} = \#e^{\text{op}}(\Gamma_{X_H^\bullet}) = \#(E_{\mathfrak{T}_N}^{\text{op}}/(H/N)).$$

Moreover, we have that

$$r_{X_H} = \#e^{\text{cl}}(\Gamma_{X_H^\bullet}) - \#v(\Gamma_{X_H^\bullet}) + 1.$$

On the other hand, since the ramification indexes of the Galois admissible covering $Y_N^\bullet \rightarrow X_H^\bullet$ at each marked points can be mono-anabelian reconstructed from N and H via the action of H/N on $E_{\mathfrak{T}_N}^{\text{op}}$, the Riemann-Hurwitz formula implies that the genus g_{X_H} can be mono-anabelian reconstructed from Π_{X^\bullet} and N .

Similar arguments to the arguments given in the proof of Lemma 3.3 imply that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . This completes the proof of the proposition. \square

Proposition 3.8. *(i) Let $\mathfrak{T}'_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell', d', y')$ and $\mathfrak{T}''_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell'', d'', y'')$ be edge-triples associated to Π_{X^\bullet} , $H_{y'}$ and $H_{y''}$ the kernels of $\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{ét}} \xrightarrow{y'} \mathbb{F}_{d'}$ and $\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{ét}} \xrightarrow{y''} \mathbb{F}_{d''}$, $f^{\bullet'} : Y^{\bullet'} \rightarrow X^\bullet$ and $f^{\bullet''} : Y^{\bullet''} \rightarrow X^\bullet$ the Galois admissible coverings corresponding to $H_{y'}$ and $H_{y''}$, respectively. Then there are natural bijections*

$$E_{\mathfrak{T}'_{\Pi_{X^\bullet}}}^{\text{cl}} \xrightarrow{\sim} E_{\mathfrak{T}''_{\Pi_{X^\bullet}}}^{\text{cl}}, \quad E_{\mathfrak{T}'_{\Pi_{X^\bullet}}}^{\text{op}} \xrightarrow{\sim} E_{\mathfrak{T}''_{\Pi_{X^\bullet}}}^{\text{op}}$$

which fit into the following commutative diagrams

$$\begin{array}{ccc}
 E_{\mathfrak{S}'_{\Pi_{X^\bullet}}}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{S}'_{\Pi_{X^\bullet}}}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}) \\
 \downarrow & & \parallel \\
 E_{\mathfrak{S}''_{\Pi_{X^\bullet}}}^{\text{cl}} & \xrightarrow{\vartheta_{\mathfrak{S}''_{\Pi_{X^\bullet}}}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X^\bullet}), \\
 \\
 E_{\mathfrak{S}'_{\Pi_{X^\bullet}}}^{\text{op}} & \xrightarrow{\vartheta_{\mathfrak{S}'_{\Pi_{X^\bullet}}}^{\text{op}}} & e^{\text{op}}(\Gamma_{X^\bullet}) \\
 \downarrow & & \parallel \\
 E_{\mathfrak{S}''_{\Pi_{X^\bullet}}}^{\text{op}} & \xrightarrow{\vartheta_{\mathfrak{S}''_{\Pi_{X^\bullet}}}^{\text{op}}} & e^{\text{op}}(\Gamma_{X^\bullet}),
 \end{array}$$

respectively. Moreover, the above bijections can be mono-anabelian reconstructed from Π_{X^\bullet} , $H_{y'}$, and $H_{y''}$.

(ii) The set of closed edges $e^{\text{cl}}(\Gamma_{X^\bullet})$ of Γ_{X^\bullet} and the set of open edges $e^{\text{op}}(\Gamma_{X^\bullet})$ of Γ_{X^\bullet} can be mono-anabelian reconstructed from Π_{X^\bullet} .

Proof. We only treat the case of closed edges. (i) Let $\alpha' \in E_{\mathfrak{S}'_{\Pi_{X^\bullet}}}^{\text{cl},*}$ and $\alpha'' \in E_{\mathfrak{S}''_{\Pi_{X^\bullet}}}^{\text{cl},*}$. Write $Y_{\alpha'}^\bullet \rightarrow Y_{\alpha'}^{\bullet'}$ and $Y_{\alpha''}^\bullet \rightarrow Y_{\alpha''}^{\bullet''}$ for the Galois admissible coverings corresponding to α' and α'' , $H_{\alpha'}$ and $H_{\alpha''}$ for the open subgroups of Π_{X^\bullet} corresponding to $Y_{\alpha'}^\bullet$ and $Y_{\alpha''}^\bullet$ (i.e., the kernels of α' and α''), respectively. Then we obtain that

$$Y_{\alpha'}^\bullet \times_{X^\bullet} Y_{\alpha''}^\bullet$$

is a connected pointed stable curve corresponding to the open subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Proposition 3.7 implies that the cardinality of the set of nodes of $Y_{\alpha'}^\bullet \times_{X^\bullet} Y_{\alpha''}^\bullet$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''}$ and Π_{X^\bullet} . Then (i) follows from Proposition 3.6 and Remark 2.2.1.

