On Cachazo-Douglas-Seiberg-Witten Conjecture for Simple Lie Algebras (Talk by Shrawan Kumar)

Abstract: Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex numbers. Consider the exterior algebra $R := \wedge(\mathfrak{g} \oplus \mathfrak{g})$ on two copies of $\mathfrak{g}$. Then, the algebra R is bigraded with the two copies of $\mathfrak{g}$ sitting in bidegrees (1,0) and (0,1) respectively. To distinguish, we denote them by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

The diagonal adjoint action of $\mathfrak{g}$ gives rise to a $\mathfrak{g}$-algebra structure on R compatible with the bigrading. We isolate three 'standard' copies of the adjoint representation $\mathfrak{g}$ in the total degree 2 component R^2. The $\mathfrak{g}$-module map $\partial : \mathfrak{g} \to \wedge^2(\mathfrak{g})$, $x \mapsto \partial x = \sum_i [x, e_i] \wedge f_i$, considered as a map to $\wedge^2(\mathfrak{g}_1)$ will be denoted by c_1, and similarly, $c_2 : \mathfrak{g} \to \wedge^2(\mathfrak{g}_2)$, and $c_3 : \mathfrak{g} \to \mathfrak{g}_1 \otimes \mathfrak{g}_2$, $x \mapsto \sum_i [x, e_i] \otimes f_i$, where $\{e_i\}_{i \leq i \leq N}$ is any basis of $\mathfrak{g}$ and $\{f_i\}_{1 \leq i \leq N}$ is the dual basis of $\mathfrak{g}$ with respect to the Killing form. We denote by C_i the image of c_i.

Let J be the (bigraded) ideal of R generated by the three copies C_1, C_2, C_3 of $\mathfrak{g}$ (in R^2) and define the bigraded $\mathfrak{g}$-algebra $A := R/J$. The Killing form gives rise to a $\mathfrak{g}$-invariant $S \in A^{1,1}$.

Motivated by supersymmetric gauge theory, Cachazo-Douglas-Seiberg-Witten made the following conjecture.

Conjecture (i) The subalgebra $A^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants in A is generated, as an algebra, by the element S.

(ii) $S^h = 0$.

(iii) $S^{h-1} \neq 0$.

The aim of this talk is to give a uniform proof of the above conjecture part (i). In addition, we give a conjecture, the validity of which would imply part (ii) of the above conjecture.

The main ingredients in the proof are: Garland's result on the Lie algebra cohomology of $\hat{\mathfrak{u}} := \mathfrak{g} \otimes t\mathbb{C}[t]$; Kostant's result on the 'diagonal' cohomolgy of $\hat{\mathfrak{u}}$ and its connection with abelian ideals in a Borel subalgebra of $\mathfrak{g}$; and a certain deformation of the singular cohomology of the infinite Grassmannian introduced by Belkale-Kumar.