Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 61–88

EXTREMAL LENGTH OF VECTOR MEASURES

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Abstract. We define the extremal length of vector measures. We show the reciprocal relation between extremal distance and extremal width of vector measures.

1. Introduction

The notion of extremal length and modulus of curve families has been studied extensively and gives a lot of applications to complex analysis and potential theory. In particular, the coincidence between modulus and p-capacity plays an important role. On the other hand, the interest on degenerate elliptic equations is increasing in these years (see e.g. [6]). As far as we know, however, there has been no such result for a modulus and a capacity associated with degenerate elliptic equations. The main aim of this paper is to define a modulus of vector measures and to establish a coincidence as before for a certain degenerate elliptic operator.

Throughout this paper we work with the Euclidean space $\mathbf{R}^n$ with $n \geq 2$. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Let Ω be a domain in $\mathbf{R}^n$ and let $\mathscr{A} = \mathscr{A}(x)$ be a positive definite symmetric $n \times n$-matrix with measurable components a_{ij} such that

$$c_0^{-2} w(x)^{2/p} |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq c_0^2 w(x)^{2/p} |\xi|^2$$

for any vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ and $x \in \Omega$, where $c_0 \geq 1$ is a constant and $0 < w < \infty$ is a weight which indicates the degeneracy of $\mathscr{A}$. It is natural to define the capacity by

$$\inf_{u \in \mathscr{D}} \int_\Omega \biggl(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \biggr)^{p/2} dx,$$

where $\mathscr{D}$ is a certain class of smooth functions, or more generally precise functions. If $\{a_{ij}\}$ is the identity matrix, then the above capacity coincides with the classical

1991 Mathematics Subject Classification: Primary 31B15.

The first author was supported in part by Japanese Ministry of Education, Science and Culture Grant-in-Aid for Scientific Research (B) (No. 09440062).

p-capacity. We shall define a generalized modulus closely related to the above capacity. In particular, we shall generalize the classical extremal distance and width so that the reciprocal relationship between them remains in force.

First we recall the definition of the modulus of a system of measures developed by Fuglede [4, Chapter I]. Let f be a nonnegative Borel measurable function and let μ be a nonnegative Borel measure. If $\int f\,d\mu \geq 1$, then we write $f \wedge \mu$. Let $\mathscr{E}$ be a system of nonnegative Borel measures. If $f \wedge \mu$ for all $\mu \in \mathscr{E}$, then we write $f \wedge \mathscr{E}$. We define the modulus $M_p(\mathscr{E})$ of $\mathscr{E}$ by

$$M_p(\mathscr{E}) = \inf\left\{ \int f^p\,dx : f \geq 0,\ f \wedge \mathscr{E} \right\},$$

where the infimum is understood to be infinity if there is no feasible f.

We generalize the above notions in connection with weights in the Muckenhoupt A_p class. Hereafter let w be a weight in the Muckenhoupt A_p class and let $\|f\|_{p,w} = \left(\int f^p w\,dx\right)^{1/p}$. Let $L^p_w(\Omega)$ be the space of all functions f on Ω such that $\|f\|_{p,w} < \infty$. The A_p-condition will be required for many properties analogous to those for the unweighted case, e.g. approximation of $f \in L^p_w(\Omega)$ by a smooth function (see Propositions 3–5 in Section 2). See also Kilpeläinen [7]. We define the weighted modulus $M_{p,w}(\mathscr{E})$ by

$$M_{p,w}(\mathscr{E}) = \inf\left\{ \int f^p w\,dx : f \geq 0,\ f \wedge \mathscr{E} \right\},$$

where the infimum is understood to be infinity if there is no feasible f. Observe that if $w = w'$ a.e., then $M_{p,w}(\mathscr{E}) = M_{p,w'}(\mathscr{E})$. Hereafter, we assume that w is defined everywhere and $0 < w < \infty$. Let $w^{-1/p}\mathscr{E} = \{w^{-1/p}\mu : \mu \in \mathscr{E}\}$. Then, by definition, $M_{p,w}(\mathscr{E}) = M_p(w^{-1/p}\mathscr{E})$ and $M_p(\mathscr{E}) = M_{p,w}(w^{1/p}\mathscr{E})$. In particular, $M_{p,w}(\mathscr{E}) = 0$ if and only if $M_p(w^{-1/p}\mathscr{E}) = 0$. Note that if $w = w'$ a.e., then the systems $w^{-1/p}\mathscr{E}$ and ${w'}^{-1/p}\mathscr{E}$ may differ, though $M_p(w^{-1/p}\mathscr{E}) = M_{p,w}(\mathscr{E}) = M_{p,w'}(\mathscr{E}) = M_p({w'}^{-1/p}\mathscr{E})$. It is essential that w is defined everywhere and $0 < w < \infty$. (If w could be ∞ on some set, then $w^{-1/p}\mathscr{E}$ might include a zero measure and $M_p(w^{-1/p}\mathscr{E}) = \infty$.) If $M_{p,w}(\mathscr{E}_0) = 0$, then we say that $\mathscr{E}_0$ is (p,w)-exceptional (abbreviated to (p,w)-exc.). If a statement concerning nonnegative Borel measures fails to hold only for (p,w)-exc. set of measures in $\mathscr{E}_0$, then we say that it holds (p,w)-a.e.

Fuglede [4, Chapter I] proved several fundamental properties of $M_{p,w}$. By $|E|$ we denote the Lebesgue measure of E.

(i) $M_{p,w}(\mathscr{E}) \leq M_{p,w}(\mathscr{E}')$ if $\mathscr{E} \subset \mathscr{E}'$.
(ii) $M_{p,w}(\mathscr{E}) \leq \sum M_{p,w}(\mathscr{E}_j)$ if $\mathscr{E} = \bigcup \mathscr{E}_j$.
(iii) If $\overline{\mu}$ is the completion of μ and $|E| = 0$, then $\overline{\mu}(E) = 0$ for (p,w)-a.e. μ.
(iv) If $f \in L^p_w(\mathbf{R}^n)$, then f is $\overline{\mu}$-integrable for (p,w)-a.e. μ.

(v) If $\|f_i - f\|_{p,w} \to 0$, there exists a subsequence f_{i_j} such that $\int |f_{i_j} - f|\,d\overline{\mu} \to 0$ for (p,w)-a.e. μ.

(vi) $M_{p,w}(\mathscr{E}) = 0$ if and only if there is $f \geq 0$ with $\|f\|_{p,w} < \infty$ such that $\int f\,d\mu = \infty$ for all $\mu \in \mathscr{E}$.

(vii) There is $f \geq 0$ such that $f \wedge \mu$ for (p,w)-a.e. $\mu \in \mathscr{E}$ and $M_{p,w}(\mathscr{E}) = \|f\|_{p,w}^p$.

In view of the property (vii) we have

$$M_{p,w}(\mathscr{E}) = \inf\left\{\int f^p w\,dx : f \geq 0,\ f \wedge \mathscr{E}\ (p,w)\text{-a.e.}\right\}.$$

Now we define the modulus of a system of vector measures. Let ν be a vector measure whose components ν_i are signed measures. The total variation $|\nu|$ of ν is defined by

$$|\nu|(E) = \sup \sum_j \left(\sum_{i=1}^n \nu_i(E_j)^2\right)^{1/2} \qquad \text{for Borel sets } E,$$

where the supremum is taken over all finite partitions $\{E_j\}$ of E into Borel sets. The total variation $|\nu|$ is a nonnegative measure. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a vector valued function. If $\int |\xi_i| d|\nu_i| < \infty$ for $i = 1, \ldots, n$, then we define $\int \xi \cdot d\nu = \sum_{i=1}^n \int \xi_i\,d\nu_i$. It is known that $|\int \xi \cdot d\nu| \leq \int |\xi|\,d|\nu|$. See [9, Chapter 13] for details. We give the notion of exceptional sets of vector measures.

Definition. Let $\mathscr{F}_0$ be a set of vector measures ν. We put $|\mathscr{F}_0| = \{|\nu| : \nu \in \mathscr{F}_0\}$. If $M_{p,w}(|\mathscr{F}_0|) = 0$, then we say that $\mathscr{F}_0$ is (p,w)-exceptional (abbreviated to (p,w)-exc.). If a statement concerning vector measures ν fails to hold only for (p,w)-exc. system $\mathscr{F}_0$, then we say that it holds (p,w)-a.e.

Let $\mathscr{A} = \mathscr{A}(x)$ be a positive definite symmetric $n \times n$-matrix with measurable components a_{ij} such that

$$c_0^{-2} w(x)^{2/p} |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq c_0^2 w(x)^{2/p} |\xi|^2$$

for any vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ and $x \in \Omega$, where $c_0 \geq 1$ is a constant. Let $\mathscr{B} = \mathscr{B}(x) = \{b_{ij}\}$ be the inverse matrix of $\mathscr{A}$. It is easy to see that $\mathscr{B}$ satisfies

$$c_0^{-2} w(x)^{-2/p} |\xi|^2 \leq \sum_{i,j} b_{ij}(x) \xi_i \xi_j \leq c_0^2 w(x)^{-2/p} |\xi|^2.$$

For simplicity we let $\mathscr{A}[\xi] = ({}^t\xi \mathscr{A} \xi)^{1/2} = \left(\sum_{i,j} a_{ij}(x) \xi_i \xi_j\right)^{1/2}$ for $\xi \in \mathbf{R}^n$; and if $\xi = \xi(x)$ is a vector valued measurable function on Ω, then we let $\mathscr{A}_p(\xi) = \left(\int_\Omega \mathscr{A}[\xi]^p\,dx\right)^{1/p}$. Similarly, $\mathscr{B}[\xi]$ and $\mathscr{B}_p(\xi)$ are defined. Observe that $\mathscr{A}$

and $\mathscr{B}$ can be written as $\mathscr{A} = \sqrt{\mathscr{A}}^{\,2}$ and $\mathscr{B} = \sqrt{\mathscr{B}}^{\,2}$ with positive symmetric matrices $\sqrt{\mathscr{A}}$ and $\sqrt{\mathscr{B}}$. Then $\mathscr{A}[\xi] = |\sqrt{\mathscr{A}}\,\xi|$, $\mathscr{B}[\xi] = |\sqrt{\mathscr{B}}\,\xi|$, $\mathscr{A}_p(\xi) = \bigl(\int_\Omega |\sqrt{\mathscr{A}}\,\xi|^p\,dx\bigr)^{1/p}$ and $\mathscr{B}_p(\xi) = \bigl(\int_\Omega |\sqrt{\mathscr{B}}\,\xi|^p\,dx\bigr)^{1/p}$. We remark that

$$
\begin{aligned}
c_0^{-1} w(x)^{1/p}|\xi| &\le |\sqrt{\mathscr{A}}\,\xi| \le c_0 w(x)^{1/p}|\xi|, \\
c_0^{-1} w(x)^{-1/p}|\xi| &\le |\sqrt{\mathscr{B}}\,\xi| \le c_0 w(x)^{-1/p}|\xi|.
\end{aligned}
\tag{1.1}
$$

In the same way as in the remark on $M_p(w^{-1/p}\mathscr{E})$, we emphasize that the matrices $\mathscr{A}$ and $\mathscr{B}$ are defined everywhere and satisfy the above inequalities. If $\mathscr{A}$ is not defined for some set of measure 0, then $\mathscr{A}$ should be defined to be the identity matrix on that set.

Definition. Let $\xi = (\xi_1, \dots, \xi_n)$ be a vector valued function and let $\nu = (\nu_1, \dots, \nu_n)$ be a vector measure defined on Ω. We write $\xi \wedge \nu$ if $\int \xi \cdot d\nu \ge 1$. Let $\mathscr{F}$ be a set of complete vector measures. We write $\xi \wedge \mathscr{F}$ (p,w)-a.e. if $\xi \wedge \nu$ for (p,w)-a.e. $\nu \in \mathscr{F}$. We define

$$
\begin{aligned}
M_{\mathscr{A},p}(\mathscr{F}) &= \inf\{\mathscr{A}_p(\xi)^p : \xi \wedge \mathscr{F}\ (p,w)\text{-a.e.}\}, \\
M_{\mathscr{B},p'}(\mathscr{F}) &= \inf\{\mathscr{B}_{p'}(\xi)^{p'} :\ \xi \wedge \mathscr{F}\ (p', w^{1-p'})\text{-a.e.}\},
\end{aligned}
$$

where the infima are understood to be infinity if there is no feasible ξ.