(ii) By Lemma 3.2 (i) and Lemma 3.3, p and $\Pi_{X^\bullet}^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Then there is an edge-triple

$$\mathfrak{S}'''_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell''', d''', y''')$$

associated to Π_{X^\bullet} which can be mono-anabelian reconstructed from Π_{X^\bullet} , where $y''' \in \text{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{F}_{d'''})$. Thus, (ii) follows from (i) and Proposition 2.2. This completes the proof of the proposition. \square

3.3. Reconstructions of dual semi-graphs.

Proposition 3.9. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup.*

(i) *The natural maps*

$$v(\Gamma_{X_H^\bullet}) \rightarrow v(\Gamma_{X^\bullet}), \quad e^{\text{cl}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{cl}}(\Gamma_{X^\bullet}), \quad \text{and} \quad e^{\text{op}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{op}}(\Gamma_{X^\bullet})$$

induced by the Galois admissible covering $X_H^\bullet \rightarrow X^\bullet$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(ii) *Suppose that H is normal. Then the natural action of Π_{X^\bullet}/H on $v(\Gamma_{X_H^\bullet})$ (resp. $e^{\text{cl}}(\Gamma_{X_H^\bullet})$, $e^{\text{op}}(\Gamma_{X_H^\bullet})$) induced by the natural action of Π_{X^\bullet}/H on X_H^\bullet can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.*

Proof. (i) By Lemma 3.2 (i), we may choose a prime number ℓ such that $\ell \neq p$ and $(\ell, [\Pi_{X^\bullet} : H]) = 1$. By Proposition 3.5, we obtain that $V_{X^\bullet, \ell}^*$ and $V_{X, \ell}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Moreover, by Proposition 3.7, we obtain that the data g_{X_H} , n_{X_H} , r_{X_H} , $\#v(\Gamma_{X_H^\bullet})$, H^{top} , and $H^{\text{ét}}$ can be mono-anabelian reconstructed from Π_{X^\bullet} and H . Then by applying similar arguments to the arguments given in the proof of Proposition 3.5 (i) (i.e., by replacing $\Pi_{X^\bullet}^{\text{ét}}$ and $\Pi_{X^\bullet}^{\text{top}, \ell}$ by $H^{\text{ét}}$ and $H^{\text{top}, \ell}$, respectively), we obtain that $V_{X_H, \ell}^*$ and $V_{X_H, \ell}$ can be also mono-anabelian reconstructed from Π_{X^\bullet} and H .

For each $\alpha \in V_{X^\bullet, \ell}^*$ and each $\alpha_H \in V_{X_H, \ell}^*$, we write $Q_\alpha \subseteq \Pi_{X^\bullet}$ and $Q_{\alpha_H} \subseteq H$ for the kernels of $\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{ét}} \xrightarrow{\alpha} \mathbb{F}_\ell$ and $H \rightarrow H^{\text{ét}} \xrightarrow{\alpha_H} \mathbb{F}_\ell$, respectively. Then, by Remark 2.1.3, we have that

$$[\alpha_H] \mapsto [\alpha],$$

where $[\alpha]$ and $[\alpha_H]$ denote the images of α and α_H in $V_{X^\bullet, \ell}^*$ and $V_{X_H, \ell}^*$, respectively, if and only if one of the following holds: (1) there exists $\alpha'_H \in V_{X_H, \ell}^*$ such that $\alpha_H \sim \alpha'_H$ and $Q_{\alpha'_H} = Q_\alpha \cap H$, where $Q_{\alpha'_H}$ denotes the kernel of $H \rightarrow H^{\text{ét}} \xrightarrow{\alpha'_H} \mathbb{F}_\ell$; (2) there exists $\alpha''_H \in V_{X_H, \ell}$ such that $\alpha_H \sim \alpha''_H$ and

$$\#v(\Gamma_{X_{Q_\alpha \cap Q_{\alpha''_H}}^\bullet}) = \ell \#v(\Gamma_{X_{Q_\alpha \cap H}^\bullet}),$$

where $Q_{\alpha''_H}$ denotes the kernel of $H \rightarrow H^{\text{ét}} \xrightarrow{\alpha''_H} \mathbb{F}_\ell$. Thus, Proposition 3.7 implies that the natural map $v(\Gamma_{X_H^\bullet}) \rightarrow v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Next, let us prove that the natural maps of sets of edges can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. We only treat the case of closed edges. By Lemma 3.2 (i), we may choose group-theoretically prime numbers ℓ and d distinct from p satisfying (i) (ii) of 2.2.2. Moreover, by applying Lemma 3.2 (iv), Lemma 3.3, and Lemma 3.4, we may choose group-theoretically a homomorphism $y : \Pi_{X^\bullet}^{\text{ét}} \rightarrow \mathbb{F}_d$ satisfying (iii) of 2.2.2 (i.e., a homomorphism $y : \Pi_{X^\bullet}^{\text{ét}} \rightarrow \mathbb{F}_d$ satisfying $\#(v(\Gamma_{X_{H_y}^\bullet})) = \#(v(\Gamma_{X^\bullet}))$, where $H_y \subseteq \Pi_{X^\bullet}$ denotes the kernel of $\Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}^{\text{ét}} \xrightarrow{y} \mathbb{F}_d$.) This means that we may choose group-theoretically an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, y)$$

associated to Π_{X^\bullet} such that $(\ell, [\Pi_{X^\bullet} : H]) = (d, [\Pi_{X^\bullet} : H]) = 1$. Moreover, we denote by

$$y_H : H^{\text{ét}} \rightarrow H^{\text{ét}} / \text{Im}(H \cap H_y) \cong \mathbb{F}_d.$$

Then we obtain an edge-triple

$$\mathfrak{T}_H \stackrel{\text{def}}{=} (\ell, d, y_H)$$

associated to H .

By applying Proposition 3.6 and Proposition 3.8, we obtain that

$$E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}, \star}, E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^{\text{cl}} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from Π_{X^\bullet} and H_y . Moreover, by Proposition 3.6, Proposition 3.7 and similar arguments to the arguments given in the proof of Proposition 3.8 (i.e., by replacing X^\bullet , Π_{X^\bullet} by X_H^\bullet , H , respectively), we obtain that

$$E_{\mathfrak{T}_H}^{\text{cl}, \star}, E_{\mathfrak{T}_H}^{\text{cl}} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_H^\bullet})$$

can be mono-anabelian reconstructed from H and $H \cap H_y$.

