On b-function, spectrum and multiplier ideals

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(Some details are in arXiv:0705.1434.)

b-function of a hypersurface

b-function (or Bernstein-Sato polynomial).
$\mathcal{D}_X$: ring of linear differential operators.

$\mathcal{D}_X[s]f^s \subset \mathcal{O}_X[\frac{1}{f}][s]f^s$,

where $\partial_i f^s = s(\partial_i f)f^{s-1}$ with $\partial_i = \partial/\partial x_i$.
Def. $b_f(s)$ the monic poly. of least degree s.t.

$b_f(s)f^s = P(x,\partial,s)f^{s+1}$ in $\mathcal{O}_X[\frac{1}{f}][s]f^s$,

with $P(x,\partial,s) \in \mathcal{D}_X[s]$. Locally, it is the minimal polynomial of s on $\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}$.

Define $b_{f,x}(s)$ replacing $\mathcal{D}_X$ with $\mathcal{D}_{X,x}$.

Observation. $i_f : X \to \tilde{X} := X \times \mathbf{C}$ by graph.

$\tilde{M} := i_{f+}\mathcal{O}_X$ $(= \mathcal{D}_{\tilde{X}}\delta(f-t) = \mathcal{O}_{X\times\mathbf{C}}[\frac{1}{f-t}]/\mathcal{O}_{X\times\mathbf{C}})$,

This is a free $\mathcal{O}_X[\partial_t]$-module of rank 1 with generator $\delta(f-t) = \frac{1}{f-t}$ satisfying

$\partial_i\delta(f-t) = -(\partial_i f)\partial_t\delta(f-t), \quad t\delta(f-t) = f\delta(t-f)$.

Then $f^s \leftrightarrow \delta(f-t)$ with $s = -\partial_t t$, and

$\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f-t)$ (as $\mathcal{D}_X[s]$-modules).

The action of t on $\mathcal{D}_X[s]f^s$ is $\mathcal{D}_X$-linear and

$t : s \mapsto s+1$, i.e. $ts = (s+1)t$.

V-filtration.
Def. V is a filtration of Kashiwara-Malgrange if V is exhaustive, separated, and satisfies for any $\alpha \in \mathbf{Q}$:
(i) $V^\alpha\tilde{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\tilde{M}$.
(ii) $tV^\alpha\tilde{M} \subset V^{\alpha+1}\tilde{M}$ and $=$ holds for $\alpha \gg 0$.
(iii) $\partial_t V^\alpha\tilde{M} \subset V^{\alpha-1}\tilde{M}$.
(iv) $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha\tilde{M}$.
V is unique if it exists.
Thm (Kashiwara, Malgrange). There exists V on $\tilde{M}$ $= i_{f+}M$ for any holonomic M. (V indexed by $\mathbf{C}$).

Relation with the b-function. If X is X affine or Stein and relatively compact, the multiplicity of a root α of $b_f(-s)$ is given by

$\deg(\text{min. poly}(s+\alpha \,|\, \mathrm{Gr}_V^\alpha(\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}))$,

where $\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f-t)$ with $s = -\partial_t t$.

Here $V^\alpha\tilde{M}$, $\mathcal{D}_X[s]f^{s+i}$ are 'lattices' of $\tilde{M} = \mathcal{O}_X[\partial_t]$,
i.e. $V^\alpha\tilde{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^\beta\tilde{M} \quad (\alpha \gg i \gg \beta)$.
Moreover, existence of V $\Leftrightarrow$ existence of $b_f(s)$

Relation with vanishing cycle functors.
Let $\rho : X_t \to X_0$ be a 'good' retraction
(using a resolution of singularities of (X, X_0)),
where $X_t = f^{-1}(t)$ with $t \ne 0$ sufficiently near 0.
Def. (Nearby and vanishing cycles)

$\psi_f\mathbf{C}_X = \mathbf{R}\rho_*\mathbf{C}_{X_t}, \quad \varphi_f\mathbf{C}_X = \psi_f\mathbf{C}_X/\mathbf{C}_{X_0}$.

If F_x denote the Milnor fiber around $x \in X_0$, then $(\mathcal{H}^j\psi_f\mathbf{C}_X)_x = H^j(F_x,\mathbf{C})$, $(\mathcal{H}^j\varphi_f\mathbf{C}_X)_x = \tilde{H}^j(F_x,\mathbf{C})$.

Def. $\psi_f M = \bigoplus_{0<\alpha\le 1}\mathrm{Gr}_V^\alpha\tilde{M}$,

$\varphi_f M = \bigoplus_{0\le\alpha<1}\mathrm{Gr}_V^\alpha\tilde{M}$.

