Recent Developments on Universal Forms

Myung-Hwan Kim
Dept. of Mathematical Sciences, Seoul National Univ.
Seoul 151-747, Korea, E-mail: mhkim@math.snu.ac.kr

Abstract. Lagrange's *four square theorem* states: the integral quadratic form $x^2+y^2+z^2+u^2$ represents all positive integers. This celebrated theorem had been generalized in many different directions such as Waring's problem and Pythagoras numbers, to name a few. One interesting generalization was made by Ramanujan in the early 20th century, who found and listed all 55 inequivalent positive definite integral quaternary diagonal quadratic forms that represent all positive integers. Dickson called such forms *universal* and confirmed Ramanujan's list. Later, Willerding added 124 quaternary non-diagonal universal forms to the list. Then in late 90's, Conway and Schneeberger announced so called *the fifteen theorem*, which characterizes the universality by the representability of a finite set of positive integers, the largest of which is 15. Using this criterion, they corrected several mistakes in the Willerding's list and announced the new and complete list of the 204 quaternary universal forms, up to equivalence. Then came Bhargava's generalization: for any infinite set S of positive integers there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all elements of S. As a byproduct, he found S_0 for some interesting sets S, for example, the set of all primes, the set of all positive odd integers, and so on. Conway-Schneeberger's and Bhargava's results shed new light in the global theory of representations of quadratic forms.

In this expository talk, these results and related works are introduced and generalized in two directions - universal forms over algebraic integers and universal forms representing higher rank forms - with some interesting applications.