For each $\beta \in E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl},*}$ and each $\beta_H \in E_{\mathfrak{X}_H}^{\text{cl},*}$, we write $P_\beta \subseteq H_y \subseteq \Pi_X^\bullet$ and $P_{\beta_H} \subseteq H \cap H_y \subseteq H$ for the kernels of β and β_H , respectively. Then, by Remark 2.2.3, we observe that

$$[\beta_H] \mapsto [\beta],$$

where $[\beta_H]$ and $[\beta]$ denote the images of β_H and β in $E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl}}$ and $E_{\mathfrak{X}_H}^{\text{cl}}$, respectively, if and only if one of the following holds: (1) there exists $\beta'_H \in E_{\mathfrak{X}_H}^{\text{cl},*}$ such that $\beta_H \sim \beta'_H$ and $P_{\beta'_H} = P_\beta \cap H$; (2) there exists $\beta''_H \in E_{\mathfrak{X}_H}^{\text{cl},*}$ such that $\beta''_H \sim \beta_H$ and

$$\#e^{\text{cl}}(\Gamma_{X_{P_\beta \cap P_{\beta''_H}}^\bullet}) = \ell \#e^{\text{cl}}(\Gamma_{X_{P_\beta \cap H}}^\bullet),$$

where $P_{\beta''_H}$ denotes the kernel of β''_H . Thus, Proposition 3.7 implies that the natural map $e^{\text{cl}}(\Gamma_{X_H^\bullet}) \rightarrow e^{\text{cl}}(\Gamma_X)$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_X^\bullet$.

(ii) follows from (i), Remark 2.1.2, and Remark 2.2.2. This completes the proof of the proposition. \square

Next, we prove that the dual semi-graphs can be mono-anabelian reconstructed from the admissible fundamental groups.

Proposition 3.10. (i) *The dual semi-graph Γ_{X^\bullet} can be mono-anabelian reconstructed from Π_X^\bullet .*

(ii) *For each open subgroup $H \subseteq \Pi_X^\bullet$, the natural map of dual semi-graphs*

$$\Gamma_{X_H^\bullet} \rightarrow \Gamma_{X^\bullet}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_X^\bullet$. Moreover, if $H \subseteq \Pi_X^\bullet$ is an open normal subgroup, then the action of Π_X^\bullet/H on $\Gamma_{X_H^\bullet}$ induced by the action of Π_X^\bullet/H on X_H^\bullet can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_X^\bullet$.

Proof. By Lemma 3.2 and Lemma 3.3, we may choose an edge-triple

$$\mathfrak{X}_{\Pi_X^\bullet} \stackrel{\text{def}}{=} (\ell, d, y)$$

associated to Π_X^\bullet . Write H_y for the kernel of $\Pi_X^\bullet \twoheadrightarrow \Pi_X^{\text{ét}} \xrightarrow{y} \mathbb{F}_d$ and Y^\bullet for the pointed stable curve over k corresponding to H_y . Then Proposition 3.6 implies that the sets

$$E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl}}, E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{op}}$$

can be mono-anabelian reconstructed from H_y and Π_X^\bullet .

Let $e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet})$ be an arbitrary edge and $v(e)$ the set of vertices on which e abuts. We only treat the case of closed edges.

Let $\beta \in E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl},*}$. Write $Y_\beta^\bullet \rightarrow Y^\bullet$ for the Galois admissible covering corresponding to β , H_β for the kernel of β which is the open normal subgroup of H_y corresponding to Y_β^\bullet , and $\Gamma_{Y_\beta^\bullet}$ for the dual semi-graph of Y_β^\bullet . Let $m_1 = \#v(\Gamma_{X^\bullet}) - 2$, $m_2 = \#v(\Gamma_{X^\bullet}) - 1$, and $i \in \{1, 2\}$. We observe that $\beta \in E_{\mathfrak{X}_{\Pi_X^\bullet}, e_\beta}^{\text{cl},*, m_i}$ if and only if

$$\#v(\Gamma_{Y_\beta^\bullet}) = \#v(\Gamma_{Y^\bullet}) - m_i + \ell m_i = \#v(\Gamma_{X^\bullet}) - m_i + \ell m_i.$$

Since Proposition 3.9 implies that $v(\Gamma_{Y_\beta^\bullet})$ and $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from H_α and Π_X^\bullet , we have that $E_{\mathfrak{X}_{\Pi_X^\bullet}, e_\beta}^{\text{cl},*, m_i}$ can be mono-anabelian reconstructed from H_y and Π_X^\bullet . By Lemma 2.3, for each $m \in \mathbb{Z}_{\geq 0}$, if $E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl},*, m} \neq \emptyset$, then the composition of maps

$$E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl},*, m} \hookrightarrow E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl},*} \twoheadrightarrow E_{\mathfrak{X}_{\Pi_X^\bullet}}^{\text{cl}} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet})$$

is a surjection. In particular, we have that $E_{\mathfrak{S}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star, m} \neq \emptyset$ if $E_{\mathfrak{S}_{\Pi_{X^\bullet}}^{\text{cl}, \star, m}} \neq \emptyset$.

Let $\alpha \in E_{\mathfrak{S}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star, n}$ be arbitrary element, where $n = m_2$ if $E_{\mathfrak{S}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star, m_2} \neq \emptyset$ (i.e., e is contained in a unique irreducible component of X^\bullet), and that $n = m_1$ if $E_{\mathfrak{S}_{\Pi_{X^\bullet}, e}}^{\text{cl}, \star, m_2} = \emptyset$ (i.e., e is contained in two different irreducible components of X^\bullet). Proposition 3.9 (i) implies that the natural map

$$f_{H_\alpha}^{\text{ver}} : v(\Gamma_{Y_\alpha^\bullet}) \rightarrow v(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from $H_\alpha \hookrightarrow \Pi_{X^\bullet}$. Then we have

$$v(e) = \{v \in v(\Gamma_{X^\bullet}) \mid \#(f_{H_\alpha}^{\text{ver}})^{-1}(v) = 1\}.$$

This means that Γ_{X^\bullet} can be mono-anabelian reconstructed from Π_{X^\bullet} . This completes the proof of (i).