Thm (Kashiwara, Malgrange).
For a regular holonomic $\mathcal{D}_X$-module M, we have

$\mathrm{DR}_X\psi_f(M) = \psi_f\mathrm{DR}_{\tilde{X}}(M)[-1]$,

$\mathrm{DR}_X\varphi_f(M) = \varphi_f\mathrm{DR}_{\tilde{X}}(M)[-1]$,

and $\exp(-2\pi i\partial_t t)$ on the left-hand side corresponds to the monodromy T on the right-hand side.

Def. $R_f = \{\text{roots of } b_f(-s)\} \subset \mathbf{Q}_{>0}$,

$m_{f,\alpha}$ = multiplicity of $\alpha \in R_f$,

$\alpha_f = \min R_f > 0$ (by Kashiwara).

We have similarly $R_{f,x}$, $m_{f,x,\alpha}$, $\alpha_{f,x}$ for $b_{f,x}(s)$.
Thm (Kashiwara, Malgrange).
(i) $e^{-2\pi i R_f} = \{$eigenvalues of T on $H^j(F_x,\mathbf{C})$ for any $x \in X_0, j \in \mathbf{Z}\}$,
(ii) $m_{f,\alpha} \le \min\{i \mid N^i\psi_{f,\lambda}\mathbf{C}_X = 0\} \quad (\lambda = e^{-2\pi i\alpha})$.
Here $\psi_{f,\lambda} = \mathrm{Ker}(T_s - \lambda) \subset \psi_f$, $N = \log T_u$.

Microlocal b-function.
Define $\tilde{R}_f, \tilde{m}_{f,\alpha}, \tilde{\alpha}_f$ with $b_f(s)$ replaced by

$\tilde{b}_f(s) := b_f(s)/(s+1)$ (microlocal b-function).

This coincides with the monic polynomial of the least degree satisfying

$\tilde{b}_f(s)\delta(f-t) = \tilde{P}\partial_t^{-1}\delta(f-t)$ with $\tilde{P} \in \mathcal{D}_X[s,\partial_t^{-1}]$.

Rem. If f has isol. sing. then $\tilde{\alpha}_{f,x}$ coincides with the min. exp. (Malgrange, Varchenko, Scherk-Steenbrink).

Thm. $\tilde{R}_f \subset [\tilde{\alpha}_f, n-\tilde{\alpha}_f], \quad \tilde{m}_{f,\alpha} \le n - \tilde{\alpha}_f - \alpha + 1$,
where $n = \dim X$.
(The proof uses the filtered duality for φ_f.)
Prop. (Kashiwara). If f is weighted-homog. with weights $w_1,\dots,w_n$ and $\mathrm{Sing}\, X_0 = \{0\}$, then

$\tilde{R}_f = E_f$ $(:= \{\text{exponents}\})$, $\quad \tilde{\alpha}_f = \sum w_i$,

$\max\tilde{R}_f = n - \tilde{\alpha}_f, \quad \tilde{m}_{f,\alpha} = 1 \ (\alpha \in \tilde{R}_f)$.

Ex. If $f = \sum_i x_i^2$, then $\tilde{\alpha}_f = n/2$ and the assertion follows from the above theorem.

Spectrum of a hypersurface

Spectrum (Steenbrink 1976, 1987)
$f : X \to \mathbf{C}$ with X smooth.

F_x : Milnor fiber around $x \in X_0 = f^{-1}(0)$.
Def. $\mathrm{Sp}(f,x) = \mathrm{Sp}(X_0,x) = \sum_{\alpha>0} n_{f,\alpha} t^\alpha$ with
$n_{f,\alpha} = \sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(F_x,\mathbf{C})_\lambda$,
where $p = [n-\alpha], \lambda = e^{-2\pi i\alpha}$ $(n = \dim X)$.
Here F is the Hodge filtration on
$\widetilde{H}^j(F_x,\mathbf{C})_\lambda := \mathrm{Ker}(T_s - \lambda)$ with $T = T_s T_u$.
Def. $E_f = \{\alpha \mid n_{f,\alpha} \ne 0\} \subset \mathbf{Q}_{>0}$ (exponents).

Isolated singularity case.
Prop. (Symmetry, positivity).
$n_{f,\alpha} = n_{f,n-\alpha} \ge 0$ $\;(f$ isol. sing.)
Thm. (Scherk-Steenbrink, Varchenko).
$\mathrm{Sp}(f+g,(x,y)) = \mathrm{Sp}(f,x)\mathrm{Sp}(g,y) \in \mathbf{Q}[t^{1/e}]$
Thm (Steenbrink). If $\mathrm{Sing}\, X_0 = \{0\}$ and f is weighted homogeneous with positive weights $w_1,\dots,w_n$ (i.e. $f = \sum_\nu c_\nu x^\nu$ with $\sum_i w_i\nu_i = 1$), then
$\mathrm{Sp}(f,x) = \prod_i (t - t^{w_i})/(t^{w_i} - 1)$.