Remark. In general we cannot remove '(p,w)-a.e.' and '$(p', w^{1-p'})$-a.e.' in the above definition. In fact, let us consider the line segment L between $(0,\dots,0)$ and $(1,0,\dots,0)$. Let $\mathscr{F} = \{dx_1|_L, -dx_1|_L\}$. Then there is no vector valued function ξ such that $\xi \wedge \mathscr{F}$. On the other hand, $|\mathscr{F}| = \{ds|_L\}$ and it is easy to see that $M_{p,w}(|\mathscr{F}|) = M_{\mathscr{A},p}(\mathscr{F}) = 0$. For a special $\mathscr{F}$ we can remove '(p,w)-a.e.' and '$(p', w^{1-p'})$-a.e.' See (1.2) below.

The monotonicity of $M_{\mathscr{A},p}$ and $M_{\mathscr{B},p'}$ is clear from the definition. However, some of the known properties for $M_{p,w}$ do not directly extend to $M_{\mathscr{A},p}$ and $M_{\mathscr{B},p'}$. For example, the classical proof of the countable subadditivity does not apply to $M_{\mathscr{A},p}$ and $M_{\mathscr{B},p'}$. The following proposition can be proved in a way common to the classical case and the present case. For completeness the proof will be given in Section 2.

Proposition 1. *Let $\mathscr{F}_j$ be an increasing sequence of vector measures such that $\mathscr{F} = \bigcup_j \mathscr{F}_j$. Then*

$$M_{\mathscr{A},p}(\mathscr{F}_j) \uparrow M_{\mathscr{A},p}(\mathscr{F}).$$

Definition. Let K_0 and K_1 be disjoint compact subsets of $\mathbf{R}^n$ such that $K_0 \cap \overline{\Omega} \ne \emptyset$ and $K_1 \cap \overline{\Omega} \ne \emptyset$. We let

$$
\begin{aligned}
\Gamma &= \Gamma(K_0, K_1, \Omega) \\
&= \{\gamma : \text{curves in } \Omega \text{ starting from } K_0 \cap \overline{\Omega} \text{ and terminating at } K_1 \cap \overline{\Omega}\}.
\end{aligned}
$$

We know that the family of all non-rectifiable curves is (p,w)-exc. Thus for (p,w)-a.e. $\gamma\in\Gamma$ we have naturally a vector measure $dx|_\gamma=(dx_1,\dots,dx_n)|_\gamma$ and a measure $ds|_\gamma=|dx|\,|_\gamma$. We write $d\Gamma=\{dx|_\gamma:\gamma\in\Gamma\}$ and $|d\Gamma|=\{ds|_\gamma:\gamma\in\Gamma\}$. More generally, for a positive definite symmetric matrix valued measurable function $Q=\{q_{ij}\}$ we let $|Qdx|\,|_\gamma=\sqrt{\sum_{i,j}q_{ij}\,dx_i\,dx_j}\,\Big|_\gamma$ and $|Qd\Gamma|=\{|Qdx|\,|_\gamma:\gamma\in\Gamma\}$.

Hereafter, K_0 and K_1 will denote disjoint compact sets such that $K_0\cap\overline{\Omega}\neq\emptyset$ and $K_1\cap\overline{\Omega}\neq\emptyset$, unless otherwise specified. We can consider moduli $M_{p,w}(|d\Gamma|)$, $M_{\mathscr{A},p}(d\Gamma)$ and $M_p(|\sqrt{\mathscr{B}}\,d\Gamma|)$. The reciprocals of these quantities are called generalized extremal distances. By definition we may assume that $K_0,K_1\subset\overline{\Omega}$ by taking the intersection with $\overline{\Omega}$.

We introduce a generalized capacity of condenser. A real function u on Ω is called a (p,w)-precise function if u is absolutely continuous on a (p,w)-a.e. curve in Ω and $\nabla u\in L^p_w(\Omega)$. (From the first requirement ∇u exists a.e. in Ω. For details see Proposition 2 in Section 2.) Let $\mathscr{D}(K_0,K_1,\Omega)$ be the family of all (p,w)-precise functions u on Ω such that $u(x)$ tends to 0 as $x\to K_0\cap\overline{\Omega}$ along (p,w)-a.e. curve in Ω and that $u(x)$ tends to 1 as $x\to K_1\cap\overline{\Omega}$ along (p,w)-a.e. curve in Ω. We define the capacity

$$C_{\mathscr{A},p}(K_0,K_1,\Omega)=\inf_{u\in\mathscr{D}(K_0,K_1,\Omega)}\mathscr{A}_p(\nabla u)^p.$$

Let $\mathscr{D}^*=\mathscr{D}^*(K_0,K_1,\Omega)$ be the family of all (p,w)-precise functions u on Ω such that $u=0$ on the intersection of Ω and a neighborhood of K_0 and $u=1$ on the intersection of Ω and a neighborhood of K_1. Similarly, we define the capacity

$$C^*_{\mathscr{A},p}(K_0,K_1,\Omega)=\inf_{u\in\mathscr{D}^*(K_0,K_1,\Omega)}\mathscr{A}_p(\nabla u)^p.$$

By definition $\mathscr{D}(K_0,K_1,\Omega)\supset\mathscr{D}^*(K_0,K_1,\Omega)\neq\emptyset$ and so

$$C_{\mathscr{A},p}(K_0,K_1,\Omega)\le C^*_{\mathscr{A},p}(K_0,K_1,\Omega)<\infty.$$

We shall show that the capacity $C_{\mathscr{A},p}(K_0,K_1,\Omega)$ agrees with the modulus $M_{\mathscr{A},p}$.

Theorem 1.

$$C_{\mathscr{A},p}(K_0,K_1,\Omega)=M_{\mathscr{A},p}(d\Gamma)=M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big)<\infty.$$

Remark. Let $\mathscr{A}'$ be another matrix function such that $\mathscr{A}'=\mathscr{A}$ a.e. and let $\mathscr{B}'$ be the inverse matrix of it. Then $C_{\mathscr{A},p}(K_0,K_1,\Omega)=C_{\mathscr{A}',p}(K_0,K_1,\Omega)$ and $M_{\mathscr{A},p}(d\Gamma)=M_{\mathscr{A}',p}(d\Gamma)$ by definition. In general, the systems $\big|\sqrt{\mathscr{B}}\,d\Gamma\big|$ and $\big|\sqrt{\mathscr{B}'}\,d\Gamma\big|$ may differ. However, the above theorem says that their moduli always coincide. A similar remark will apply to Theorem 3 below.

We shall show that the capacity $C^*_{\mathscr{A},p}(K_0, K_1, \Omega)$ is related to the modulus $M_{\mathscr{B},p'}$. Let

$$\nabla\mathscr{D}^* = \nabla\mathscr{D}^*(K_0, K_1, \Omega) = \{\nabla u : u \in \mathscr{D}^*(K_0, K_1, \Omega)\}$$

and consider $M_{\mathscr{B},p'}(\nabla\mathscr{D}^*)$. We observe that '$(p', w^{1-p'})$-a.e.' in the definition of $M_{\mathscr{B},p'}$ can be removed for $\mathscr{F} = \nabla\mathscr{D}^*$. In view of Hölder's inequality,

$$\int_\Omega f|\nabla u|\,dx \le \|\nabla u\|_{p,w}\,\|f\|_{p',w^{1-p'}} < \infty$$

for $u \in \mathscr{D}^*$ and $f \in L^{p'}_{w^{1-p'}}$. Hence the property (vi) of $M_{p,w}$ implies that only the empty set in $\nabla\mathscr{D}^*$ is $(p', w^{1-p'})$-exc. Therefore

$$M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) = \inf\{\mathscr{B}_{p'}(\xi)^{p'} : \xi \wedge \nabla\mathscr{D}^*\}. \tag{1.2}$$

With the aid of (1.2) and the minimax theorem of Ky Fan [3] (see also [1, Theorem 2.4.1]), we shall prove the following theorem.

Theorem 2. *If* $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) = 0$, *then we have* $M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) = \infty$; *if* $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) > 0$, *then*

$$C^*_{\mathscr{A},p}(K_0, K_1, \Omega)^{1/p} M_{\mathscr{B},p'}(\nabla\mathscr{D}^*)^{1/p'} = 1.$$

In particular, $0 < M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) \le \infty$ *in any case.*

Let us consider families of sets separating K_0 and K_1. More generally, we consider functions separating K_0 and K_1. For this purpose we let $BV(\Omega)$ be the space of functions u on Ω of bounded variation. Here, a function u is said to be of bounded variation if u is locally integrable on Ω and its distributional derivative ∇u is a finite vector measure on Ω (see [8, Chapter 6], [5, Chapter 1], [12, Chapter 5] and [2, Chapter 5]). The total variation of ∇u is denoted by $\|u\|_{BV}$ and is calculated by

$$\|u\|_{BV} = \sup\left\{\int_\Omega u \operatorname{div}\xi\,dx : \xi \in C_0^\infty(\Omega; \mathbf{R}^n),\ |\xi| \le 1\right\}.$$

The perimeter of a set E relative to Ω is defined by $P_\Omega(E) = \|\chi_{E\cap\Omega}\|_{BV}$ if $\chi_{E\cap\Omega} \in BV(\Omega)$ and $P_\Omega(E) = \infty$ if $\chi_{E\cap\Omega} \notin BV(\Omega)$.

Definition. Let Ω be bounded. We let $S = S(K_0, K_1, \Omega)$ be the family of functions $u \in BV(\Omega)$ such that $u = 0$ on the intersection of Ω and a neighborhood of K_0 and $u = 1$ on the intersection of Ω and a neighborhood of K_1. Let $\Sigma = \Sigma(K_0, K_1, \Omega)$ be the family of characteristic functions χ_E of sets $E \subset \Omega$ such that $P_\Omega(E) < \infty$ and $E = U \cap \Omega$ for some open set $U \subset \mathbf{R}^n$ with $K_0 \cap \overline{U} = \emptyset$ and $K_1 \subset U$.

Remark. If Ω is unbounded, then unbounded sets "separating" K_0 and K_1 in Ω may be considered. In general, unbounded sets can have infinite perimeter. Thus the above definition of $S(K_0, K_1, \Omega)$ and $\Sigma(K_0, K_1, \Omega)$ has to be generalized. Such a generalization may be done by using the notions of "locally bounded variation" and "locally bounded perimeter" (see [2, Chapter 5]). For simplicity we shall restrict ourselves to a bounded domain Ω in this paper.