Similar arguments to the arguments given in the proof above imply that, for each open subgroup $H \subseteq \Pi_{X^\bullet}$, the dual semi-graph

$$\Gamma_{X_H^\bullet}$$

can be mono-anabelian reconstructed from Π_{X^\bullet} and H . Then (ii) follows from Proposition 3.9. \square

3.4. Main theorem. Now, we prove the main theorem of the present paper.

Theorem 3.11. *Let X^\bullet be an arbitrary pointed stable curve (i.e., we do not assume that X^\bullet satisfies Condition A) of type (g_X, n_X) over an algebraically closed field of positive characteristic and Π_{X^\bullet} the admissible fundamental group of X^\bullet . Then the topological data*

$$\{(g_\Gamma, n_\Gamma)\}_{\Gamma \in \text{Sub}(\Gamma_{X^\bullet})}, \{(g_{\Gamma \setminus L}, n_{\Gamma \setminus L})\}_{\Gamma \setminus L \in \text{CSub}(\Gamma_{X^\bullet})}$$

and the combinatorial data

$$\text{Sub}(\Gamma_{X^\bullet}), \text{CSub}(\Gamma_{X^\bullet}), \text{Sub}(\Pi_{X^\bullet}), \text{CSub}(\Pi_{X^\bullet})$$

associated to X^\bullet defined in Definition 1.4 can be mono-anabelian reconstructed from Π_{X^\bullet} .

Proof. Since Π_{X^\bullet} is topologically finitely generated, there exists a set of open normal subgroups $\{H_i\}_{i \in \mathbb{N}}$ (e.g. characteristic subgroups) of Π_{X^\bullet} such that the following conditions are satisfied: (1) $H_1 \stackrel{\text{def}}{=} \Pi_{X^\bullet}$; (2) $H_i \supseteq H_{i+1}$ for each $i \in \mathbb{N}$; (3) $\varprojlim_{i \in \mathbb{N}} \Pi_{X^\bullet}/H_i = \Pi_{X^\bullet}$. By Remark 1.5.2, we may assume that $X_{H_2}^\bullet$ satisfies Condition A.

First, we claim the following:

- (1) For each $i \in \mathbb{N}$, the dual semi-graph $\Gamma_{X_{H_i}^\bullet}$ of $X_{H_i}^\bullet$ corresponding to H_i can be mono-anabelian reconstructed from H_i ;
- (2) For each $i \in \mathbb{N}$, the natural map of dual semi-graphs

$$\Gamma_{X_{H_i}^\bullet} \rightarrow \Gamma_{X^\bullet}$$

induced by the admissible covering $X_{H_i}^\bullet \rightarrow X^\bullet$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$, and the natural action of Π_{X^\bullet}/H_i on $\Gamma_{X_{H_i}^\bullet}$ induced by the natural action of Π_{X^\bullet}/H_i on $X_{H_i}^\bullet$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$.

Proposition 3.10 (i) implies that, for each $i \geq 2$, $\Gamma_{X_{H_i}^\bullet}$ can be mono-anabelian reconstructed from H_i . Moreover, Remark 2.1.2 and Remark 2.2.2 imply that, for each $i \geq 2$, the natural action of Π_{X^\bullet}/H_i on $\Gamma_{X_{H_i}^\bullet}$ induced by the natural action of Π_{X^\bullet}/H_i on $X_{H_i}^\bullet$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$. For each $i, j \geq 2$ such that

$j > i$, by applying Proposition 3.10 (ii), we may identify naturally $\Gamma_{X_{H_j}^\bullet}/(H_i/H_j)$ with $\Gamma_{X_{H_i}^\bullet}$. Moreover, we may identify naturally $\Gamma_{X_{H_j}^\bullet}/H_j$ with $\Gamma_{X_{H_i}^\bullet}/H_i$. Thus, we may put

$$\Gamma_{X^\bullet} \stackrel{\text{def}}{=} \Gamma_{X_{H_2}^\bullet}/H_2.$$

Then we obtain a natural map

$$\Gamma_{X_i^\bullet} \rightarrow \Gamma_{X_i^\bullet}/H_i = \Gamma_{X_2^\bullet}/H_2 = \Gamma_{X^\bullet}, \quad i \geq 2,$$

which can be mono-anabelian reconstructed from $H_i \hookrightarrow \Pi_{X^\bullet}$. This completes the proof of the claim.

Since Γ_{X^\bullet} can be mono-anabelian reconstructed from Π_{X^\bullet} , by 1.3.2 and 1.3.3, the claim implies that the combinatorial data

$$\text{Sub}(\Gamma_{X^\bullet}), \text{CSub}(\Gamma_{X^\bullet}), \text{Sub}(\Pi_{X^\bullet}), \text{CSub}(\Pi_{X^\bullet})$$

associated to X^\bullet can be mono-anabelian reconstructed from Π_{X^\bullet} . In particular, $\text{Ver}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} . On the other hand, we have that

$$n_\Gamma \stackrel{\text{def}}{=} \#(e^{\text{op}}(\Gamma)), \quad n_{\Gamma \setminus L} \stackrel{\text{def}}{=} n_\Gamma + 2\#(L)$$

can be mono-anabelian reconstructed from Π_{X^\bullet} (see 1.2.3 and 1.2.4). Moreover, the Betti numbers r_{X_Γ} and $r_{X_{\Gamma \setminus L}}$ of the dual semi-graphs of X_Γ^\bullet and $X_{\Gamma \setminus L}^\bullet$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Since $\text{Ver}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from Π_{X^\bullet} , [T2, Theorem 0.2] implies that $\{(g_v, n_v)\}_{v \in v(\Gamma_{X^\bullet})}$ can be mono-anabelian reconstructed from Π_{X^\bullet} . Then

$$g_\Gamma \stackrel{\text{def}}{=} r_{X_\Gamma} + \sum_{v \in v(\Gamma)} g_v, \quad g_{\Gamma \setminus L} \stackrel{\text{def}}{=} r_{X_{\Gamma \setminus L}} + \sum_{v \in v(\Gamma \setminus L)} g_v$$

can be mono-anabelian reconstructed from Π_{X^\bullet} . We complete the proof of the theorem. \square

The following corollary follows from Theorem 3.11.