Nondegenerate Newton boundary case.
If $n = 2$ and f has nondegenerate Newton boundary ∂P_f s.t. $\mathbf{R}^2_{\ge 0} \setminus P_f$ is bounded, then by Steenbrink
$E_f \cap (0,1] = \bigcup_\sigma E_\sigma^{\le 1}$ (using symmetry)
with $E_\sigma^{\le 1} = \{L_\sigma(u) \mid u \in \mathbf{Z}^2_{>0} \cap (\{0\} \cup \sigma)^{\text{conv.hull}}\}$,
where L_σ is a linear function s.t. $L_\sigma^{-1}(1) \supset \sigma$.
Generalization. For $n > 2$, the filtration V on $\Omega_X^n/df \wedge \Omega_X^{n-1}$ is induced by the Newton filtration, and there is a combinatorial description by Steenbrink.

Multiplier ideals and an extension theorem

Multiplier ideals. Z subvariety of a smooth X.
$\mathcal{J}(X,\alpha Z)$: multiplier ideal ($\alpha \in \mathbf{Q}_{>0}$), where
$g \in \mathcal{J}(X,\alpha Z) \Leftrightarrow |g|^2/(\sum |f_i|^2)^\alpha$ locally integrable
with $f_1,\dots,f_r$ local generators of the ideal of Z, or
$\mathcal{J}(X,\alpha Z) = \rho_*\omega_{\tilde{X}/X}(-\sum_i [\alpha m_i]\tilde{D}_i))$
with $\rho : (\tilde{X},\tilde{D}) \to (X,Z)$ an embedded resolution s.t. $\rho^{-1}\mathcal{I}_Z$ generates the ideal $\mathcal{I}_{\tilde{D}}$ of $\tilde{D} = \sum_i m_i \tilde{D}_i$.
Def. $\alpha \in \mathrm{JN}(Z) := \{$Jumping numbers of $Z\} \Leftrightarrow$
$\mathcal{G}(X,\alpha Z) := \mathcal{J}(X,(\alpha-\varepsilon)Z)/\mathcal{J}(X,\alpha Z) \ne 0\,(0<\varepsilon\ll 1)$.

Extension of multiplier ideals
Assume $X = Y \times \mathbf{C}^r$, $D = f^{-1}(0)$ and $\lambda^* f = f$ with
$\lambda \cdot (y,z_1,\dots,z_r) = (y,\lambda^{w_1}z_1,\dots,\lambda^{w_1}z_r) \in Y \times \mathbf{C}^r$,
for $\lambda \in \mathbf{C}^*$, where $w_i > 0$.
For $y \in Y = Y \times \{0\} \subset X$, let
$G^{>\alpha}\mathcal{O}_{X,y} = \{g \in \mathcal{O}_{X,y} \mid v(g) > \alpha\}$ with
$v(\sum a_\nu z^\nu) = \min\{\sum_i w_i(\nu_i+1) \mid a_\nu \ne 0\}$. Then
Thm. $\mathcal{J}(X,\alpha D)_y = (j_*\mathcal{J}(X',\alpha D'))_y \cap G^{>\alpha}\mathcal{O}_{X,y}$,
where $j : X' := X \setminus (Y \times \{0\}) \to X$, $D' := D \cap X'$.

Cor. If D is the affine cone of a divisor of degree d on $\mathbf{P}^{n-1}$, then
$\mathcal{J}(X,\alpha D) = I_0^k$ with $k = [d\alpha] - n + 1$ for $\alpha < \alpha'_f$,
where I_0 is the ideal of 0, and $\alpha'_f := \min_{x\ne 0}\{\alpha_{f,x}\}$.
(For hyperplane arrangements this is due to Mustata.)
Cor. $\dim F^{n-1}H^{n-1}(F_0,\mathbf{C})_{\mathbf{e}(-k/d)} = \binom{k-1}{n-1}$,
for $0 < \frac{k}{d} < \alpha'_f$. (Same for P instead of F.)
Cor. $\alpha_f = \min(\alpha'_f, \frac{n}{d})$ (using $\alpha_f = \min \mathrm{JN}(D)$).