We define the following families:

$$\begin{aligned}
\nabla S &= \nabla S(K_0, K_1, \Omega) = \{\nabla u : u \in S(K_0, K_1, \Omega)\},\\
|\nabla S| &= |\nabla S(K_0, K_1, \Omega)| = \{|\nabla u| : u \in S(K_0, K_1, \Omega)\},\\
\nabla \Sigma &= \nabla \Sigma(K_0, K_1, \Omega) = \{\nabla \chi_E : \chi_E \in \Sigma(K_0, K_1, \Omega)\},\\
|\nabla \Sigma| &= |\nabla \Sigma(K_0, K_1, \Omega)| = \{|\nabla \chi_E| : \chi_E \in \Sigma(K_0, K_1, \Omega)\}.
\end{aligned}$$

More generally, we let

$$\begin{aligned}
\big|\sqrt{\mathscr{A}}\,\nabla S\big| &= \big\{\big|\sqrt{\mathscr{A}}\,\nabla u\big| : u \in S\big\},\\
\big|\sqrt{\mathscr{A}}\,\nabla \Sigma\big| &= \big\{\big|\sqrt{\mathscr{A}}\,\nabla \chi_E\big| : \chi_E \in \Sigma\big\}.
\end{aligned}$$

We consider $M_{\mathscr{B},p'}(\nabla S)$, $M_{\mathscr{B},p'}(\nabla \Sigma)$, $M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla S\big|\big)$, $M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla \Sigma\big|\big)$. The reciprocals of these quantities are called generalized extremal widths. Obviously, $\nabla\Sigma \subset \nabla S$, so that

$$M_{\mathscr{B},p'}(\nabla S) \ge M_{\mathscr{B},p'}(\nabla \Sigma) \quad \text{and} \quad M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla S\big|\big) \ge M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla \Sigma\big|\big).$$

There is a relationship between S and $\mathscr{D}^*$ in Theorem 2. Since Ω is bounded and $w \in A_p$, it follows that $\int_\Omega w^{1-p'}\,dx < \infty$, so that by Hölder's inequality

$$\int_\Omega |\nabla u|\,dx \le \left(\int_\Omega |\nabla u|^p w\,dx\right)^{1/p} \left(\int_\Omega w^{1-p'}\,dx\right)^{1/p'} < \infty \tag{1.3}$$

for any (p,w)-precise functions u. Hence $\mathscr{D}^* \subset S$ and $0 < M_{\mathscr{B},p'}(\nabla \mathscr{D}^*) \le M_{\mathscr{B},p'}(\nabla S)$. Moreover, we have the following theorem.

Theorem 3. *Let Ω be bounded. Then*

$$\begin{aligned}
M_{\mathscr{B},p'}(\nabla S) = M_{\mathscr{B},p'}(\nabla \Sigma) &= M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla S\big|\big)\\
&= M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla \Sigma\big|\big) = M_{\mathscr{B},p'}(\nabla \mathscr{D}^*) > 0.
\end{aligned}$$

We are interested in whether or not

$$C_{\mathscr{A},p}(K_0, K_1, \Omega) = C^*_{\mathscr{A},p}(K_0, K_1, \Omega).$$

If this is true, then Theorems 1–3 give a relationship between generalized extremal distances and extremal widths. We shall reduce this question to the problem of continuity of extremal distances. For this purpose we need the uniform continuity of $w(x)^{1/p}\sqrt{\mathscr{B}(x)}$, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Omega$ and $|x-y| < \delta$, then

$$\left| w(x)^{1/p}\sqrt{\mathscr{B}(x)} - w(y)^{1/p}\sqrt{\mathscr{B}(y)} \right| < \varepsilon.$$

We remark that this continuity implies neither the continuity of $\sqrt{\mathscr{B}(x)}$ nor $w(x)$.

Theorem 4. *Suppose that $w(x)^{1/p}\sqrt{\mathscr{B}(x)}$ is uniformly continuous on Ω. Then $M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big)$ has the continuity property. That is, let K_0^j and K_1^j be sequences of compact sets such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^{\infty} K_0^j$, and $K_1 = \bigcap_{j=0}^{\infty} K_1^j$. If $\Gamma_j = \Gamma(K_0^j, K_1^j, \Omega)$, then*

$$M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma_j\big|\big) \downarrow M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big).$$

As a result of Theorems 1–4 we obtain the following reciprocal relation between extremal distance and extremal width.

Theorem 5. *Suppose that $w(x)^{1/p}\sqrt{\mathscr{B}(x)}$ is uniformly continuous on Ω. Then*

$$C_{\mathscr{A},p}(K_0, K_1, \Omega) = C^*_{\mathscr{A},p}(K_0, K_1, \Omega).$$

Moreover, suppose Ω is bounded. If $M_{\mathscr{A},p}(d\Gamma) = 0$, then $M_{\mathscr{B},p'}(\nabla\Sigma) = \infty$; if $M_{\mathscr{A},p}(d\Gamma) > 0$, then

$$M_{\mathscr{A},p}(d\Gamma)^{1/p} M_{\mathscr{B},p'}(\nabla\Sigma)^{1/p'} = 1.$$

The above theorem yields the classical reciprocal relation. Let $\mathscr{A} = w^{2/p}E$ with the identity matrix E. Then $\sqrt{\mathscr{A}} = w^{1/p}E$, $\mathscr{B} = w^{-2/p}E$, $\sqrt{\mathscr{B}} = w^{-1/p}E$, $M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big) = M_{p,w}\big(\big|d\Gamma\big|\big)$ and $M_{p'}(|\sqrt{\mathscr{A}}\,\nabla\Sigma|) = M_{p',w^{1-p'}}(|\nabla\Sigma|)$. Hence we obtain

Corollary. *Let Ω be bounded. If $M_{p,w}(|d\Gamma|) = 0$, then $M_{p',w^{1-p'}}(|\nabla\Sigma|) = \infty$; if $M_{p,w}(|d\Gamma|) > 0$, then*

$$M_{p,w}(|d\Gamma|)^{1/p} M_{p',w^{1-p'}}(|\nabla\Sigma|)^{1/p'} = 1.$$

Acknowledgment. The authors would like to thank the referee for valuable comments.

2. Preliminaries

Proof of Proposition 1. Since $M_{\mathscr{A},p}$ is monotone, we may assume that

$$\lim M_{\mathscr{A},p}(\mathscr{F}_j) = \alpha < \infty,$$

and we have only to show that $M_{\mathscr{A},p}(\mathscr{F}) \le \alpha$. By definition there is ξ_j such that $\xi_j \wedge \mathscr{F}_j$ (p,w)-a.e. such that

$$M_{\mathscr{A},p}(\mathscr{F}_j) \le \mathscr{A}_p(\xi_j)^p \le M_{\mathscr{A},p}(\mathscr{F}_j) + \frac{1}{j}.$$

Let $j \le i$. We have $\frac{1}{2}(\xi_i+\xi_j)\wedge\mathscr{F}_j$ (p,w)-a.e. Hence $\mathscr{A}_p\big(\frac{1}{2}(\xi_i+\xi_j)\big) \ge M_{\mathscr{A},p}(\mathscr{F}_j)$. Let us invoke the Clarkson inequalities. Suppose $p \ge 2$. Then

$$\mathscr{A}_p\big(\tfrac{1}{2}(\xi_i-\xi_j)\big)^p \le \tfrac{1}{2}\mathscr{A}_p(\xi_i)^p + \tfrac{1}{2}\mathscr{A}_p(\xi_j)^p - \mathscr{A}_p\big(\tfrac{1}{2}(\xi_i+\xi_j)\big)^p.$$

The first two terms on the right hand side tend to α and the limit of the last term is not less than α as $i,j \to \infty$. Hence

$$\big\|\tfrac{1}{2}(\xi_i-\xi_j)\big\|_{p,w} \le c_0\mathscr{A}_p\big(\tfrac{1}{2}(\xi_i-\xi_j)\big)^p \to 0.$$

Hence there is ξ such that $\big\|\frac{1}{2}(\xi-\xi_j)\big\|_{p,w} \to 0$ and $\mathscr{A}_p(\xi)^p = \alpha$. The same is true of the case $1 < p \le 2$. By the property (v) of $M_{p,w}$, taking a subsequence, if necessary, we obtain

$$\int |\xi-\xi_j|\,d|\nu| \to 0 \qquad \text{for } (p,w)\text{-a.e. } \nu \in \mathscr{F}.$$

Hence

$$\int \xi\cdot d\nu = \int \xi_j\cdot d\nu + \int (\xi-\xi_j)\cdot d\nu \ge \liminf \int \xi_j\cdot d\nu \ge 1 \qquad \text{for } (p,w)\text{-a.e. } \nu \in \mathscr{F},$$

so that

$$M_{\mathscr{A},p}(\mathscr{F}) \le \mathscr{A}_p(\xi)^p = \alpha.$$

The proposition is proved.

Let us clarify the meaning of ∇u for a (p,w)-precise function. To this end we introduce ACL (=absolutely continuous on lines) functions. We say that u is ACL if u is absolutely continuous on each component of the part in Ω of a.e. (= almost every with respect to $(n-1)$-dimensional measure) line parallel to each coordinate axis. By definition a (p,w)-precise function u is ACL.

Proposition 2. *Let u be an ACL function on Ω. Then the Dini derivatives $\partial u/\partial x_i$ exist a.e. in Ω and are measurable for $i = 1, \ldots, n$. Moreover, if $\partial u/\partial x_i$ are locally integrable, then they coincide with the distributional derivatives of u.*

Proof. In view of Nikodym [10, Lemme on p. 133], we see that u is measurable on Ω. For the first assertion it suffices to prove the existence and the measurability of $\partial u/\partial x_i$ on a closed cube $Q \subset \Omega$. We may assume that $i = 1$. We find $\delta > 0$ such that $(x_1 + h, x_2, \ldots, x_n) \in Q$ whenever $x = (x_1, \ldots, x_n) \in Q$ and $|h| < \delta$. Let

$$u_h(x) = \frac{u(x_1 + h, x_2, \ldots, x_n) - u(x_1, \ldots, x_n)}{h}$$

for $|h| < \delta$. Since $u(\cdot, x_2, \ldots, x_n)$ is absolutely continuous for a.e. $(x_2, \ldots, x_n)$, it follows that $\lim_{h\to 0} u_h$ exists a.e. on Q and is measurable.

Now suppose that $\partial u/\partial x_i$ is locally integrable on Ω. Take $\varphi \in C_0^\infty(\Omega)$. Then φu is ACL on Ω and

$$0 = \int_{-\infty}^{+\infty} \frac{\partial(\varphi u)}{\partial x_i}\, dx_i = \int_{-\infty}^{+\infty} \left(\varphi \frac{\partial u}{\partial x_i} + u \frac{\partial \varphi}{\partial x_i}\right) dx_i,$$

whence

$$\int_\Omega \varphi \frac{\partial u}{\partial x_i}\, dx = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx.$$

This means $\partial u/\partial x_i$ is the distributional derivative of u.

Observe that the last assertion of the proposition applies to a (p, w)-precise function u since $w \in A_p$, and so $L_w^p(\Omega) \subset L_{\mathrm{loc}}^1(\Omega)$. See [2, Section 4.9] for another proof of differentiability of a precise function. Note that a precise function is defined as a quasicontinuous function in [2].

In the rest of this section we shall deal with approximation. Let ψ be a nonnegative, radially symmetric smooth function supported in the unit ball $\{|y| < 1\}$ such that $\int \psi\, dy = 1$. Let $\psi_r(x) = r^{-n}\psi(x/r)$ and consider $f * \psi_r$ for a locally integrable function f on $\mathbf{R}^n$. It is well known that $f * \psi_r(x) \to f(x)$ as $r \to 0$ and $|f * \psi_r(x)| \le cMf(x)$ for a.e. $x \in \mathbf{R}^n$, where Mf is the maximal function of f and c is a positive constant depending only on ψ. Hence, if $f \in L_w^p(\mathbf{R}^n)$, then the maximal inequality $\|Mf\|_{p,w} \le c\|f\|_{p,w}$ and the dominated convergence theorem imply that $\|f * \psi_r - f\|_{p,w} \to 0$ as $r \to 0$. Thus $f \in L_w^p(\mathbf{R}^n)$ is approximated by smooth functions.

We would like to establish such an approximation property for functions defined only on an open set $G \ne \mathbf{R}^n$. To this end we use a positive smooth function α on G which vanishes on ∂G.

Lemma 2.1. *There exists $\alpha = \alpha_G \in C^\infty(G)$ such that*

(i) $0 < \alpha \le 1$,
(ii) $|\nabla \alpha| \le \frac{1}{2}$,
(iii) $\alpha(x) \le \frac{1}{2}\operatorname{dist}(x, \partial G)$.