Corollary 3.12. *We maintain the notation introduced in Theorem 3.11. Let W^\bullet be a pointed stable curve of type (g_W, n_W) over an algebraically closed field of positive characteristic and Π_{W^\bullet} the admissible fundamental group of W^\bullet . Then we can detect group-theoretically whether or not there exists a sub-semi-graph Γ of Γ_{X^\bullet} (resp. a semi-graph associated to a sub-semi-graph Γ of Γ_{X^\bullet} and a set of edges L of Γ) such that $(g_W, n_W) = (g_\Gamma, n_\Gamma)$ (resp. $(g_W, n_W) = (g_{\Gamma \setminus L}, n_{\Gamma \setminus L})$) and $\Pi_{W^\bullet} \xrightarrow{\sim} \Pi_{\widehat{\Gamma}}$ (resp. $\Pi_{W^\bullet} \xrightarrow{\sim} \Pi_{\widehat{\Gamma \setminus L}}$).*

4. THE SET $\overline{\Pi}_{g,n}^{\text{ord}}$

4.1. **The definitions of $\overline{\Pi}_{g,n}^{\text{ord}}$ and $\pi_{g,n}^{\text{adm}} : \overline{M}_{g,n}^{\text{ord}} \twoheadrightarrow \overline{\Pi}_{g,n}^{\text{ord}}$.** We maintain the notation introduced in 0.2 and 1.3.

4.1.1. Let $q \in \overline{M}_{g,n}$, X_q^\bullet a pointed stable curve of type (g, n) corresponding to a geometric point over q , and Π_q the admissible fundamental group of X_q^\bullet . Let $\Gamma_{X_q^\bullet}$ be the dual semi-graph of X_q^\bullet . Since the isomorphism class of $\Gamma_{X_q^\bullet}$ does not depend on the choices of geometric points over q , we write Γ_q for $\Gamma_{X_q^\bullet}$, and say Γ_q the *dual semi-graph associated to q* . Moreover, let $\widehat{\Gamma}_q$ be the dual semi-graph of the universal admissible covering of X_q^\bullet associated to Π_q , and $\pi_q : \widehat{\Gamma}_q \twoheadrightarrow \Gamma_q$ the surjective map of semi-graphs. We put

$$\text{Edg}_e^{\text{op}}(\Pi_q) \stackrel{\text{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in \pi_q^{-1}(e)}, \quad e \in e^{\text{op}}(\Gamma_q),$$

which can be mono-anabelian reconstructed from Π_q by Theorem 3.11.

Let $o_{\Pi_q} : \{\text{Edg}_e^{\text{op}}(\Pi_q)\}_{e \in e^{\text{op}}(\Gamma_q)} \xrightarrow{\sim} \{1, \dots, n\}$ be a bijective map. We shall say that

$$(\Pi_q, o_{\Pi_q})$$

is an *ordered admissible fundamental group* of q . Let $(\Pi_{q'}, o_{\Pi_{q'}})$ be an ordered admissible fundamental group of q' . Let $\phi : \Pi_{q'} \xrightarrow{\sim} \Pi_q$ be an isomorphism of profinite groups and $\phi^{\text{op}} : \{\text{Edg}_e^{\text{op}}(\Pi_{q'})\}_{e \in e^{\text{op}}(\Gamma_{q'})} \xrightarrow{\sim} \{\text{Edg}_e^{\text{op}}(\Pi_q)\}_{e \in e^{\text{op}}(\Gamma_q)}$ the bijective map which is mono-anabelian reconstructed from ϕ by Theorem 3.11. An isomorphism of ordered admissible fundamental groups is a pair

$$(\phi, \phi^{\text{op}}) : (\Pi_{q'}, o_{\Pi_{q'}}) \xrightarrow{\sim} (\Pi_q, o_{\Pi_q})$$

such that $o_{\Pi_q} \circ \phi^{\text{op}} = o_{\Pi_{q'}}$.

4.1.2. We denote by

$$\overline{\Pi}_{g,n}^{\text{ord}}$$

the set of isomorphism classes of ordered admissible fundamental groups of $q \in \overline{M}_{g,n}$. Moreover, Theorem 3.11 implies that $\overline{\Pi}_{g,n}^{\text{ord}}$ can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$.

Let $o_q : e^{\text{op}}(\Gamma_q) \xrightarrow{\sim} \{1, \dots, n\}$ be a bijective map. Then (X_q^\bullet, o_{X_q}) is an ordered pointed stable curve of type (g, n) , where $o_{X_q} : D_{X_q} \xrightarrow{\sim} \{1, \dots, n\}$ is the bijective map induced by o_q . Moreover, since o_q does not depend on the choices of geometric points over q , we have $(q, o_q) \in \overline{M}_{g,n}^{\text{ord}}$. We put

$$\tilde{\pi}_{g,n}^{\text{adm}} : \overline{M}_{g,n} \rightarrow \overline{\Pi}_{g,n}, \quad q \mapsto [\Pi_q],$$

and put

$$\tilde{\pi}_{g,n}^{\text{adm,ord}} : \overline{M}_{g,n}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}, \quad (q, o_q) \mapsto [(\Pi_q, o_{\Pi_q})],$$

where o_{Π_q} denotes the bijective map induced by o_q via the natural bijection (1.3.4)

$$\{\text{Edg}_e^{\text{op}}(\Pi_q)\}_{e \in e^{\text{op}}(\Gamma_q)} \xrightarrow{\sim} \text{Edg}^{\text{op}}(\Pi_q)/\Pi_q \xrightarrow{\sim} e^{\text{op}}(\Gamma_q),$$

and $[(\Pi_q, o_{\Pi_q})]$ denotes the isomorphism class of (Π_q, o_{Π_q}) . Then we obtain the following result:

Theorem 4.1. *Denote by $\overline{\Pi}_{g,n}^{\text{ord}}$ the set of isomorphism classes of ordered admissible fundamental groups of $q \in \overline{M}_{g,n}$. Then there are natural surjective maps*

$$\tilde{\pi}_{g,n}^{\text{adm,ord}} : \overline{M}_{g,n}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}, \quad (q, o_q) \mapsto [(\Pi_q, o_{\Pi_q})],$$

and

$$\overline{\Pi}_{g,n}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}, \quad [(\Pi_q, o_{\Pi_q})] \mapsto [\Pi_q],$$

which fit into the following commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}^{\text{ord}} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm,ord}}} & \overline{\Pi}_{g,n}^{\text{ord}} \\ \downarrow & & \downarrow \\ \overline{M}_{g,n} & \xrightarrow{\tilde{\pi}_{g,n}^{\text{adm}}} & \overline{\Pi}_{g,n}. \end{array}$$

Moreover, $\overline{\Pi}_{g,n}^{\text{ord}}$ can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$.

4.2. **Clutching maps I.** We maintain the notation introduced in 4.1.

4.2.1. Let $R \stackrel{\text{def}}{=} \{r_1, \dots, r_{n_1}\}$ and $S \stackrel{\text{def}}{=} \{s_1, \dots, s_{n_2}\}$ be distinct subsets of $\{1, \dots, n\}$ such that $r_1 < \dots < r_{n_1}$, that $s_1 < \dots < s_{n_2}$, and that $n_1 + n_2 = n$. Recall that we have the following clutching morphism for moduli stacks ([K, Definition 3.8]):

$$\alpha'_{g_1, g_2, R, S} : \overline{\mathcal{M}}_{g_1, n_1+1}^{\text{ord}} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{g_2, n_2+1}^{\text{ord}} \rightarrow \overline{\mathcal{M}}_{g, n}^{\text{ord}}$$

where $g = g_1 + g_2$. We see that $\alpha'_{g_1, g_2, R, S}$ induces the following continuous map of topological spaces:

$$\tilde{\alpha}_{g_1, g_2, R, S} : \overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}} \rightarrow \overline{M}_{g, n}^{\text{ord}}$$

where $\overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}}$ denotes the product as topological spaces.

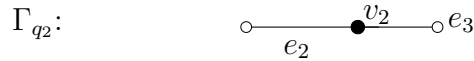
4.2.2. Let $i \in \{1, 2\}$, $[(\Pi_{q_i}, o_{\Pi_{q_i}})] \in \overline{\Pi}_{g_i, n_i+1}^{\text{ord}}$, and $[(\Pi_q, o_{\Pi_q})] \in \overline{\Pi}_{g, n}^{\text{ord}}$. Moreover, let Γ_{q_i} and Γ_q be the dual semi-graphs associated to q_i and q , respectively.

Let $e_i \in e^{\text{op}}(\Gamma_{q_i})$. We shall say that Γ_q is *glued by* Γ_{q_1} and Γ_{q_2} along e_1 and e_2 if the following conditions are satisfied:

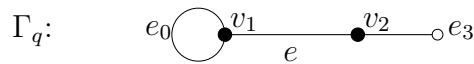
(i) There exists an isomorphism $\alpha_{q_i}^{\text{sg}} : \Gamma_{q_i} \xrightarrow{\sim} \Gamma'_{q_i}$, where $\Gamma'_{q_i} \in \text{Sub}(\Gamma_q)$ is a sub-semi-graph of Γ_q (Definition 1.1 (e)) such that $\alpha_{q_1}^{\text{sg}}(e_1) = \alpha_{q_2}^{\text{sg}}(e_2) = e \in e^{\text{cl}}(\Gamma_q)$.

(ii) If we regard Γ_{q_i} as a sub-semi-graph of Γ_q via the isomorphism $\alpha_{q_i}^{\text{sg}}$ (i.e., we identify Γ_{q_i} with Γ'_{q_i}), then the following conditions hold: $v(\Gamma_{q_1}) \cup v(\Gamma_{q_2}) = v(\Gamma_q)$, $(e^{\text{op}}(\Gamma_{q_1}) \cup e^{\text{op}}(\Gamma_{q_2})) \setminus \{e\} = e^{\text{op}}(\Gamma_q)$, $e^{\text{cl}}(\Gamma_{q_1}) \cup e^{\text{cl}}(\Gamma_{q_2}) \cup \{e\} = e^{\text{cl}}(\Gamma_q)$, $v(\Gamma_{q_1}) \cap v(\Gamma_{q_2}) = \emptyset$, $e^{\text{cl}}(\Gamma_{q_1}) \cap e^{\text{cl}}(\Gamma_{q_2}) = \emptyset$, and $e^{\text{op}}(\Gamma_{q_1}) \cap e^{\text{op}}(\Gamma_{q_2}) = \{e\}$.

Example 4.2. Let Γ_{q_i} , $i \in \{1, 2\}$, be the following semi-graph:



Then Γ_q is as follows, which is glued by Γ_{q_1} and Γ_{q_2} along e_1 and e_2 :



4.2.3. Now, let us define a clutching map for $\overline{\Pi}_{g,n}^{\text{ord}}$ corresponding to $\tilde{\alpha}_{g_1, g_2, R, S}$. First, we note that $o_{\Pi_{q_i}}$ induces a bijection

$$o_{q_i} : e^{\text{op}}(\Gamma_{q_i}) \xrightarrow{\sim} \{1, \dots, n_i + 1\}.$$

Then we write $e_{n_i+1} \stackrel{\text{def}}{=} o_{q_i}^{-1}(n_i + 1)$.