Relations in the hypersurface case

Assume $D = f^{-1}(0)$ hypersurface.
Thm [Budur]. If f has isol. sing., then for $\alpha \in (0,1)$
$n_{f,\alpha} = \dim \mathcal{G}(X,\alpha D)_x$ (so $\mathrm{JN}(D) \cap (0,1) = E_f \cap (0,1)$).
Def. V induced by $\mathcal{O}_X \subset \mathcal{O}_X[\partial_t]\delta(f-t)$. Then
Thm [Budur, S]. $\mathcal{J}(X,\alpha D) = V^\alpha\mathcal{O}_X$ if $\alpha \notin \mathrm{JN}(D)$.
$\mathcal{J}(X,\alpha D) = V^{\alpha+\varepsilon}\mathcal{O}_X$, $V^\alpha\mathcal{O}_X = \mathcal{J}(X,(\alpha-\varepsilon)D)$
for any $\alpha \in \mathbf{Q}_{>0}$ where $0 < \varepsilon \ll 1$.
Rem. V : left-continuous, i.e. $V^\alpha\mathcal{O}_X = V^{\alpha-\varepsilon}\mathcal{O}_X$,
$\mathcal{J}(X,\alpha D)$: right-cont., $\mathcal{J}(X,\alpha D) = \mathcal{J}(X,(\alpha+\varepsilon)D)$.

As a corollary we reprove results of Ein, Lazarsfeld, Smith and Varolin, and of Lichtin, Yano and Kollár.
Cor. (i) $\mathrm{JN}(D) \cap (0,1] \subset R_f$. (ii) $\alpha_f = \min \mathrm{JN}(D)$.
Set $\alpha'_{f,x} = \min_{y\ne x}\{\alpha_{f,y}\}$. Then
Thm. If $\xi f = f$ for a vector field ξ, then
$R_f \cap (0,\alpha'_{f,x}) = \mathrm{JN}(D) \cap (0,\alpha'_{f,x})$.
Rem. This does not hold without the assumption on ξ nor without restricting to $(0,\alpha'_{f,x})$.
Periodicity. $\mathrm{JN}(D) = (\mathrm{JN}(D) \cap (0,1]) + \mathbf{N}$,
since $f\mathcal{J}(X,\alpha D) = \mathcal{J}(X,(\alpha+1)D)$ $(\alpha > 0)$.

Malgrange's formula (isol. sing. case).
Def. Brieskorn lattice (where X is analytic)
$H''_f = \Omega^n_{X,x}/df \wedge d\Omega^{n-2}_{X,x}$ $\;(n = \dim X)$,
$\widetilde{H}''_f = \sum_{i\ge 0}(t\partial_t)^i H''_f$ (saturation).
These are $\mathbf{C}\{t\}$-modules with reg. sing. connection, and ∂_t^{-1} is well-defined on H''_f.
Thm (Malgrange).
$\tilde{b}_f(s) = \min \mathrm{poly}(-\partial_t t \mid \widetilde{H}''_f/t\widetilde{H}''_f)$.
Rem. Kashiwara's formula on b-function can be proved using this and Brieskorn's calculation.

Asymptotic Hodge structure (isol. sing. case).
Thm (Varchenko, Scherk-Steenbrink).
$F^pH^{n-1}(F_x,\mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha H''_f$,
under $H^{n-1}(F_x,\mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha \mathcal{G}_f$ with $\mathcal{G}_f := H''_f[\partial_t]$,
where $p = [n-\alpha], \lambda = e^{-2\pi i\alpha}$ $(n = \dim X)$.
Rem. This is generalized to the non-isol. sing. case (an origin of the theory of mixed Hodge modules).
Cor. $\widetilde{R}_f \subset \bigcup_{k=0}^{n-1}(E_f - k)$, $\tilde{\alpha}_f = \min \widetilde{R}_f = \min E_f$.
The proof uses the following

Reformulation of Malgrange formula (isol. sing.)
$\widetilde{F}^pH^{n-1}(F_x,\mathbf{C})_\lambda := \mathrm{Gr}_V^\alpha \widetilde{H}''_f$ under the isomorphism
$H^{n-1}(F_x,\mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha H''_f[\partial_t]$ $(p = [n-\alpha], \lambda = e^{-2\pi i\alpha})$.

Then
$\widetilde{m}_{f,\alpha} = \deg(\text{min poly}(N \,|\, \mathrm{Gr}^p_{\widetilde{F}} H^{n-1}(F_x, \mathbf{C})_\lambda))$.

Rem. $E_f \cap (0,1) = \mathrm{JN}(D) \cap (0,1)$ in the isol. sing. case (by Budur).

Rem. If f is weighted homog. with isol. sing, then
$\widetilde{F} = F$ and $\widetilde{R}_f = E_f$ (by Kashiwara).

Ex. If $f = x^5 + y^4 + x^3y^2$, then
$E_f = \{\frac{i}{5} + \frac{j}{4} : 1 \le i \le 4,\ 1 \le j \le 3\}$,
$\widetilde{R}_f = E_f \cup \{\frac{11}{20}\} \setminus \{\frac{31}{20}\}$.
This is the simplest examples s.t. $E_f \ne \widetilde{R}_f$.