Proof. Let $\{G_j\}_{j=1}^{\infty}$ be an increasing sequence of open sets such that $\overline{G_j} \Subset G_{j+1}$ and $G_j \uparrow G$. For each $j \geq 1$ we find a nonnegative function $\beta_j \in C_0^{\infty}(G)$ such that $\beta_j > 0$ on $G_j \setminus G_{j-1}$ and $\beta_j = 0$ on $(G \setminus G_{j+1}) \cup G_{j-2}$, where we put $G_0 = G_{-1} = \emptyset$ conventionally. Multiplying by a positive constant, if necessary, we may assume that $\beta_j \leq \frac{1}{3}$, $|\nabla \beta_j| \leq \frac{1}{6}$ and $\beta_j(x) \leq \frac{1}{6}\operatorname{dist}(x, \partial G)$. Then the function $\alpha = \sum_{j=1}^{\infty} \beta_j$ has the required properties.

Lemma 2.2. *Let $|y| \leq 1$ and $0 < r < 1$. Then the mapping $T_{y,r}$: $x \to x + r\alpha(x)y$ gives a C^{∞} homeomorphism of G onto itself.*

Proof. If $y = 0$, then there is nothing to prove. Suppose $y \neq 0$. In view of (iii) of Lemma 2.1 we see that $T_{y,r}$ is a mapping from G to G. Suppose $T_{y,r}(x) = T_{y,r}(x')$ for $x, x' \in G$. Then $x - x' = r\big(\alpha(x) - \alpha(x')\big)y$, so that by (ii) of Lemma 2.1 $|x - x'| \leq \frac{1}{2} r|x - x'|$, and hence $x = x'$. Thus $T_{y,r}$ is injective. Let l be a component of the intersection of G and a line in the direction of y. Then, by the definition of $T_{y,r}$ and (iii) of Lemma 2.1, l is invariant under $T_{y,r}$. In particular, $T_{y,r}$ is a surjection onto G. Observe that the Jacobian of $T_{y,r}$ is equal to $1 + r\nabla\alpha \cdot y \geq 1 - \frac{1}{2}r > 0$ by (ii) of Lemma 2.1. Therefore, the inverse mapping $T_{y,r}^{-1}$ is C^{∞} too.

As before Lemma 2.1, let ψ be a nonnegative, radially symmetric smooth function supported in the unit ball $\{|y| < 1\}$ such that $\int \psi\, dy = 1$. For a locally integrable function f on G we define

$$(2.1)\quad (f)_r(x) = \int_{|y|<1} f\big(x + r\alpha(x)y\big)\psi(y)\, dy = \frac{1}{\big(r\alpha(x)\big)^n} \int_{\mathbf{R}^n} f(y)\psi\left(\frac{y - x}{r\alpha(x)}\right) dy$$

for $x \in G$. Observe that $(f)_r \in C^{\infty}(G)$. It is easy to see that $(f)_r(x) \to f(x)$ as $r \to 0$ and $|(f)_r(x)| \leq cMf(x)$ for a.e. $x \in G$. Hence the dominated convergence theorem yields

Proposition 3. *Let $f \in L_w^p(G)$. Then $\|(f)_r - f\|_{p,w} \to 0$ as $r \to 0$.*

With some calculation we can show the following proposition (cf. [9, Theorem 4.6]).

Proposition 4. *Let u be a (p, w)-precise function over G. Then*

$$\|\nabla(u)_r - \nabla u\|_{p,w} \to 0 \qquad \text{as } r \to 0.$$

Let $G = \Omega$ and apply Proposition 4 to $C^*_{\mathscr{A},p}(K_0, K_1, \Omega)$. Suppose $u \in \mathscr{D}^* = \mathscr{D}^*(K_0, K_1, \Omega)$. By definition, if $r > 0$ is sufficiently small, then $(u)_r \in \mathscr{D}^*$. Hence, by Proposition 4 and truncation, we have

Proposition 5.

$$C^*_{\mathscr{A},p}(K_0, K_1, \Omega) = \inf_{\substack{u \in \mathscr{D}^* \cap C^{\infty}(\Omega) \\ 0 \leq u \leq 1}} \mathscr{A}_p(\nabla u)^p.$$

3. Proof of Theorem 1

The proof of Theorem 1 will be based on the following three lemmas. Let us recall $\Gamma = \Gamma(K_0, K_1, \Omega)$. Without loss of generality we may assume that $K_0, K_1 \subset \overline{\Omega}$.

Lemma 3.1.

$$M_{\mathscr{A},p}(d\Gamma) \le C_{\mathscr{A},p}(K_0, K_1, \Omega) < \infty.$$

Proof. Clearly $\mathscr{D}(K_0, K_1, \Omega) \neq \emptyset$ so that $C_{\mathscr{A},p}(K_0, K_1, \Omega) < \infty$. Take $u \in \mathscr{D}(K_0, K_1, \Omega)$. It is known (cf. [9, §§4.3–4]) that

$$\int_\gamma \nabla u \cdot dx = u\bigl(e(\gamma)\bigr) - u\bigl(s(\gamma)\bigr) = 1 \qquad \text{for } (p,w)\text{-a.e. curve } \gamma,$$

where $s(\gamma)$ and $e(\gamma)$ denote the starting point and the end point of γ, respectively. Hence $\nabla u \wedge d\Gamma$ (p,w)-a.e., so that $\mathscr{A}_p(\nabla u)^p \ge M_{\mathscr{A},p}(d\Gamma)$. Taking the infimum with respect to u, we obtain the lemma.

Lemma 3.2.

$$M_p\bigl(\bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|\bigr) \le M_{\mathscr{A},p}(d\Gamma) < \infty.$$

Proof. By Lemma 3.1 $M_{\mathscr{A},p}(d\Gamma) < \infty$. Let $\xi \wedge d\Gamma$ (p,w)-a.e. Then for (p,w)-a.e. $\gamma \in \Gamma$ we have

$$1 \le \int_\gamma \xi \cdot dx = \int_\gamma \sqrt{\mathscr{A}}\, \xi \cdot \sqrt{\mathscr{B}}\ dx \le \int_\gamma \bigl|\sqrt{\mathscr{A}}\, \xi\bigr| \bigl|\sqrt{\mathscr{B}}\ dx\bigr|.$$

Thus $\bigl|\sqrt{\mathscr{A}}\, \xi\bigr| \wedge \bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|$ (p,w)-a.e. Taking the infimum with respect to ξ, we obtain

$$M_{\mathscr{A},p}(d\Gamma) = \inf \mathscr{A}_p(\xi)^p = \inf \int_\Omega \bigl|\sqrt{\mathscr{A}}\, \xi\bigr|^p\, dx \ge M_p\bigl(\bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|\bigr).$$

The lemma follows.

Lemma 3.3.

$$C_{\mathscr{A},p}(K_0, K_1, \Omega) \le M_p\bigl(\bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|\bigr) < \infty.$$

Proof. As was observed in the introduction,

$$M_{p,w}\bigl(w^{1/p}\bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|\bigr) = M_p\bigl(\bigl|\sqrt{\mathscr{B}}\, d\Gamma\bigr|\bigr) < \infty.$$

Let ϱ be a nonnegative function in $L^p_w(\Omega)$ such that

$$\int_\gamma \varrho w^{1/p} \bigl|\sqrt{\mathscr{B}}\ dx\bigr| \ge 1 \qquad \text{for every curve } \gamma \in \Gamma. \tag{3.1}$$

For each $x \in \Omega$ we let Γ_0^x be the family of curves starting in K_0 and ending at x. Define

$$u(x) = \inf_{\gamma \in \Gamma_0^x} \int_\gamma \varrho w^{1/p} \left| \sqrt{\mathscr{B}}\, dx \right|.$$

We claim

(i) u is (p,w)-precise in Ω;

(ii) for a.e. $x \in \Omega$

$$|\sqrt{\mathscr{A}(x)}\, \nabla u(x)| \le \varrho(x) w(x)^{1/p}; \tag{3.2}$$

(iii) $\lim u(x) = 0$ as $x \to K_0$ along (p,w)-a.e. curve $\gamma \in \Gamma$;

(iv) $\liminf u(x) \ge 1$ as $x \to K_1$ along (p,w)-a.e. curve $\gamma \in \Gamma$.

By (iv) of the properties of modulus stated in Section 1 and (1.1) we have

$$\int_\gamma \varrho w^{1/p} \left| \sqrt{\mathscr{B}}\, dx \right| \le c_0 \int_\gamma \varrho\, ds < \infty \tag{3.3}$$

for (p,w)-a.e. curve γ in Ω. Hence for (p,w)-a.e. curve γ

$$|u(b) - u(a)| \le \int_{\widetilde{ab}} \varrho w^{1/p} \left| \sqrt{\mathscr{B}}\, dx \right| \le c_0 \int_{\widetilde{ab}} \varrho\, ds \qquad \text{for any points } a, b \in \gamma, \tag{3.4}$$

where $\widetilde{ab}$ is the arc on γ connecting a and b. This implies that u is absolutely continuous on γ for (p,w)-a.e. curve γ. We know that (p,w)-a.e. segment γ in Ω parallel to one of the coordinate axes satisfies (3.4) and hence by Fubini's theorem $|\partial u/\partial x_i| \le c_0 \varrho$ a.e. in Ω for each $i = 1, \dots, n$. Hence $\nabla u \in L^p_w(\Omega)$ and u is a (p,w)-precise function on Ω.

In order to show (3.2) we take a countable dense set $\{\xi_j\}$ in the unit sphere $\{x : |x| = 1\}$. Then we see that (3.4) is satisfied along (p,w)-a.e. segment γ in Ω parallel to one of ξ_j. Hence

$$|u(a + h\xi_j) - u(a)| \le \int_0^h \varrho(a + t\xi_j) w(a + t\xi_j)^{1/p} \left| \sqrt{\mathscr{B}(a + t\xi_j)}\, \xi_j \right| dt$$

for a.e. $a \in \Omega$. Divide by h and let $h \to 0$. We have

$$|\xi_j \cdot \nabla u(a)| \le \varrho(a) w(a)^{1/p} \left| \sqrt{\mathscr{B}(a)}\, \xi_j \right| \qquad \text{for a.e. } a \in \Omega. \tag{3.5}$$

If $|\nabla u(a)| = 0$, then (3.2) is true at $x = a$. Suppose $|\nabla u(a)| > 0$. Then $\left| \sqrt{\mathscr{A}(a)} \nabla u(a) \right| > 0$. Take a sequence $\{\xi_{j_k}\}$ tending to $\mathscr{A}(a)\nabla u(a)/|\mathscr{A}(a)\nabla u(a)|$. Then by (3.5)

$$\begin{aligned} \left| \frac{\mathscr{A}(a)\nabla u(a) \cdot \nabla u(a)}{|\mathscr{A}(a)\nabla u(a)|} \right| &\le \varrho(a) w(a)^{1/p} \left| \frac{\sqrt{\mathscr{B}(a)}\, \mathscr{A}(a)\nabla u(a)}{|\mathscr{A}(a)\nabla u(a)|} \right| \\ &= \varrho(a) w(a)^{1/p} \frac{\left| \sqrt{\mathscr{A}(a)}\, \nabla u(a) \right|}{|\mathscr{A}(a)\nabla u(a)|}. \end{aligned}$$

Since $|\mathscr{A}(a)\nabla u(a)\cdot\nabla u(a)| = \bigl|\sqrt{\mathscr{A}(a)}\,\nabla u(a)\bigr|^2$, we have (3.2) for $x=a$.