We put

$$\alpha_{g_1, g_2, R, S}^{\text{gp}} : \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} \rightarrow \overline{\Pi}_{g, n}^{\text{ord}}, \quad ([(\Pi_{q_i}, o_{\Pi_{q_i}})])_{i \in \{1, 2\}} \mapsto [(\Pi_q, o_{\Pi_q})],$$

where $[(\Pi_{q_i}, o_{\Pi_{q_i}})]$ and $[(\Pi_q, o_{\Pi_q})]$ satisfy the following conditions:

(i) Γ_q is glued by Γ_{q_1} and Γ_{q_2} along e_{n_1+1} and e_{n_2+1} .

(ii) There exists an isomorphism of profinite groups $\alpha_{q_i} : \Pi_{q_i} \xrightarrow{\sim} \Pi_{\widehat{\Gamma}'_{q_i}}$, where Γ'_{q_i} is the sub-semi-graph of Γ_q defined in 4.2.2.

(iii) Theorem 3.11 implies that the isomorphism α_{q_i} determines group-theoretically an isomorphism of semi-graphs $\Gamma_{q_i} \xrightarrow{\sim} \Gamma'_{q_i}$. Then this isomorphism coincides with the isomorphism $\alpha_{q_i}^{\text{sg}}$ defined in 4.2.2.

(iv) The bijections $o_{q_1} \sqcup o_{q_2}$ and $e^{\text{op}}(\Gamma_{q_1}) \setminus \{e_{n_1+1}\} \sqcup e^{\text{op}}(\Gamma_{q_2}) \setminus \{e_{n_2+1}\} \xrightarrow{\sim} e^{\text{op}}(\Gamma_q)$ induced by $\alpha_{q_1}^{\text{sg}} \sqcup \alpha_{q_2}^{\text{sg}}$ determine a bijection

$$o_q : e^{\text{op}}(\Gamma_q) \xrightarrow{\sim} \{1, \dots, n\}$$

which fits into the following commutative diagram

$$\begin{array}{ccc} e^{\text{op}}(\Gamma_{q_1}) \setminus \{e_{n_1+1}\} \sqcup e^{\text{op}}(\Gamma_{q_2}) \setminus \{e_{n_2+1}\} & \xrightarrow{o_{q_1}|_{e^{\text{op}}(\Gamma_{q_1}) \setminus \{e_{n_1+1}\}} \sqcup o_{q_2}|_{e^{\text{op}}(\Gamma_{q_2}) \setminus \{e_{n_2+1}\}}} & \{1, \dots, n_1\} \sqcup \{1, \dots, n_2\} \\ \downarrow & & \downarrow \\ e^{\text{op}}(\Gamma_q) & \xrightarrow{o_q} & \{1, \dots, n\}, \end{array}$$

where the vertical arrow on the right-hand side is the bijection

$$\{1, \dots, n_1\} \sqcup \{1, \dots, n_2\} \xrightarrow{\sim} \{1, \dots, n\}, \quad a, b \mapsto r_a, s_b.$$

(v) The bijection $e^{\text{op}}(\Gamma_q) \xrightarrow{\sim} \{1, \dots, n\}$ induced by o_{Π_q} coincides with o_q defined in (iv).

4.2.4. [M1, Appendix] (or [M2, Section 2] for a more general theory) implies that $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ is well-defined. Moreover, by applying Theorem 3.11 and Corollary 3.12, we see that $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ can be mono-anabelian reconstructed from $\overline{\Pi}_{g, n}$. We obtain the following result:

Theorem 4.3. *There exists a map*

$$\alpha_{g_1, g_2, R, S}^{\text{gp}} : \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} \rightarrow \overline{\Pi}_{g, n}^{\text{ord}}, \quad ([(\Pi_{q_i}, o_{\Pi_{q_i}})])_{i \in \{1, 2\}} \mapsto [(\Pi_q, o_{\Pi_q})]$$

which fits into the following diagram

$$\begin{array}{ccc} \overline{M}_{g_1, n_1+1}^{\text{ord}} \times \overline{M}_{g_2, n_2+1}^{\text{ord}} & \xrightarrow{\tilde{\alpha}_{g_1, g_2, R, S}} & \overline{M}_{g, n}^{\text{ord}} \\ \downarrow \tilde{\pi}_{g_1, n_1+1}^{\text{adm, ord}} \times \tilde{\pi}_{g_2, n_2+1}^{\text{adm, ord}} & & \downarrow \tilde{\pi}_{g, n}^{\text{adm, ord}} \\ \overline{\Pi}_{g_1, n_1+1}^{\text{ord}} \times \overline{\Pi}_{g_2, n_2+1}^{\text{ord}} & \xrightarrow{\alpha_{g_1, g_2, R, S}^{\text{gp}}} & \overline{\Pi}_{g, n}^{\text{ord}}. \end{array}$$

Moreover, $\overline{\Pi}_{g_1, n_1+1}^{\text{ord}}$, $\overline{\Pi}_{g_2, n_2+1}^{\text{ord}}$, and $\alpha_{g_1, g_2, R, S}^{\text{gp}}$ can be mono-anabelian reconstructed from $\overline{\Pi}_{g, n}$.

4.3. **Clutching maps II.** We maintain the notation introduced in 4.1.

4.3.1. Recall that we have the following clutching morphism for moduli stacks ([K, Definition 3.8]):

$$\beta' : \overline{\mathcal{M}}_{g-1, n+2}^{\text{ord}} \rightarrow \overline{\mathcal{M}}_{g, n}^{\text{ord}}.$$

We see that β' induces the following continuous map of topological spaces:

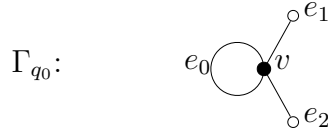
$$\tilde{\beta} : \overline{M}_{g-1, n+2}^{\text{ord}} \rightarrow \overline{M}_{g, n}^{\text{ord}}.$$

4.3.2. Let $[(\Pi_{q_0}, o_{\Pi_{q_0}})] \in \overline{\Pi}_{g-1, n+2}^{\text{ord}}$ and $[(\Pi_q, o_{\Pi_q})] \in \overline{\Pi}_{g, n}^{\text{ord}}$. Moreover, let Γ_{q_0} and Γ_q be the dual semi-graphs associated to q_0 and q , respectively.