Rem. Let $f = g + h \in \mathbf{C}[x]$ with g weighted homog. of weights $w_1, \dots, w_n$ and $h = x^\nu$ s.t. $\sum_i w_i\nu_i > 1$ and $h \notin (\partial g)$. Then $E_f \ne \widetilde{R}_f$.

Hyperplane arrangement case

Hyperplane arrangements.

$X = \mathbf{C}^n \supset D$ a central hyperplane arrangement
(Here, central means the affine cone of $Z \subset \mathbf{P}^{n-1}$).
$D = f^{-1}(0)$ with f reduced, $d = \deg f > n$.
Assume D is not the pull-back of $D' \subset \mathbf{C}^{n'}$ $(n' < n)$.

Thm. (i) $\max R_f < 2 - \frac{1}{d}$. (ii) $m_1 = n$.

Proof of (i) uses a generalization of Malgrange's formula and a partial generalization of a solution of Aomoto's conjecture (Esnault, Schechtman, Viehweg, Terao, Varchenko)

Generalization of Malgrange's formula.

Thm. $\exists\, P$ the pole order filtration on $H^{n-1}(F_0, \mathbf{C})_\lambda$ s.t. if $(\alpha + \mathbf{N}) \cap R'_f = \emptyset$ (where $R'_f = \cup_{x \ne 0} R_{f,x}$), then
$\alpha \in R_f \Leftrightarrow \mathrm{Gr}^p_P H^{n-1}(F_0, \mathbf{C})_\lambda \ne 0$,
with $p = [n - \alpha], \lambda = e^{-2\pi i\alpha}$ $(n = \dim X)$.

Rem. This reduces the proof of Thm. (i) to
$P^i H^{n-1}(F_0, \mathbf{C})_\lambda = H^{n-1}(F_0, \mathbf{C})_\lambda$
for $i = n - 1$ if $\lambda = 1, e^{2\pi i/d}$, and $i = n - 2$ otherwise.

Generic case. If D is a generic central hyperplane arrangement, then
$b_f(s) = (s + 1)^{n-1} \prod_{j=n}^{2d-2} (s + \frac{j}{d})$
by U. Walther (except for the multiplicity of -1).
He uses a completely different method.

Non-generic case. In case $n = 3$, $d \le 7$, and $\max\{\mathrm{mult}_x D \,|\, x \ne 0\} = 3$, we have
$b_f(s) = (s + 1) \prod_{i=2}^{4} (s + \frac{i}{3}) \prod_{j=3}^{r} (s + \frac{j}{d})$
where $r = 2d - 2$ or $2d - 3$.

b-function of a subvariety

b-functions.

X smooth $\supset Z$ closed subvariety.
$f = (f_1, \dots, f_r)$ generators of the ideal of Z.
Define the action of t_j on $\mathcal{O}_X[\frac{1}{f_1 \cdots f_r}][s_1, \dots, s_r] \prod_i f_i^{s_i}$ by $t_j(s_i) = s_i + 1$ if $i = j$, and $t_j(s_i) = s_i$ otherwise.
Put $s_{i,j} := s_i t_i^{-1} t_j$, $s = \sum_i s_i$.
Then $b_f(s)$ is the monic poly. of least degree s.t.
$b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_i f_i^{s_i}$,
where P_k belong to the ring generated by $\mathcal{D}_X$ and $s_{i,j}$.

Rem. We can replace $\prod_i f_i^{s_i}$ with $\prod_i \delta(t_i - f_i)$, using the direct image by the graph of $f : X \to \mathbf{C}^r$.
Then the existence of $b_f(s)$ follows from the theory of the V-filtration of Kashiwara and Malgrange.
This b-function has appeared in work of Sabbah and Gyoja for the study of b-functions of several variables.

Thm [Budur, Mustata, S]. Let $c = \mathrm{codim}_X Z$.
Then $b_f(s - c)$ depends only on Z and is independent of the choice of X, $f = (f_1, \dots, f_r)$ and also of r.

Equivalent definition.

$b_f(s)$ is the monic polynomial of least degree s.t.
$b_f(s) \prod_i f^{s_i} \in \sum_{|c|=1} \mathcal{D}_X[\mathbf{s}] \prod_{c_i<0} \binom{s_i}{-c_i} \prod_i f_i^{s_i + c_i}$,
where $c = (c_1, \dots, c_r) \in \mathbf{Z}^r$ with $|c| := \sum_i c_i = 1$.
Here $\mathcal{D}_X[\mathbf{s}] = \mathcal{D}_X[s_1, \cdots, s_r]$.
This is used in the monomial ideal case.

Remark. The well-definedness does not hold without the term $\prod_{c_i<0} \binom{s_i}{-c_i}$.