We have observed that a (p,w)-a.e. curve satisfies (3.3). For the proof of (iii) and (iv) take $\gamma\in\Gamma$ satisfying (3.3) and express it in parameter: $x_\gamma(t)$, $t_0<t<t_1$, such that $x_\gamma(t)\to K_0$ as $t\to t_0$ and $x_\gamma(t)\to K_1$ as $t\to t_1$. By the definition of $u(x)$ and (3.3)

$$0\le u\bigl(x_\gamma(t)\bigr)\le \int_{t_0}^{t}\varrho w^{1/p}\bigl|\sqrt{\mathscr{B}}\,dx\bigr|\le c_0\int_{t_0}^{t}\varrho\,ds\to 0$$

as $t\to t_0$. Thus $\lim u(x)=0$ as $x\to K_0$ along γ. We have (iii). Let us prove (iv) by contradiction. Suppose $\alpha=\liminf_{t\to t_1}u\bigl(x_\gamma(t)\bigr)<1$ and let $\varepsilon=1-\alpha>0$. By definition there is t, $t_0<t<t_1$, such that

$$\bigl|u\bigl(x_\gamma(t)\bigr)-\alpha\bigr|<\frac{\varepsilon}{3}\qquad\text{and}\qquad\int_t^{t_1}\varrho\,ds<\frac{\varepsilon}{3c_0}.$$

By the definition of $u(x)$ there is $\gamma'\in\Gamma_0^x$ with $x=x_\gamma(t)$ such that

$$\int_{\gamma'}\varrho w^{1/p}\bigl|\sqrt{\mathscr{B}}\,dx\bigr|<u(x)+\frac{\varepsilon}{3}.$$

Let γ'' be the subcurve of γ corresponding to (t,t_1). Then $\gamma'+\gamma''\in\Gamma$ and by (3.3)

$$\int_{\gamma'+\gamma''}\varrho w^{1/p}\bigl|\sqrt{\mathscr{B}}\,dx\bigr|<u(x)+\frac{\varepsilon}{3}+c_0\int_t^{t_1}\varrho\,ds<\alpha+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=1,$$

which contradicts (3.1). Thus (iv) follows.

Now let $\tilde u=\min\{u,1\}$. Then $\tilde u\in\mathscr{D}(K_0,K_1,\Omega)$. Hence by the definition of $C_{\mathscr{A},p}$ we have

$$C_{\mathscr{A},p}(K_0,K_1,\Omega)\le\mathscr{A}_p(\nabla\tilde u)^p\le\mathscr{A}_p(\nabla u)^p=\int_\Omega\bigl|\sqrt{\mathscr{A}}\,\nabla u\bigr|^p\,dx\le\int_\Omega\varrho^p w\,dx,$$

where (3.2) is used in the last inequality. Taking the infimum with respect to ϱ, we obtain

$$C_{\mathscr{A},p}(K_0,K_1,\Omega)\le M_{p,w}\bigl(w^{1/p}\bigl|\sqrt{\mathscr{B}}\,d\Gamma\bigr|\bigr)=M_p(|\sqrt{\mathscr{B}}\,d\Gamma|).$$

The lemma follows.

Combining Lemmas 3.1–3, we obtain Theorem 1.

4. Proof of Theorem 2

Proof of Theorem 2. We observe that $\mathscr{D}^*$ is a convex set, $\Xi = \{\xi : \mathscr{B}_{p'}(\xi) \le 1\}$ is a weakly compact convex set and

$$\Phi(u,\xi) = -\int_\Omega \nabla u \cdot \xi \, dx$$

is a bilinear functional on $\mathscr{D}^* \times \Xi$ such that $\Phi(u,\cdot)$ is continuous with respect to the weak topology of Ξ. Hence the minimax theorem ([3]) yields

$$\sup_{u\in\mathscr{D}^*} \inf_{\xi\in\Xi} \Phi(u,\xi) = \inf_{\xi\in\Xi} \sup_{u\in\mathscr{D}^*} \Phi(u,\xi),$$

in other words

$$\inf_{u\in\mathscr{D}^*} \sup_{\xi\in\Xi} \int_\Omega \nabla u \cdot \xi \, dx = \sup_{\xi\in\Xi} \inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx. \tag{4.1}$$

Observe that

$$\sup_{\xi\in\Xi} \int_\Omega \nabla u \cdot \xi \, dx = \mathscr{A}_p(\nabla u). \tag{4.2}$$

In fact, Hölder's inequality yields

$$\int_\Omega \nabla u \cdot \xi \, dx = \int_\Omega \sqrt{\mathscr{A}}\, \nabla u \cdot \sqrt{\mathscr{B}}\, \xi \, dx \le \mathscr{A}_p(\nabla u)\mathscr{B}_{p'}(\xi) \le \mathscr{A}_p(\nabla u)$$

for $\xi \in \Xi$. Conversely, let $\xi = \mathscr{A}_p(\nabla u)^{1-p} \big|\sqrt{\mathscr{A}}\, \nabla u\big|^{p-2} A\nabla u$. Then $\mathscr{B}_{p'}(\xi) = 1$ and $\int_\Omega \nabla u \cdot \xi \, dx = \mathscr{A}_p(\nabla u)$. Thus (4.2) follows, and (4.1) becomes

$$C^*_{\mathscr{A},p}(K_0,K_1,\Omega)^{1/p} = \sup_{\xi\in\Xi} \inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx. \tag{4.3}$$

Suppose $C^*_{\mathscr{A},p}(K_0,K_1,\Omega) = 0$. Then $\inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx = 0$ for any $\xi \in \Xi$, so that there is no ξ with $\mathscr{B}_{p'}(\xi) < \infty$ and $\xi \wedge \mathscr{D}^*$. Thus $M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) = \infty$, and the theorem follows in this case. Now suppose $C^*_{\mathscr{A},p}(K_0,K_1,\Omega) > 0$. We claim

$$\sup_{\xi\in\Xi} \inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx = \sup\Big\{ \mathscr{B}_{p'}(\xi)^{-1} : \inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx \ge 1 \Big\}. \tag{4.4}$$

Let α and β be the left hand side and the right hand side of (4.4), respectively. By definition

$$\mathscr{B}_{p'}(\xi) \le 1 \qquad \Longrightarrow \qquad \inf_{u\in\mathscr{D}^*} \int_\Omega \nabla u \cdot \xi \, dx \le \alpha.$$

By homogeneity this is equivalent to

$$\inf_{u\in\mathscr{D}^*}\int_\Omega \nabla u\cdot\xi\,dx>1 \qquad\Longrightarrow\qquad \mathscr{B}_{p'}(\xi)>\frac{1}{\alpha}.$$

This means that $\beta\le\alpha$. Similarly we can prove $\alpha\le\beta$ and we have (4.4). By definition the right hand side of (4.4) is equal to

$$\left(\inf\left\{\mathscr{B}_{p'}(\xi):\inf_{u\in\mathscr{D}^*}\int_\Omega \nabla u\cdot\xi\,dx\ge 1\right\}\right)^{-1}=M_{\mathscr{B},p'}(\nabla\mathscr{D}^*)^{-1/p'},$$

where (1.2) is used. Hence (4.3) and (4.4) yield

$$C^*_{\mathscr{A},p}(K_0,K_1,\Omega)^{1/p}=M_{\mathscr{B},p'}(\nabla\mathscr{D}^*)^{-1/p'},$$

which yields the theorem.

5. Proof of Theorem 3

In this section we let Ω be bounded. The proof of Theorem 3 will be done by the following three lemmas.

Lemma 5.1.

$$M_{\mathscr{B},p'}(\nabla S)\ge M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla S\big|\big)\ge M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big), \tag{5.1}$$

$$M_{\mathscr{B},p'}(\nabla S)\ge M_{\mathscr{B},p'}(\nabla\Sigma)\ge M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big). \tag{5.2}$$

Proof. Let us prove the first inequality of (5.1). We may assume that $M_{\mathscr{B},p'}(\nabla S)<\infty$. Let $\xi\wedge\nabla u$ for $(p',w^{1-p'})$-a.e. $\nabla u\in\nabla S$. Then there is a $(p',w^{1-p'})$-exc. subsystem $\mathscr{U}_0$ of ∇S such that $\xi\wedge\nabla u$ holds for every $\nabla u\in\nabla S\setminus\mathscr{U}_0$. We have $M_{p',w^{1-p'}}(|\mathscr{U}_0|)=0$. We claim that $M_{p'}\big(\big\{\big|\sqrt{\mathscr{A}}\,\nabla u\big|:\nabla u\in\mathscr{U}_0\big\}\big)=0$. In fact, for a given $\varepsilon>0$, there is $\varrho\ge0$ such that $\varrho\wedge|\mathscr{U}_0|$ and $\int\varrho^{p'}w^{1-p'}\,dx<\varepsilon$. By (1.1)

$$1\le\int\varrho|\nabla u|=\int\varrho w^{-1/p}w^{1/p}|\nabla u|\le c_0\int\varrho w^{-1/p}\big|\sqrt{\mathscr{A}}\,\nabla u\big|$$

for $\nabla u\in\mathscr{U}_0$. Hence

$$M_{p'}\big(\big\{\big|\sqrt{\mathscr{A}}\,\nabla u\big|:\nabla u\in\mathscr{U}_0\big\}\big)\le\int(c_0\varrho w^{-1/p})^{p'}\,dx=c_0^{p'}\int\varrho^{p'}w^{1-p'}\,dx<c_0^{p'}\varepsilon.$$

The arbitrariness of ε gives $M_{p'}\big(\big\{\big|\sqrt{\mathscr{A}}\,\nabla u\big|:\nabla u\in\mathscr{U}_0\big\}\big)=0$. For $\nabla u\in\nabla S\setminus\mathscr{U}_0$ we have

$$1\le\int_\Omega\xi\cdot\nabla u=\int_\Omega\sqrt{\mathscr{B}}\,\xi\cdot\sqrt{\mathscr{A}}\,\nabla u\le\int_\Omega\big|\sqrt{\mathscr{B}}\,\xi\big|\big|\sqrt{\mathscr{A}}\,\nabla u\big|$$

so that $|\sqrt{\mathscr{B}}\,\xi| \wedge \big|\sqrt{\mathscr{A}}\,\nabla S\big|$ p'-a.e. Taking the infimum with respect to ξ, we obtain

$$M_{\mathscr{B},p'}(\nabla S) = \inf \mathscr{B}_{p'}(\xi)^{p'} = \inf \int_\Omega \big|\sqrt{\mathscr{B}}\,\xi\big|^{p'}\,dx \ge M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla S\big|\big).$$

Thus the first inequality of (5.1) follows. The second inequality of (5.1) and the first inequality of (5.2) are trivial. In the same way as above, we have the second inequality of (5.2). The lemma follows.

Lemma 5.2.

$$M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big) \ge M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) > 0.$$

Proof. We may assume that $M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big) < \infty$. In view of Theorem 2, it is sufficient to show that $C^*_{\mathscr{A},p}(K_0,K_1,\Omega) > 0$ and

$$C^*_{\mathscr{A},p}(K_0,K_1,\Omega)^{1/p} M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big)^{1/p'} \ge 1. \tag{5.3}$$

Let $u \in \mathscr{D}^* \cap C^\infty(\Omega)$ and $0 \le u \le 1$. Put $N_t = \{x \in \Omega : u(x) > t\}$. We claim that $\chi_{N_t} \in \Sigma$ for a.e. t, $0 < t < 1$. By the Sard theorem $\{x \in \Omega : u(x) = t\}$ is a smooth manifold for a.e. t. For such t Gauss' divergence theorem yields

$$\int_{N_t} \operatorname{div}\varphi\,dx = -\int_{u=t} \varphi\cdot\mathbf{n}\,dS \qquad \text{for } \varphi \in C_0^\infty(\Omega;\mathbf{R}^n),$$

where $\mathbf{n} = \nabla u/|\nabla u|$ is the normal to $\{u = t\}$ directed into N_t. By definition the above identity implies $\nabla\chi_{N_t} = \mathbf{n}dS|_{u=t}$. By the coarea formula (see e.g. [8, Theorem 1.2.4])

$$\int_0^1 \|\chi_{N_t}\|_{BV}\,dt = \int_0^1 dt \int_{u=t} dS = \int_\Omega |\nabla u|\,dx.$$

The last integral is convergent by (1.3) and hence $\chi_{N_t} \in \Sigma$ for a.e. t.