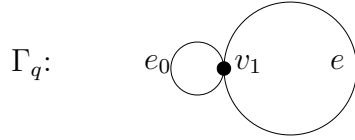
Let $e_1, e_2 \in e^{\text{op}}(\Gamma_{q_0})$. We shall say that Γ_q is *glued by Γ_{q_0} along e_1 and e_2* if there exists a closed edge $e \in e^{\text{cl}}(\Gamma_q)$ such that $\Gamma_q \setminus \{e\}$ is a semi-graph associated to Γ_q and $\{e\}$, and that there exists an isomorphism

$$\beta_{q_0}^{\text{sg}, \circ} : \Gamma_{q_0} \setminus \{e_1, e_2\} \xrightarrow{\sim} \Gamma_q \setminus \{e\}.$$

Example 4.4. Let Γ_{q_0} be the following semi-graph:



Then Γ_q is as follows, which is glued by Γ_{q_0} along e_1 and e_2 :



4.3.3. Now, let us define a clutching map for $\overline{\Pi}_{g, n}^{\text{ord}}$ corresponding to $\tilde{\beta}$. First, we note that $o_{\Pi_{q_0}}$ induces a bijection

$$o_{q_0} : e^{\text{op}}(\Gamma_{q_0}) \xrightarrow{\sim} \{1, \dots, n+1, n+2\}.$$

Then we write $e_{n+i} \stackrel{\text{def}}{=} o_{q_0}^{-1}(n+i)$, $i \in \{1, 2\}$.

We put

$$\beta^{\text{gp}} : \overline{\Pi}_{g-1, n+2}^{\text{ord}} \rightarrow \overline{\Pi}_{g, n}^{\text{ord}}, \quad [(\Pi_{q_0}, o_{\Pi_{q_0}})] \mapsto [(\Pi_q, o_{\Pi_q})],$$

where $[(\Pi_{q_0}, o_{\Pi_{q_0}})]$ and $[(\Pi_q, o_{\Pi_q})]$ satisfy the following conditions:

- (i) There exists a closed edge $e \in e^{\text{cl}}(\Gamma_q)$ such that Γ_q is glued by Γ_{q_0} along e_{n+1} and e_{n+2} .
- (ii) There exists an isomorphism of profinite groups $\beta_{q_0} : \Pi_{q_0} \xrightarrow{\sim} \Pi_{\widehat{\Gamma_{q_0} \setminus \{e\}}}$.
- (iii) Write $\Gamma_{q \setminus e}$ for the dual semi-graph of $X_{\Gamma_q \setminus \{e\}}^\bullet$ (1.2.4). Theorem 3.11 implies that β_{q_0} induces group-theoretically an isomorphism of semi-graphs $\Gamma_{q_0} \xrightarrow{\sim} \Gamma_{q \setminus e}$. Then the restriction of this isomorphism to $\Gamma_{q_0} \setminus \{e_{n+1}, e_{n+2}\}$ coincides with the isomorphism $\beta_{q_0}^{\text{sg}, \circ} : \Gamma_{q_0} \setminus \{e_{n+1}, e_{n+2}\} \xrightarrow{\sim} \Gamma_q \setminus \{e\}$ defined in 4.3.2.

(iv) The bijections o_{q_0} and $e^{\text{op}}(\Gamma_{q_0}) \setminus \{e_{n+1}, e_{n+2}\} \xrightarrow{\sim} e^{\text{op}}(\Gamma_q)$ induced by $\beta_{q_0}^{\text{sg}, \circ}$ determine a bijection

$$o_q : e^{\text{op}}(\Gamma_q) \xrightarrow{\sim} \{1, \dots, n\}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} e^{\text{op}}(\Gamma_{q_0}) \setminus \{e_{n+1}, e_{n+2}\} & \xrightarrow{o_{q_0}|_{e^{\text{op}}(\Gamma_{q_0}) \setminus \{e_{n+1}, e_{n+2}\}}} & \{1, \dots, n\} \\ \downarrow & & \parallel \\ e^{\text{op}}(\Gamma_q) & \xrightarrow{o_q} & \{1, \dots, n\} \end{array}$$

(v) The bijection $e^{\text{op}}(\Gamma_q) \xrightarrow{\sim} \{1, \dots, n\}$ induced by o_{Π_q} coincides with o_q defined in (iv).

4.3.4. [M1, Appendix] (or [M2, Section 2] for a more general theory) implies that β^{gp} is well-defined. Moreover, by applying Theorem 3.11 and Corollary 3.12, we see that β^{gp} can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$. We obtain the following result:

Theorem 4.5. *There exists a map*

$$\beta^{\text{gp}} : \overline{\Pi}_{g-1, n+2}^{\text{ord}} \rightarrow \overline{\Pi}_{g,n}^{\text{ord}}, \quad [(\Pi_{q_0}, o_{\Pi_{q_0}})] \mapsto [(\Pi_q, o_{\Pi_q})],$$

which fits into the following diagram

$$\begin{array}{ccc} \overline{M}_{g-1, n+2}^{\text{ord}} & \xrightarrow{\beta} & \overline{M}_{g,n}^{\text{ord}} \\ \tilde{\pi}_{g-1, n+2}^{\text{adm, ord}} \downarrow & & \tilde{\pi}_{g,n}^{\text{adm, ord}} \downarrow \\ \overline{\Pi}_{g-1, n+2}^{\text{ord}} & \xrightarrow{\beta^{\text{gp}}} & \overline{\Pi}_{g,n}^{\text{ord}} \end{array}$$

Moreover, $\overline{\Pi}_{g-1, n+2}^{\text{ord}}$ and β^{gp} can be mono-anabelian reconstructed from $\overline{\Pi}_{g,n}$.

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