We have the induced filtration V by
$\mathcal{O}_X \subset i_{f+}\mathcal{O}_X = \mathcal{O}_X[\partial_1, \dots, \partial_r] \prod_i \delta(t_i - f_i)$.

Thm [Budur, Mustata, S].
$\mathcal{J}(X, \alpha Z) = V^\alpha \mathcal{O}_X$ if $\alpha \notin \mathrm{JN}(Z)$, and in general
$\mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon}\mathcal{O}_X$, $V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z)$.

Cor [BMS]. $\mathrm{JN}(Z) \cap [\alpha_f, \alpha_f + 1) \subset R_f$.

Thm [BMS]. If Z is reduced and CI, then
Z has rational sing. $\Leftrightarrow \alpha_f = r$ with multiplicity 1.

Spectrum of a subvariety

Spectrum.

X smooth $\supset Z$ closed subvariety with ideal $\mathcal{I}_Z$.

Def. $\mathrm{Sp}_Z \mathbf{Q}_X = \psi_t\, j_* \mathbf{Q}_{X \times \mathbf{C}^*}$ (Verdier specialization),
where $j : \mathcal{S}pec_X \mathcal{O}_X[t, t^{-1}] \to \mathcal{S}pec_X(\bigoplus_{i \in \mathbf{Z}} \mathcal{I}_Z^{-i} \otimes t^i)$
(deformation to the normal cone $N_Z X$).
Λ : irred. comp. of the fiber $(N_Z X)_z$ over $z \in Z$,
$\xi \in \Lambda$ sufficiently general with $i_\xi : \{\xi\} \to N_Z X$.

Def. $\widehat{\mathrm{Sp}}(Z, \Lambda) = \sum_{\alpha > 0} n_{\Lambda,\alpha} t^\alpha$, where
$n_{\Lambda,\alpha} = \sum_j (-1)^j \dim \mathrm{Gr}^p_F H^{j + c_\Lambda}(i_\xi^* \mathrm{Sp}_Z \mathbf{C}_X)_\lambda$,
with $p = [c_\Lambda + 1 - \alpha], \lambda = e^{-2\pi i\alpha}$ $(c_\Lambda = \dim X - \dim \Lambda)$

Def. $\mathrm{Sp}(Z, \Lambda) = \widehat{\mathrm{Sp}}(Z, \Lambda) - (-1)^{c_\Lambda} t^{c_\Lambda + 1}$.
If $(N_Z X)_x$ is irreducible (e.g. if Z is CI), set
$\widehat{\mathrm{Sp}}(Z, x) = \widehat{\mathrm{Sp}}(Z, \Lambda)$, etc. for $\Lambda = (N_Z X)_x$.
This generalizes the definition for hypersurfaces.

Rem. In general, we have
$n_{\Lambda,\alpha} = 0$ $(\alpha \le 0)$, $n_{\Lambda,\beta} \ge 0$ $(\beta \in (0, 1])$.
In the ICIS (isol. complete intersection sing.) case,

$\widetilde{n}_{x,\alpha} \ge 0$ with $\mathrm{Sp}(Z,x) = \sum_\alpha \widetilde{n}_{x,\alpha} t^\alpha$,
but symmetry and semicontinuity do not hold.

In the ICIS case, our definition coincides with the one by Ebeling and Steenbrink except for $n_{x,\alpha}$ with $\alpha \in \mathbf{Z}$. Indeed, they take generic 1-parameter smoothings

$f : X' \to \mathbf{C}$ of Z, $\quad g : X'' \to \mathbf{C}$ of X',

and consider $\varphi_f \psi_g \mathbf{Q}_{X''}[n]$ (where $n = \dim Z$) with

$0 \to \widetilde{H}^n(F_f, \mathbf{C}) \to \varphi_f \psi_g \mathbf{Q}_{X''}[n] \to H^{n+1}(F_g, \mathbf{C}) \to 0$,

since $\psi_g \mathbf{Q}_{X''}|_{X' \setminus \{0\}} = \mathbf{Q}$, $(\psi_g \mathbf{Q}_{X''})_0 = \mathbf{R}\Gamma(F_g, \mathbf{C})$.
The action of the monodromy on $H^{n+1}(F_g, \mathbf{C})$ is associated to the functor φ_f, and is the identity.