Since $M_{p'}\big(\big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|\big) < \infty$, there is $\varrho \ge 0$ such that $\varrho \wedge \big|\sqrt{\mathscr{A}}\,\nabla\Sigma\big|$ and $\|\varrho\|_{p'} < \infty$. Take such a ϱ. Then

$$\int_{u=t} \varrho\big|\sqrt{\mathscr{A}}\,\mathbf{n}\big|\,dS \ge 1 \qquad \text{for a.e. } t.$$

Hence the coarea formula yields

$$\begin{aligned}1 &\le \int_0^1 dt \int_{u=t} \varrho\big|\sqrt{\mathscr{A}}\,\mathbf{n}\big|\,dS = \int_0^1 dt \int_{u=t} \varrho\frac{\big|\sqrt{\mathscr{A}}\,\nabla u\big|}{|\nabla u|}\,dS \\ &= \int_\Omega \varrho\big|\sqrt{\mathscr{A}}\,\nabla u\big|\,dx \le \mathscr{A}_p(\nabla u)\,\|\varrho\|_{p'}\,.\end{aligned}$$

Taking the infimum with respect to ϱ and u, we obtain $C^*_{\mathscr{A},p}(K_0,K_1,\Omega) > 0$ and (5.3) by Proposition 5. The lemma is proved.

The proof of Theorem 3 is completed by the following lemma.

Lemma 5.3.

$$M_{\mathscr{B},p'}(\nabla\mathscr{D}^*) \geq M_{\mathscr{B},p'}(\nabla S).$$

This is the most difficult part and the proof will be divided into several steps. In these steps, we shall vary the compact sets K_0, K_1 and the open set Ω, so we shall write

$$M_{\mathscr{B},p'}(K_0,K_1,\Omega) = M_{\mathscr{B},p'}\big(\nabla S(K_0,K_1,\Omega)\big),$$
$$M^*_{\mathscr{B},p'}(K_0,K_1,\Omega) = M_{\mathscr{B},p'}\big(\nabla\mathscr{D}^*(K_0,K_1,\Omega)\big).$$

The outline of the proof is as follows: We prove, in Lemma 5.4, the monotonicity of $M^*_{\mathscr{B},p'}(K_0,K_1,\Omega)$ with respect to Ω and introduce an approximation property with respect to Ω. Lemma 5.5 shows the existence of an open subset of Ω with this approximation property. The approximation with respect to K_0 and K_1 stated in Lemma 5.6 is rather straightforward from Proposition 1. Lemma 5.7 includes the crucial part of the proof of Lemma 5.3, which deals with the case $K_0, K_1 \subset \Omega$ under an additional assumption. The proof uses the approximation property of BV functions. Finally, by enlarging Ω, we reduce the general case $K_0, K_1 \subset \overline{\Omega}$ to Lemma 5.7 and complete the proof of Lemma 5.3.

Lemma 5.4. *Let Ω_1 be an open subset of Ω such that $K_0, K_1 \subset \overline{\Omega_1}$. Then*

$$M^*_{\mathscr{B},p'}(K_0,K_1,\Omega_1) \geq M^*_{\mathscr{B},p'}(K_0,K_1,\Omega).$$

Proof. We may assume $M^*_{\mathscr{B},p'}(K_0,K_1,\Omega_1) < \infty$. Let $\xi \wedge \nabla\mathscr{D}^*(K_0,K_1,\Omega_1)$ and set $\overline{\xi} = \xi$ on Ω_1 and $\overline{\xi} = 0$ on $\Omega \setminus \Omega_1$. Then

$$\int_\Omega \overline{\xi}\cdot\nabla u\,dx = \int_{\Omega_1} \xi\cdot\nabla u|_{\Omega_1}\,dx \geq 1$$

for $u \in \mathscr{D}^*(K_0,K_1,\Omega)$ since $u|_{\Omega_1} \in \mathscr{D}^*(K_0,K_1,\Omega_1)$. Hence

$$\int_{\Omega_1} B[\xi]^{p'}\,dx = \mathscr{B}_{p'}(\overline{\xi})^{p'} \geq M^*_{\mathscr{B},p'}(K_0,K_1,\Omega).$$

Taking the infimum with respect to ξ, we obtain the lemma.

In view of the above monotonicity we introduce the following definition.

Definition. Let $K_0, K_1 \subset \Omega$. We say that Ω can be approximated from inside with respect to $M^*_{\mathscr{B},p'}(K_0,K_1,\Omega)$ if for each $\varepsilon > 0$ there is an open set Ω' such that $K_0, K_1 \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ and $M^*_{\mathscr{B},p'}(K_0,K_1,\Omega) \leq M^*_{\mathscr{B},p'}(K_0,K_1,\Omega') \leq M^*_{\mathscr{B},p'}(K_0,K_1,\Omega) + \varepsilon$.

Lemma 5.5. *Let $K_0, K_1 \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$. Suppose that $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_1) < \infty$. Then there is Ω_2 such that $\overline{\Omega_1} \subset \Omega_2 \subset \Omega$ and Ω_2 can be approximated from inside with respect to $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_2)$.*

Proof. For small $r > 0$ we let

$$\Omega(r) = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}.$$

Observe that $\Omega(r) \uparrow \Omega$ as $r \downarrow 0$. There is an $r_1 > 0$ such that $\overline{\Omega_1} \subset \Omega(r_1)$. Put $\varphi(r) = M^*_{\mathscr{B},p'}\big(K_0, K_1, \Omega(r)\big)$. By Lemma 5.4 $\varphi(r)$ is a nondecreasing function with $0 < \varphi(r) \le \varphi(r_1) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_1) < \infty$ for $0 < r \le r_1$, whence $\varphi(r)$ is continuous from the right at r outside a countable set of r. Hence, there is r_2, $0 < r_2 < r_1$, such that $\lim_{r \downarrow r_2} \varphi(r) = \varphi(r_2)$, i.e.,

$$M^*_{\mathscr{B},p'}\big(K_0, K_1, \Omega(r)\big) \downarrow M^*_{\mathscr{B},p'}\big(K_0, K_1, \Omega(r_2)\big) \qquad \text{as } r \downarrow r_2.$$

If $\Omega_2 = \Omega(r_2)$, we observe that $\overline{\Omega_1} \subset \Omega_2 \subset \Omega$ and Ω_2 can be approximated from inside with respect to $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_2)$.

Lemma 5.6. *Let $K_0, K_1 \subset \Omega$ and let $\{K_0^j\}$ and $\{K_1^j\}$ be decreasing sequences of compact sets such that $\Omega \supset K_0^j \downarrow K_0$ and $\Omega \supset K_1^j \downarrow K_1$. Then*

$$M_{\mathscr{B},p'}(K_0^j, K_1^j, \Omega) \uparrow M_{\mathscr{B},p'}(K_0, K_1, \Omega).$$

Proof. It is easy to see that

$$\nabla S(K_0^j, K_1^j, \Omega) \uparrow \nabla S(K_0, K_1, \Omega).$$

Hence the lemma follows from Proposition 1.

Lemma 5.7. *Let $K_0, K_1 \subset \Omega$ and suppose Ω can be approximated from inside with respect to $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega)$. Then*

$$M_{\mathscr{B},p'}(K_0, K_1, \Omega) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega).$$

Proof. Let $\varepsilon > 0$. By assumption there exists Ω_1 such that $K_0, K_1 \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ and

$$M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_1) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega) + \varepsilon. \tag{5.4}$$

By Lemma 5.6 there are compact sets K_0^1 and K_1^1 such that $K_0 \subset \operatorname{int} K_0^1 \subset K_0^1 \subset \Omega$, $K_1 \subset \operatorname{int} K_1^1 \subset K_1^1 \subset \Omega$, and

$$M_{\mathscr{B},p'}(K_0, K_1, \Omega) \le M_{\mathscr{B},p'}(K_0^1, K_1^1, \Omega) + \varepsilon. \tag{5.5}$$

We claim

$$M_{\mathscr{B},p'}(K_0^1, K_1^1, \Omega) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_1). \tag{5.6}$$

Then the required inequality follows from (5.4), (5.5), (5.6) and the arbitrariness of $\varepsilon > 0$.

Let us prove (5.6). We may assume that $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega_1) < \infty$. Let $\xi \wedge \nabla \mathscr{D}^*(K_0, K_1, \Omega_1)$. We extend ξ by $\overline{\xi} = \xi$ on Ω_1 and $\overline{\xi} = 0$ elsewhere. Let ξ_r be the symmetric mollification, i.e.,

$$\xi_r(x) = \int_{|y|<1} \overline{\xi}(x + ry)\psi(y)\, dy,$$

where ψ is a nonnegative, radially symmetric smooth function supported in the unit ball $\{|y| < 1\}$ such that $\int \psi\, dy = 1$. Let

$$\Omega_2 = \left\{x \in \Omega : \operatorname{dist}(x, \partial\,\Omega) > \tfrac{1}{2} \operatorname{dist}(\overline{\Omega_1}, \partial\,\Omega)\right\}$$

and $r_0 = \min\left\{\operatorname{dist}(K_0, \partial K_0^1), \operatorname{dist}(K_1, \partial K_1^1), \frac{1}{2}\operatorname{dist}(\overline{\Omega_1}, \partial\,\Omega)\right\} > 0$. Let $0 < r < r_0$. We shall see that

$$\xi_r \in C_0^\infty(\Omega_2; \mathbf{R}^n) \qquad \text{and} \qquad \xi_r \wedge \nabla \mathscr{D}^*(K_0^1, K_1^1, \Omega). \tag{5.7}$$

The first assertion of (5.7) follows from the definition. Suppose $u \in \mathscr{D}^*(K_0^1, K_1^1, \Omega)$. Then

$$\begin{aligned} \int_\Omega \xi_r \cdot \nabla u\, dx &= \int_{|y|<1} \psi(y)\, dy \int_\Omega \overline{\xi}(x + ry) \cdot \nabla u(x)\, dx \\ &= \int_{|y|<1} \psi(y)\, dy \int_{\Omega_1} \xi(x) \cdot \nabla u(x - ry)\, dx. \end{aligned}$$

Observe that $u(x-ry)|_{\Omega_1} \in \mathscr{D}^*(K_0, K_1, \Omega_1)$ for $|y| < 1$. As $\xi \wedge \nabla \mathscr{D}^*(K_0, K_1, \Omega_1)$, it follows that $\int_{\Omega_1} \xi(x) \cdot \nabla u(x - ry)\, dx \ge 1$, so that $\int_\Omega \xi_r \cdot \nabla u\, dx \ge 1$. Hence (5.7) follows.

Next we claim that

$$\xi_r \wedge \nabla S(K_0^1, K_1^1, \Omega). \tag{5.8}$$

It is well known that a BV function is approximated by smooth functions (see e.g. [8, Theorem 6.1.2]). Let $u \in S(K_0^1, K_1^1, \Omega)$. Then we find a sequence of smooth functions $u_j \in C^\infty(\Omega)$ such that $u_j = 0$ on a neighborhood of K_0^1, $u_j = 1$ on a neighborhood of K_1^1, $u_j \to u$ in $L^1(\Omega)$, $\|u_j\|_{BV} \to \|u\|_{BV}$ and $\nabla u_j \to \nabla u$ weakly, i.e.,

$$\int_\Omega \varphi \cdot \nabla u_j\, dx \to \int_\Omega \varphi \cdot \nabla u$$

for any $\varphi \in C_0^\infty(\Omega; \mathbf{R}^n)$. Note that the sign '$dx$' does not appear on the right hand side, since ∇u is a measure which need not be absolutely continuous with respect to the Lebesgue measure. Observe that it may happen that $\int_\Omega |\nabla u_j|^p w\,dx = \infty$. To avoid this difficulty, take $f \in C_0^\infty(\Omega)$ such that $0 \le f \le 1$ on Ω and $f = 1$ on Ω_2. Then we see that $fu_j \in C_0^\infty(\Omega)$ and hence $fu_j \in \mathscr{D}^*(K_0^1, K_1^1, \Omega)$. Hence (5.7) yields

$$1 \le \int_\Omega \xi_r \cdot \nabla(fu_j)\,dx = \int_\Omega \xi_r \cdot \nabla u_j\,dx \to \int_\Omega \xi_r \cdot \nabla u.$$

Thus (5.8) follows.