$\mathcal{I}_Z$: ideal sheaf of $Z \subset X$. For $z \in Z$ and $\beta \in (0,1] \cap \mathbf{Q}$

$\bar{\mathcal{A}} := \bigoplus_{j \ge 0} \mathcal{I}_Z^j / \mathcal{I}_Z^{j+1}$, $\quad \bar{\mathcal{A}}(z) := \bar{\mathcal{A}} / \mathbf{m}_{Z,z} \bar{\mathcal{A}}$,

$\mathcal{M}(\beta) := \bigoplus_{i \ge 0} \mathcal{G}(X, (\beta + i)Z)$,

$\mathcal{M}(\beta, z) := \mathcal{M}(\beta) / \mathbf{m}_{Z,z} \mathcal{M}(\beta)$,

with $(\mathcal{I}_Z^j / \mathcal{I}_Z^{j+1})\, \mathcal{G}(X, (\beta+i)Z) \subset \mathcal{G}(X, (\beta+i+j)Z)$.
For $z \in Z$ and Λ irred. comp. of $(N_Z X)_z = \mathrm{Spec}\, \bar{\mathcal{A}}(z)$,

$\mu_{\Lambda,\beta} := \dim_{\mathbf{C}(\Lambda)} \mathcal{M}(\beta, z) \otimes_{\bar{\mathcal{A}}(z)} \mathbf{C}(\Lambda)$,

where $\mathbf{C}(\Lambda)$ is the function field of Λ.

Thm [Dimca, Maisonobe, S]. Let $\beta \in (0,1] \cap \mathbf{Q}$.
(i) $0 \le n_{\Lambda,\beta} \le \mu_{\Lambda,\beta}$ (so $n_{\Lambda,\beta} = 0$ if $z \notin \mathrm{supp}\, \mathcal{M}(\beta)$).
(ii) $n_{\Lambda,\beta} = \mu_{\Lambda,\beta}$ if $\mathrm{supp}_{\bar{\mathcal{A}}} \mathcal{M}(\beta) \subset (N_Z X)_z$ on a nbd of the generic point of Λ.
(For hypersurfaces, this is due to Budur.)

Cor [DMS]. If $n_{\Lambda,\alpha} \ne 0$ with $\alpha \in (0,1)$, then $\exists j_0 \in \mathbf{N}$ s.t. $\alpha + j \in \mathrm{JN}(Z)$ for any integer $j \ge j_0$.

Thm [DMS]. If T is a transversal slice to a stratum of a good Whitney stratification and $r = \mathrm{codim}\, T$,

$\widehat{\mathrm{Sp}}(Z, \Lambda) = (-t)^r\, \widehat{\mathrm{Sp}}(Z \cap T, \Lambda)$.

Monomial ideal case

Monomial ideals.
$\mathbf{a} \subset \mathbf{C}[x] := \mathbf{C}[x_1, \dots, x_n]$ a monomial ideal. Let

$\Gamma_\mathbf{a} = \{u \in \mathbf{N}^n \mid x^u \in \mathbf{a}\}$, $\quad \mathbf{1} = (1, \dots, 1)$,

$P_\mathbf{a}$: the convex hull of $\Gamma_\mathbf{a} \subset \mathbf{R}^n_{\ge 0}$,

$U(\alpha) := \{\nu \in \mathbf{N}^n \mid \nu + \mathbf{1} \in (\alpha + \varepsilon) P_\mathbf{a}\ (0 < \varepsilon \ll 1)\}$.

Thm (Multiplier ideals) [Howald].

$J(X, \alpha Z) = \sum_{\nu \in U(\alpha)} \mathbf{C}\, x^\nu$.

Cor. $\mathrm{JN}(Z) = \{\phi(\nu) \mid \nu \in \mathbf{Z}^n_{>0}\}$.
Here $\phi(\nu) = L_\sigma(\nu)$ if $\nu \in \mathrm{Cone}(0, \sigma) := \bigcup_{\lambda \ge 0} \lambda\sigma$.

For a maximal face σ of $P_\mathbf{a}$, set

L_σ : linear function s.t. $L_\sigma|_\sigma = 1$,

c_σ : smallest positive integer s.t. $c_\sigma L_\sigma \in \mathbf{Z}[x]$.

$e_\sigma = |G'_\sigma / G_\sigma|$ with $G'_\sigma = \mathbf{Z}^n \cap L_\sigma^{-1}(0)$ and G_σ generated by $\nu - \nu'$ for $\nu, \nu' \in \Gamma_\mathbf{a} \cap \sigma$.

Thm (Spectrum) [DMS]. There is a one-to-one correspondence between the max. compact faces σ of $P_\mathbf{a}$ and the irred. comp. Λ of $(N_Z X)_0$, and

$\widehat{\mathrm{Sp}}(Z, \Lambda) = \sum_{i=1}^{c_\sigma} e_\sigma t^{i/c_\sigma}$.

For a face σ of $P_\mathbf{a}$, set

V_σ : the linear subspace generated by σ,

M_σ : the subsemigroup generated by $u - v$ with $u \in \Gamma_\mathbf{a}$, $v \in \Gamma_\mathbf{a} \cap \sigma$,

$M'_\sigma = v_0 + M_\sigma$ with $v_0 \in \Gamma_\mathbf{a} \cap \sigma$ (indep. of v_0),

$R_\sigma = \{L_\sigma(u) \mid u \in ((M_\sigma \setminus M'_\sigma) + \mathbf{1}) \cap V_\sigma\}$,

$R_\mathbf{a} = \{\text{roots of } b_\mathbf{a}(-s)\}$.

Thm (b-function) [BMS]. $R_\mathbf{a} = \bigcup_\sigma R_\sigma$ with σ not contained in any coordinate hyperplanes.

Rem. R_σ may depend on the other σ'.
(i) If $\mathbf{a} = (xy^5, x^3y^2, x^5y)$, then $R_\mathbf{a} = R_\sigma \cup R_{\sigma'}$,

$R_\sigma = \{\frac{5}{13}, \frac{i}{13}\ (7 \le i \le 17), \frac{19}{13}\}$, $\quad R_{\sigma'} = \{\frac{j}{7}\ (3 \le j \le 9)\}$.

So $R_\sigma = \{\frac{3i+2j}{13}\ (1 \le i \le 3, 1 \le j \le 5)\}$ with $L_\sigma(i,j) = \frac{3i+2j}{13}$.
(ii) If $\mathbf{a} = (xy^5, x^3y^2, x^4y)$, then

$R_\sigma = \{\frac{i}{13}\ (5 \le i \le 17)\}$, $\quad R_{\sigma'} = \{\frac{j}{5}\ (2 \le j \le 6)\}$.

So $R_\sigma \ne \{\frac{3i+2j}{13}\ (1 \le i \le 3, 1 \le j \le 5)\}$ with $\frac{19}{13}$ shifted to $\frac{6}{13}$.

Comparison. $D := f^{-1}(0)$ for $f = \sum c_\nu x^\nu \in \mathbf{C}[x]$ with non-degenerate Newton boundary $\partial P_f = \partial P_\mathbf{a}$.
Assume $Z_{\mathrm{red}} = \{0\}$ s.t. $\mathrm{Sing}\, D = \{0\}$. Then

$$\begin{array}{ccc} \mathrm{JN}(D) \cap (0,1) & \overset{(1)}{=} & \mathrm{JN}(Z) \cap (0,1) \\ {\scriptstyle (2)}\| & & \cap\, {\scriptstyle (3)} \\ E_f \cap (0,1) & \overset{(4)}{\subset} & \bigcup_\Lambda E_{Z,\Lambda} \cap (0,1) \end{array}$$

where $E_{Z,\Lambda} = \{\alpha \mid n_{\Lambda,\alpha} \ne 0\}$.
(1) Howald, (2) Budur, (1)+(2) Howald, Steenbrink.
(3)(4) DMS. (In general (3)(4) are not equality.)

Ex. If $\mathbf{a} = (x_1^{m_1}, \dots, x_n^{m_n})$, then

$\mathrm{JN}(Z) = \{\sum_i \frac{a_i}{m_i} \mid a_i \in \mathbf{Z}_{>0}\}$,

$\widehat{\mathrm{Sp}}(Z, 0) = \sum_{i=1}^{c_\sigma} e_\sigma\, t^{i/c_\sigma}$,

$b_\mathbf{a}(s) = \big[\prod_{(a_1,\dots,a_n) \in E} \big(s + \sum_i \frac{a_i}{m_i}\big)\big]_{\mathrm{red}}$,

where

$c_\sigma = \mathrm{LCM}(m_1, \dots, m_n)$,

$e_\sigma = m_1 \cdots m_n / c_\sigma$,

$E = \{(a_1, \dots, a_n) \in \mathbf{N}^n \mid a_i \in [1, m_i]\}$.

Ex. If $f = \sum_i x_i^{m_i}$ and $D = f^{-1}(0)$, then

$\mathrm{JN}(D) \cap (0,1] = \{\sum_i \frac{a_i}{m_i} \mid a_i \in \mathbf{Z}_{>0}\} \cap (0,1]$,

with $\mathrm{JN}(D) = (\mathrm{JN}(D) \cap (0,1]) + \mathbf{N}$,

$\mathrm{Sp}(D, 0) = \prod_i (t - t^{1/m_i}) / (t^{1/m_i} - 1)$,

$\widetilde{b}_f(s) = \big[\prod_{(a_1,\dots,a_n) \in \widetilde{E}} \big(s + \sum_i \frac{a_i}{m_i}\big)\big]_{\mathrm{red}}$,

where

$\widetilde{E} = \{(a_1, \dots, a_n) \in \mathbf{N}^n \mid a_i \in [1, m_i - 1]\}$.