By (5.8) we have $M_{\mathscr{B},p'}(K_0^1, K_1^1, \Omega) \le \mathscr{B}_{p'}(\xi_r)^{p'}$, and by the approximation property, the right hand side tends to $\mathscr{B}_{p'}(\xi)^{p'}$ as $r \downarrow 0$. Hence, taking the infimum with respect to ξ we obtain (5.6). Thus the lemma is proved.

Proof of Lemma 5.3. We may assume that

$$M^*_{\mathscr{B},p'}(K_0, K_1, \Omega) = M_{\mathscr{B},p'}(\nabla \mathscr{D}^*) < \infty.$$

If necessary, taking the intersection with $\overline{\Omega}$, we may assume that $K_0, K_1 \subset \overline{\Omega}$. Let us take decreasing sequences $\{K_0^j\}$ and $\{K_1^j\}$ of compact sets such that $K_0^{j+1} \subset \operatorname{int} K_0^j$, $K_1^{j+1} \subset \operatorname{int} K_1^j$, $K_0^j \downarrow K_0$ and $K_1^j \downarrow K_1$. Let $\Omega^j = \Omega \cup (\operatorname{int} K_0^j) \cup (\operatorname{int} K_1^j)$. We claim

$$\nabla S(K_0^j, K_1^j, \Omega^j) \uparrow \nabla S(K_0, K_1, \Omega). \tag{5.9}$$

Take $u \in S(K_0^j, K_1^j, \Omega^j)$. By $\nabla_{\Omega^j} u$ we denote the distributional derivative of u over Ω^j. By definition $\nabla_{\Omega^j} u$ is concentrated on $\Omega^j \setminus (K_0^j \cup K_1^j) \subset \Omega$ and $\nabla_{\Omega^j} u = \nabla_{\Omega^{j+1}}(u|_{\Omega^{j+1}}) = \nabla_\Omega(u|_\Omega)$. Hence $\nabla S(K_0^j, K_1^j, \Omega^j)$ is increasing and $\bigcup \nabla S(K_0^j, K_1^j, \Omega^j) \subset \nabla S(K_0, K_1, \Omega)$. Conversely, take $u \in S(K_0, K_1, \Omega)$. By definition if j is sufficiently large, then $u = 0$ on the intersection of Ω and a neighborhood of K_0^j and $u = 1$ on the intersection of Ω and a neighborhood of K_0^j. Let

$$\overline{u} = \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \operatorname{int} K_0^j, \\ 1 & \text{on } \operatorname{int} K_1^j. \end{cases}$$

Then $\overline{u} \in S(K_0^j, K_1^j, \Omega^j)$, $\nabla_{\Omega^j}\overline{u}$ is concentrated on $\Omega \setminus (K_0^j \cup K_1^j)$, and $\nabla_{\Omega^j}\overline{u} = \nabla_\Omega u$. Hence $\nabla_\Omega u \in \nabla S(K_0^j, K_1^j, \Omega^j)$. Thus (5.9) follows.

With the aid of Proposition 1 and (5.9) we have

$$M_{\mathscr{B},p'}(K_0^j, K_1^j, \Omega^j) \uparrow M_{\mathscr{B},p'}(K_0, K_1, \Omega). \tag{5.10}$$

Since $\Omega \subset \Omega^j$, it follows from Lemma 5.4 that

$$M^*_{\mathscr{B},p'}(K_0, K_1, \Omega^j) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega),$$

where the right hand side has been assumed to be finite. Hence, in view of Lemma 5.5, we can modify the above K_0^j, K_1^j and Ω^j so that Ω^j can be approximated from inside with respect to $M^*_{\mathscr{B},p'}(K_0, K_1, \Omega^j)$. Obviously $S(K_0^j, K_1^j, \Omega^j) \subset S(K_0, K_1, \Omega^j)$ and so $M_{\mathscr{B},p'}(K_0^j, K_1^j, \Omega^j) \le M_{\mathscr{B},p'}(K_0, K_1, \Omega^j)$. Now by (5.10) and Lemmas 5.4 and 5.7 we have

$$\begin{aligned} M_{\mathscr{B},p'}(K_0, K_1, \Omega) &= \lim M_{\mathscr{B},p'}(K_0^j, K_1^j, \Omega^j) \le \lim M_{\mathscr{B},p'}(K_0, K_1, \Omega^j) \\ &\le \lim M^*_{\mathscr{B},p'}(K_0, K_1, \Omega^j) \le M^*_{\mathscr{B},p'}(K_0, K_1, \Omega). \end{aligned}$$

Thus the lemma follows.

Combining all the Lemmas 5.1–5.3, we obtain Theorem 3.

6. Proof of Theorems 4 and 5

As was observed in the introduction, we have

$$M_p\big(\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big) = M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\,d\Gamma\big|\big).$$

Moreover, by (1.1)

$$c_0^{-1}\,ds \le w(x)^{1/p}\big|\sqrt{\mathscr{B}}\;dx\big| \le c_0\,ds$$

for a rectifiable curve γ, $dx = dx|_\gamma$ and $ds = |dx|$. Throughout this section we let $d\sigma = w(x)^{1/p}|\sqrt{\mathscr{B}}\;dx|$ for each rectifiable curve. We have $c_0^{-1} \le d\sigma/ds \le c_0$. The following lemma depends on the ingenious idea of Shlyk ([11]) which was digested by Ohtsuka ([9, Chapter 2]).

Lemma 6.1 (Shlyk–Ohtsuka). *Let $\varrho \in L^p_w(\mathbf{R}^n)$ be a positive lower semicontinuous function which is continuous on $\Omega \setminus (K_0 \cup K_1)$. Let K_0^j and K_1^j be sequences of compact sets such that $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^\infty K_0^j$, and $K_1 = \bigcap_{j=0}^\infty K_1^j$. Then for each $\varepsilon > 0$ we can construct a function ϱ' on Ω, $\varrho' \ge \varrho$, with the following properties*

(i) $\int_\Omega \varrho'^p w\,dx \le \int_\Omega \varrho^p w\,dx + \varepsilon$.

(ii) *Suppose for each j there is $\gamma_j \in \Gamma_j = \Gamma(K_0^j, K_1^j, \Omega)$ such that $\int_{\gamma_j} \varrho'\,d\sigma \le \alpha$. Then there exists $\widetilde{\gamma} \in \Gamma(K_0, K_1, \Omega)$ such that $\int_{\widetilde{\gamma}} \varrho\,d\sigma \le \alpha + \varepsilon$.*

Remark. The most difficult part of the lemma is the existence of $\widetilde{\gamma}$ inside Ω. It is rather easy to find a curve in the closure of Ω with the integral inequality of (ii). However, such a curve need not belong to $\Gamma(K_0, K_1, \Omega)$.

Proof of Lemma 6.1. The construction of ϱ' is as follows. Let $K^j = K_0^j \cup K_1^j$, $W^j = K^{j-1} \setminus \operatorname{int} K^j$ and $d_j = \operatorname{dist}(\partial K^{j-1}, \partial K^j) > 0$. By assumption $\inf_{W^j \cap \Omega} \varrho > 0$ and so we can find $\varepsilon_j \downarrow 0$ such that

$$\sum_{j=1}^{\infty} (1+\varepsilon_j^{-1})^p \varepsilon_j^{p+1} < \varepsilon, \tag{6.1}$$

$$c_0 \alpha \varepsilon_j < d_j \inf_{W^j \cap \Omega} \varrho. \tag{6.2}$$

We can find a sequence of compact subsets Ω_j of Ω increasing to Ω such that

$$\int_{\Omega \setminus \Omega_j} \varrho^p w \, dx < \varepsilon_j^{p+1}.$$

Let $V^j = (\Omega \setminus \Omega_j) \cap W^j$ and set

$$\varrho'(x) = \begin{cases} (1+\varepsilon_j^{-1})\varrho(x) & \text{if } x \in V^j, \\ \varrho(x) & \text{if } x \in \Omega \setminus (\cup V^j). \end{cases}$$

It is easy to see that (i) holds. In fact, observe from (6.1) that

$$\begin{aligned} \int_{\Omega} \varrho'^p w \, dx &= \sum_j \int_{V^j} [(1+\varepsilon_j^{-1})\varrho]^p w \, dx + \int_{\Omega \setminus (\cup V^j)} \varrho^p w \, dx \\ &\le \sum_j (1+\varepsilon_j^{-1})^p \varepsilon_j^{p+1} + \int_{\Omega} \varrho^p w \, dx < \varepsilon + \int_{\Omega} \varrho^p w \, dx. \end{aligned}$$

Now let us show (ii). For a moment we fix $j \ge 1$. By definition $\gamma_k \in \Gamma_j = \Gamma(K_0^j, K_1^j, \Omega)$ for $k \ge j$. Hence γ_k includes arcs γ_k' and γ_k'' such that γ_k' connects ∂K_0^j and ∂K_0^{j-1}; γ_k'' connects ∂K_1^j and ∂K_1^{j-1}. We claim that γ_k' and γ_k'' are not included in V^j. In fact, if $\gamma_k' \subset V^j$, then we would have from (6.2)

$$\alpha \ge \int_{\gamma_k} \varrho' \, d\sigma \ge \int_{\gamma_k'} \varrho' \, d\sigma \ge \varepsilon_j^{-1} \int_{\gamma_k'} \varrho \, d\sigma \ge \varepsilon_j^{-1} c_0^{-1} d_j \inf_{W^j \cap \Omega} \varrho > \alpha,$$

a contradiction. Therefore

$$\gamma_k \cap \big(\Omega_j \cap (K_0^{j-1} \setminus \operatorname{int} K_0^j)\big) \ne \emptyset \qquad \text{and} \qquad \gamma_k \cap \big(\Omega_j \cap (K_1^{j-1} \setminus \operatorname{int} K_1^j)\big) \ne \emptyset$$

for $k \ge j$. Observe that $\Omega_j \cap (K_0^{j-1} \setminus \operatorname{int} K_0^j)$ is compact. We can find a point $x_0^j \in \Omega_j \cap (K_0^{j-1} \setminus \operatorname{int} K_0^j)$ such that for some subsequence $\{\gamma_k^j\}_k$ there is a point in

γ_k^j converging to x_0^j. Hence, if we place a small closed ball $B_0^j \subset \Omega$ with center at x_0^j, then we may assume that the subsequence γ_k^j eventually intersects B_0^j. Since ϱ is continuous at $x_0^j \in \Omega \setminus K^j$, we can choose B_0^j so small that

$$\int_l \varrho\, d\sigma \le \varepsilon/2^{j+3} \qquad \text{for any segment } l \subset B_0^j. \tag{6.3}$$

In the same way we can choose a small closed ball $B_1^j \subset \Omega$ with center at $x_1^j \in \Omega_j \cap (K_1^{j-1} \setminus \operatorname{int} K_1^j)$ so that γ_k^j intersects B_1^j. We start this process from $j = 1$ and choose subsequences $\{\gamma_k^j\}_k$ inductively such that γ_k^j intersects B_0^j and B_1^j. See Figure 1.

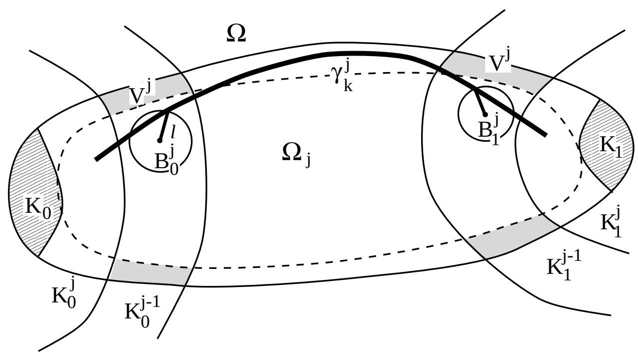

Figure 1.

Now let us consider the diagonal γ_k^k. Then γ_k^k intersects B_0^j and B_1^j for $1 \le j \le k$. To γ_k^k we add two suitable radii of B_i^j for each $i = 0, 1$ and for $1 \le j \le k$ so that we have a connected curve $\widetilde{\gamma}_k \in \Gamma(K_0^k, K_1^k, \Omega)$ crossing over $\{x_0^j, x_1^j\}_{j=1}^k$. By (6.3) we have

$$\int_{\widetilde{\gamma}_k} \varrho\, d\sigma \le \int_{\gamma_k^k} \varrho\, d\sigma + 2\sum_{j=1}^k \frac{\varepsilon}{2^{j+3}} \le \alpha + \frac{\varepsilon}{4}.$$

Let Γ_0 be the totality of all curves in $\Omega \setminus (K_0 \cup K_1)$ connecting x_0^1 and x_1^1. For $i = 0, 1$ let Γ_i^j be the totality of all curves in $\Omega \setminus (K_0 \cup K_1)$ connecting x_i^j and x_i^{j+1}. Then

$$\inf_{\gamma \in \Gamma_0} \int_\gamma \varrho\, d\sigma + \sum_{j=1}^k \inf_{\gamma \in \Gamma_0^j} \int_\gamma \varrho\, d\sigma + \sum_{j=1}^k \inf_{\gamma \in \Gamma_1^j} \int_\gamma \varrho\, d\sigma \le \int_{\widetilde{\gamma}_k} \varrho\, d\sigma \le \alpha + \frac{\varepsilon}{4}.$$

Therefore we can choose $C_0 \in \Gamma_0$ and $C_i^j \in \Gamma_i^j$ such that

$$\int_{C_0} \varrho\, d\sigma < \inf_{\gamma \in \Gamma_0} \int_\gamma \varrho\, d\sigma + \frac{\varepsilon}{2},$$

$$\int_{C_i^j} \varrho\, d\sigma < \inf_{\gamma \in \Gamma_i^j} \int_\gamma \varrho\, d\sigma + \frac{\varepsilon}{2^{j+3}}.$$

Let

$$\widetilde{\gamma} = \cdots + C_0^1 + C_0 + C_1^1 + \cdots.$$

See Figure 2.

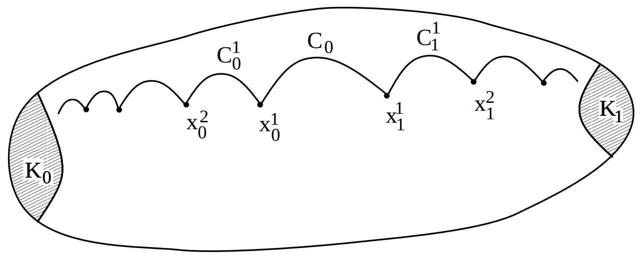

Figure 2.

Then $\widetilde{\gamma} \in \Gamma(K_0, K_1, \Omega)$ and

$$\int_{\widetilde{\gamma}} \varrho \, d\sigma \le \alpha + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + 2\sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+3}} = \alpha + \varepsilon.$$

The lemma is proved.

Proof of Theorem 4. Since $M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma\big|\big) = M_p\big(\big|\sqrt{\mathscr{B}}\, d\Gamma\big|\big)$, it is sufficient to show that

$$M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma_j\big|\big) \downarrow M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma\big|\big). \tag{6.4}$$

Let M be the right hand side and let $0 < \varepsilon < \frac{1}{2}$. By definition there is a nonnegative function $\tilde{\varrho}$ such that $\tilde{\varrho} \wedge w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma\big|$ and $\|\tilde{\varrho}\|_{p,w}^p < M + \varepsilon$. If necessary, adding a positive function on $\mathbf{R}^n$, we may assume that $\tilde{\varrho}$ is strictly positive on $\mathbf{R}^n$. Moreover, by the Vitali–Carathéodry theorem, we may assume that $\tilde{\varrho}$ is lower semicontinuous on $\mathbf{R}^n$. Let

$$(\tilde{\varrho})_r(x) = \int_{|y|<1} \tilde{\varrho}\big(x + r\alpha(x)y\big)\psi(y)\, dy$$

be as in (2.1) with $G = \Omega \setminus (K_0 \cup K_1)$. We observe that $(\tilde{\varrho})_r$ is $C^\infty(G)$ and in particular continuous on G. If r is sufficiently small, then $\|(\tilde{\varrho})_r\|_{p,w}^p < M + \varepsilon$.

By the uniform continuity of $w(x)^{1/p}\sqrt{\mathscr{B}(x)}$, we can take $\delta > 0$ such that if $x, z \in \Omega$ and $|x - z| < \delta$, then

$$\Big|w(x)^{1/p}\sqrt{\mathscr{B}(x)} - w(z)^{1/p}\sqrt{\mathscr{B}(z)}\,\Big| < c_0^{-1}\varepsilon, \tag{6.5}$$

where c_0 is the constant in (1.1). Let $0 < r < \delta$. We claim

$$(1+\varepsilon)(\tilde{\varrho})_r \wedge w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma\big|. \tag{6.6}$$

Let $\gamma \in \Gamma$. We have

$$\begin{aligned} \int_\gamma (\tilde{\varrho})_r \, d\sigma &= \int_{|y|<1} \psi(y)\, dy \int_\gamma \tilde{\varrho}\big(x + r\alpha(x)y\big)\, d\sigma(x) \\ &= \int_{|y|<1} \psi(y)\, dy \int_{T_{y,r}(\gamma)} \tilde{\varrho}(z)\, d\sigma\big(T_{y,r}^{-1}(z)\big), \end{aligned} \tag{6.7}$$

where $T_{y,r}(x) = x + r\alpha(x)y$. For a moment we fix y, $|y| < 1$. By the property of $\alpha(x)$ we see that $T_{y,r}(\gamma) \in \Gamma$ and so $\tilde{\varrho} \wedge d\sigma|_{T_{y,r}(\gamma)}$. Since $|x - z| < r < \delta$ with $z = T_{y,r}(x)$, it follows from (1.1) and (6.5) that

$$\begin{aligned} \big|w(z)^{1/p}\sqrt{\mathscr{B}(z)}\,\xi\big| &\le \big|w(x)^{1/p}\sqrt{\mathscr{B}(x)}\,\xi\big| + \big|\big(w(x)^{1/p}\sqrt{\mathscr{B}(x)} - w(z)^{1/p}\sqrt{\mathscr{B}(z)}\,\big)\xi\big| \\ &\le \big|w(x)^{1/p}\sqrt{\mathscr{B}(x)}\,\xi\big| + c_0^{-1}\varepsilon|\xi| \le (1+\varepsilon)\big|w(x)^{1/p}\sqrt{\mathscr{B}(x)}\,\xi\big| \end{aligned}$$

for any vectors $\xi \in \mathbf{R}^n$. Hence

$$(1+\varepsilon)\, d\sigma\big(T_{y,r}^{-1}(z)\big) \ge d\sigma(z) \qquad \text{for } z \in T_{y,r}(\gamma),$$

so that by (6.7) and $\tilde{\varrho} \wedge d\sigma|_{T_{y,r}(\gamma)}$,

$$(1+\varepsilon)\int_\gamma (\tilde{\varrho})_r \, d\sigma \ge \int_{|y|<1} \psi(y)\, dy \int_{T_{y,r}(\gamma)} \tilde{\varrho}(z)\, d\sigma(z) \ge 1.$$

Thus (6.6) is proved.

Now let $\varrho = (1+\varepsilon)(\tilde{\varrho})_r$ with sufficiently small $r > 0$. Then

$$\int_\gamma \varrho \, d\sigma \ge 1 \qquad \text{for any } \gamma \in \Gamma, \tag{6.8}$$

$$\int_\Omega \varrho^p w \, dx < (1+\varepsilon)^p (M + \varepsilon). \tag{6.9}$$

Let ϱ' be as in Lemma 6.1. We show that

$$\int_\gamma \varrho' \, d\sigma > 1 - 2\varepsilon \qquad \text{for } \gamma \in \Gamma_j$$

with sufficiently large j. In fact, suppose the contrary. Then there would be a sequence $\{j_k\}$ and curves $\gamma_k \in \Gamma_{j_k}$ such that

$$\int_{\gamma_k} \varrho' \, d\sigma \le 1 - 2\varepsilon.$$

By Lemma 6.1 we would find $\widetilde{\gamma} \in \Gamma$ such that

$$\int_{\widetilde{\gamma}} \varrho \, d\sigma \le 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon,$$

a contradiction to (6.8).

Now the proof is easy. We have $(1-2\varepsilon)^{-1}\varrho' \wedge w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma_j\big|$ for sufficiently large j and so

$$M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma_j\big|\big) \le \int_\Omega [(1-2\varepsilon)^{-1}\varrho']^p w \, dx \le (1-2\varepsilon)^{-p}\big((1+\varepsilon)^p(M+\varepsilon)+\varepsilon\big)$$

by (6.9). Hence, letting $j \to \infty$ and then $\varepsilon \to 0$, we obtain

$$\limsup_{j\to\infty} M_{p,w}\big(w^{1/p}\big|\sqrt{\mathscr{B}}\, d\Gamma_j\big|\big) \le M,$$

which yields (6.4). The theorem is proved.

For the proof of Theorem 5 we prepare two lemmas. With a slight abuse of notation, we say that a set $X \subset \Omega$ is (p,w)-exc. if $M_{p,w}(\Gamma_X) = 0$, where Γ_X is the family of curves in Ω which terminate at X. We say that a property holds (p,w)-a.e. if the set of points where the property fails to hold is (p,w)-exc. We shall employ the following lemmas given by Ohtsuka [9, Chapter 4].

Lemma 6.2. *Let u be a (p,w)-precise function in Ω and let Γ be a family of curves in Ω whose end points lie in Ω. Then u tends to the value of u at the end point along (p,w)-a.e. curve of Γ.*

Lemma 6.3. *Let u be a (p,w)-precise function in Ω. Then any function equal to u (p,w)-a.e. in Ω is (p,w)-precise in Ω.*

Proof of Theorem 5. Let K_0^j and K_1^j be as in Theorem 4. Take $u \in \mathscr{D}(K_0^j, K_1^j, \Omega)$. We put

$$\overline{u} = \begin{cases} 0 & \text{on } K_0^j \cap \Omega, \\ 1 & \text{on } K_1^j \cap \Omega, \\ u & \text{on } \Omega \setminus (K_0^j \cup K_1^j). \end{cases}$$

By the definition of $\mathscr{D}(K_0^j, K_1^j, \Omega)$ and Lemma 6.2 we have $u = 0$ (p,w)-a.e. on $K_0^j \cap \Omega$ and $u = 1$ (p,w)-a.e. on $K_1^j \cap \Omega$. Hence $u = \overline{u}$ (p,w)-a.e. in Ω and hence $\overline{u}$ is a (p,w)-precise function in Ω by Lemma 6.3. By definition $\overline{u} \in \mathscr{D}^*(K_0, K_1, \Omega)$. Hence $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) \le \mathscr{A}_p(\nabla \overline{u}) = \mathscr{A}_p(\nabla u)$, and taking the infimum with respect to u, we obtain $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) \le C_{\mathscr{A},p}(K_0^j, K_1^j, \Omega)$. By Theorem 1 we have $C_{\mathscr{A},p}(K_0^j, K_1^j, \Omega) = M_p\big(\big|\sqrt{\mathscr{B}}\, d\Gamma(K_0^j, K_1^j, \Omega)\big|\big)$ and $C_{\mathscr{A},p}(K_0, K_1, \Omega) = M_p\big(\big|\sqrt{\mathscr{B}}\, d\Gamma\big|\big)$. By Theorem 4 the first quantity tends to the second as $j \to \infty$. Hence $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) \le C_{\mathscr{A},p}(K_0, K_1, \Omega)$. The opposite inequality is obvious and we have $C^*_{\mathscr{A},p}(K_0, K_1, \Omega) = C_{\mathscr{A},p}(K_0, K_1, \Omega)$. The second assertion of the theorem readily follows from Theorems 1–3.

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Received 14 April 1997